Fractional diffusion for kinetic equations

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Diffusion fractionnaire pour des équations cinétiques

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Résumé

Cette thèse porte principalement sur les limites de diffusion fractionnaire pour des équations cinétiques avec équilibre à queue lourde. L'objectif est d'améliorer et de généraliser la méthode spectrale développée par Gilles Lebeau et Marjolaine Puel pour l'équation de Fokker-Planck en dimension 1 en toute dimension. Dans un travail en collaboration avec Marjolaine Puel et inspiré des travaux de Herbert Koch pour l'équation de KdV non linéaire, nous avons construit un couple-propre, solution du problème spectral associé à l'opérateur de Fokker-Planck. Ce résultat, établi d'abord en dimension 1 puis généralisé en dimension supérieure, a pour conséquence directe la limite de diffusion pour l'équation de Fokker-Planck. La valeur propre donne la bonne échelle en temps ainsi que le coefficient de diffusion, tandis que la fonction propre est utilisée comme fonction test dans la méthode des moments. Ces résultats ont été obtenus de manière indépendante aux travaux récents d'Emeric Bouin et Clément Mouhot (2022), qui ont été réalisés un problème spectral et des techniques différents.

La deuxième partie de cette thèse porte sur l'étude de propagation de régularité Gevrey pour les solutions du système de Vlasov-Navier-Stokes. Dans ce système couplé de l'équation cinétique de Vlasov et des équations macroscopiques de Navier-Stokes, nous montrons que pour une donnée initiale dans la classe de fonctions Gevrey, la solution du système va rester dans la classe Gevrey tant qu'il existe des solutions Sobolev. On utilise une méthode basée sur des estimations d'énergie et la caractérisation de la classe Gevrey par Fourier et les espaces de Sobolev. Cette méthode est inspirée des travaux de Levermore et Oliver, Kukavica et Vicol pour Euler et Velozo Ruiz pour Vlasov-Poisson.

Mots clés : limite de diffusion, diffusion fractionnaire, équations cinétiques, équation de Fokker-Planck, équation de Boltzmann linéaire, équilibre à queue lourde, problème spectral, solution-propre, méthode spectrale, régularité Gevrey, propagation de régularité, système de Vlasov-Navier-Stokes.

Abstract

This thesis mainly deals with fractional diffusion limits for kinetic equations with heavy-tail equilibrium. The objective is to improve and generalize the spectral method developed by Gilles Lebeau and Marjolaine Puel for the Fokker-Planck equation in dimension 1 to any dimension. In a work in collaboration with Marjolaine Puel and inspired by the work of Herbert Koch for the nonlinear KdV equation, we have constructed an eigenpair, solution to the spectral problem associated with the Fokker-Planck operator. This result, first established in dimension 1 then generalized in higher dimension, gave as a direct consequence the diffusion limit for the Fokker-Planck equation. The eigenvalue gives the correct scale in time as well as the diffusion coefficient, while the eigenfunction is used as a test function in the moments method. These results were obtained independently of recent work by Emeric Bouin and Clément Mouhot (2022), which used a different spectral problem and different techniques.

The second part of this thesis deals with the study of propagation of Gevrey regularity for the solutions of the Vlasov-Navier-Stokes system. In this coupled system of the kinetic equation of Vlasov and the macroscopic equations of Navier-Stokes, we show that for an initial datum in the class of Gevrey functions, the solution of the system will remain in the Gevrey class as long as Sobolev solutions exist. We use a method based on energy estimates and characterization of the Gevrey class by Fourier and Sobolev spaces. This method is inspired by the work of Levermore and Oliver, Kukavica and Vicol for Euler and Velozo Ruiz for Vlasov-Poisson.

Keywords: diffusion limit, fractional diffusion, kinetic equations, Fokker-Planck equation, linear Boltzmann equation, heavy-tail equilibrium, spectral problem, eigen-solution, spectral method, Gevrey regularity, propagation of regularity, Vlasov-Navier-Stokes system.

Structure et langue du manuscrit

Cette thèse est constituée de six chapitres. Le premier (Chapitre 0) est rédigé en langue française et constitue une introduction générale, présentant un résumé des résultats obtenus durant la thèse. Ces résultats sont ensuite exposés en détail dans les cinq chapitres suivants, qui sont rédigés en langue anglaise.

Structure and language of the manuscript

This thesis consists of six chapters. The first one (Chapter 0) is written in French and is dedicated to a general introduction, providing a summary of the results obtained during the thesis. These results are then presented in detail in the five subsequent chapters, which are written in English.

Contents

Introduction et synthèse

Cette thèse porte principalement sur les *limites de diffusion fractionnaire pour des équations cinétiques* telles que Boltzmann linéaire et Fokker-Planck, avec « *équilibre à queue lourde* » ou ce qu'on entend parfois par *équilibre à décroissance lente.* Ce sujet est l'objet des Parties I et II. Enfin, la Partie III porte sur l'étude de la *propagation de régularité Gevrey pour les solutions du système de Vlasov-Navier-Stokes* (VNS).

Ce chapitre est structuré comme suit : nous commençons par une motivation théorique pour l'étude des limites de diffusion et la présentation des différentes échelles de descriptions dans la première section. La deuxième et la troisième sections sont consacrées aux théorie cinétique et équations macroscopiques respectivement, où nous donnons des exemples d'équations classiques dans chacune, avec plus de détail pour les équations sur lesquelles nous reviendrons dans ce manuscrit. La quatrième section est dédiée au contexte historique des limites de diffusion pour les équations de Boltzmann linéaire et Fokker-Planck et à la dérivation formelle de l'équation de diffusion de ces deux dernières. Enfin dans la section cinq, nous présentons les principaux résultats obtenus pendant la thèse.

0.1 Motivations et cadre de la thèse

L'objectif de la partie principale de cette thèse est de justifier rigoureusement la dérivation de l'équation de diffusion fractionnaire (qui est une équation écrite au niveau macroscopique) du modèle cinétique de Fokker-Planck avec équilibre à queue lourde (qui est une équation écrite au niveau mésoscopique), et dans ce cadre le problème mathématique central abordé dans ce travail consiste à construire un couple-propre "fonction propre - valeur propre" pour l'opérateur de Fokker-Planck, en tenant compte de la partie advection.

Derrière cette question on trouve une motivation théorique, puisque le passage entre deux niveaux de descriptions présente une partie du sixième problème de Hilbert comme nous allons le voir dans la sous-section qui suit, et on trouve une motivation d'un point de vue application comme nous allons le voir dans la sous-section 0.4.1, puisque l'équation dérivée est beaucoup plus simple que l'équation de départ.

L'objectif de la dernière partie est d'étudier la propagation de régularité des solutions du système de VNS qui sont dans la classe de fonctions Gevrey (ou analytiques réelles), c'est-à-dire, pour une donnée initiale dans la classe Gevrey (ou analytique), est-ce que la solution du dernier système – tant qu'elle existe – va rester dans la classe Gevrey (ou va rester analytique) ? et quelle est la relation entre le rayon de régularité (ou d'analyticité) de la solution au temps $t > 0$ et au temps initial $t = 0$?

0.1.1 Le sixième problème de Hilbert

Le sixième problème de Hilbert est une motivation théorique de l'étude de limite hydrodynamique dont la question est : peut-on axiomatiser la mécanique, et en particulier : *Peut-on établir rigoureusement les équations fondamentales de la mécanique des fluides à partir d'un modèle microscopique gouverné par les lois de Newton ?*

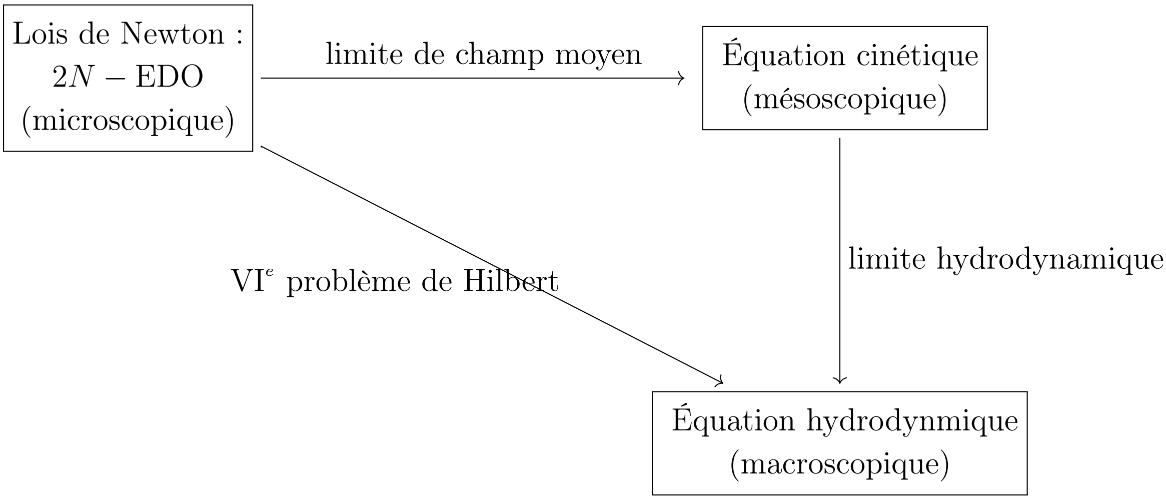
Ce célèbre problème de *physique statistique* a été étudié de façon intensive, comme en témoigne l'abondance des travaux cités dans les ouvrages de synthèse [Spo12] et [KL98].

D. Hilbert proposait de passer des équations de Newton à l'équation de Boltzmann, puis de l'équation de Boltzmann aux diverses équations de l'hydrodynamique comme il l'énonça à l'occasion du Congrès international des mathématiciens en 1900 [Hil02] : « le livre de M. Boltzmann sur les Principes de la Mécanique nous incite à établir et à discuter du point de vue mathématique d'une manière complète et rigoureuse les méthodes basées sur l'idée de passage à la limite, et qui de la conception atomique nous conduisent aux lois du mouvement des continua. »

On perd ainsi de la généralité, car l'équation de Boltzmann ne peut aboutir qu'à une classe restreinte d'équations hydrodynamiques, i.e. avec loi d'état des gaz parfaits. Ainsi, il est intéressant de regarder des **différents modèles** cinétiques.

Le passage d'un système de particules gouverné par les lois de Newton aux équations cinétiques est le cadre de ce qu'on appelle « *limite de champ moyen* », tandis que le passage des équations cinétiques aux équations hydrodynamiques s'appelle « *limite (ou approximation) hydrodynamique* ». Pour autant, chacune des deux étapes demeure extrêmement délicate.

On présente les différents passages entre les différentes échelles et équations dans le schéma suivant :



La limite de diffusion – que nous allons voir plus en détail dans la section 0.4 – rentre dans le cadre des limites hydrodynamiques. Afin de mieux comprendre ces passages entre les différents modèles, nous allons présenter les différentes échelles ou niveaux de descriptions dans la sous-section qui suit.

0.1.2 Différentes descriptions et modèles

Un gaz, ou tout autre système constitué d'un nombre très grand de particules, peut être décrit de plusieurs manières différentes et dans différentes échelles. On peut le décrire par exemple d'une manière très précise en suivant le mouvement de chaque particule, comme on peut se placer un petit peu plus loin et regarder un nuage de particules au lieu de regarder les particules de près "une par une", comme on peut aussi mesurer certaines quantités – les *observables* – qui peuvent nous fournir des informations intéressantes telles que la température, la vitesse, la pression etc.

Dans cette sous-section, nous allons présenter trois niveaux de descriptions, à savoir : microscopique, macroscopique et mésoscopique, et nous précisons dans chacun le modèle qui correspond : système d'EDO ou EDS, équations hydrodynamique et équations cinétiques. Il existe d'autres modèles, que nous n'allons pas présenter ici, et qui jouent un rôle très important dans différents domaines tel que la physique, on peut citer les *modèles quantiques* (voir [Laf19] pour la description, des exemples ainsi que le passage vers un autre modèle).

0.1.2.1 Description microscopique

L'échelle microscopique englobe toutes les notions physiques – grandeurs, éléments constituants, mouvements, phénomènes – dont la manifestation a lieu dans un domaine spatial indiscernable à l'œil nu, c'est-à-dire de dimension inférieure au dixième de millimètre[1].

Considérons par exemple la modélisation mathématique d'un gaz formé d'une assemblée de N particules identiques en interaction (où N est très grand, typiquement de l'ordre du nombre d'Avogadro $\approx 6.23 \times 10^{23}$) évoluant dans l'espace $\mathbb{R}^d$ ($d = 3$ étant le cadre le plus naturel). Ainsi, si l'on suppose que la position x_i et la vitesse v_i sont suffisantes pour décrire l'état d'une particule $p_i, i = 1, \ldots, N$, alors l'état du gaz est défini par un point représentatif dans l'espace des phases $\left(\mathbb{R}^d_x \times \mathbb{R}^d_v\right)^N$. Une telle description est dite *microscopique.*

On peut alors écrire les lois de Newton sous la forme d'un immense système de $2N$ équations différentielles du premier ordre, faisant intervenir les positions $(x_i(t))_{i=1}^N$ et les vitesses $(v_i(t))_{i=1}^N$ de toutes les particules, ainsi que les forces d'interactions $(\mathcal{F}_i(t))_{i=1}^N$:

$$\begin{cases} \dot{x}_i(t) = v_i(t), & t \geqslant 0, \\ \dot{v}_i(t) = \mathcal{F}_i(t), & t \geqslant 0. \end{cases}$$

La force $\mathcal{F}$ est donnée selon les situations, donnons deux exemples classiques :

- Le premier est le plus simple, c'est le cas où les forces sont nulles, c'est-à-dire que les particules sont uniquement soumises à leurs inerties et se déplacent "librement" en ligne droite :

$$v_i(t) = v_i(0), \qquad x_i(t) = x_i(0) + tv_i(0), \qquad \forall t \geqslant 0,\ \forall i \in \{1, \ldots, N\}. \tag{0.1.1}$$

- Le second correspond au cas où les particules interagissent deux à deux via un potentiel d'interaction $\Phi = \Phi(x)$, dans ce cas :

$$\mathcal{F}_i(t) = -\sum_{j \neq i} \nabla_x \Phi\big(x_i(t) - x_j(t)\big). \tag{0.1.2}$$

  Cette situation a lieu par exemple lorsque les particules sont chargées positivement et se repoussent mutuellement mais que le gaz est suffisamment dilué pour que les interactions faisant intervenir plus de deux particules soient négligées.

0.1.2.2 Description macroscopique

L'échelle macroscopique englobe toutes les notions physiques – grandeurs, éléments constituants, mouvements, phénomènes – dont la manifestation est observable (au contraire de microscopique), soit directement, soit par l'intermédiaire d'un instrument d'observation ou de mesure. Ainsi, le monde macroscopique est constitué de tout ce qui peut se mesurer et s'observer à notre échelle. La taille des objets décrits varie du millimètre à quelques dizaines de mètre.

[1] Le monde microscopique est constitué de ce qui est trop petit pour être observé directement.

Au niveau microscopique, la modélisation mathématique nous conduit à des systèmes de N équations différentielles. Dans une description macroscopique, on rencontre plutôt des modèles “hydrodynamiques”, donc des équations aux dérivées partielles de la physique ou de *la mécaniques des fluides* telles que les équations d'Euler et Navier-Stokes, l'équation de la chaleur ou diffusion, l'équation des ondes ...

0.1.2.3 Description mésoscopique

Le niveau mésoscopique est un niveau de description intermédiaire entre les deux niveaux précédents, c'est-à-dire le niveau microscopique et le niveau macroscopique.

En thermodynamique, l'échelle mésoscopique est suffisamment étendue pour inclure un grand nombre de particules (de telle sorte que leurs propriétés statistiques ne fluctuent pas significativement), tout en restant suffisamment fine pour que les grandeurs thermodynamiques (pression, température, etc.) restent locales (ponctuelles à l'échelle macroscopique). L'échelle mésoscopique dépend donc de la densité des corps étudiés : de l'ordre du nanomètre pour les corps condensés, jusqu'à plusieurs centaines de kilomètres pour des gaz raréfiés.

À ce niveau de description, on trouve les équations cinétiques[2] que nous allons voir un peu plus en détail dans la section qui suit.

0.2 Théorie cinétique

L'objet de la théorie cinétique est la modélisation d'un gaz (ou plasma, ou tout autre système constitué d'un grand nombre de particules) par une fonction de distribution dans l'espace des phases associé aux atomes. Nous supposerons toujours ce système constitué de particules identiques, non relativistes, non quantiques, ayant pour seuls degrés de liberté des mouvements de translation. Sous ces hypothèses simplificatrices, si le gaz est contenu dans un domaine (borné ou non) $\Omega \subset \mathbb{R}^d$ et observé sur un intervalle de temps $[0, T]$ (ou $[0, +\infty[$), le modèle associé est une fonction $f(t, x, v)$ positive définie sur $[0, T] \times \Omega \times \mathbb{R}^d$. La distribution $f(t, x, v)$, représente la densité de particules qui se trouve au temps t à la position x avec une vitesses v.

Dans cette description, les grandeurs macroscopiques mesurables (les “observables”) peuvent êtres exprimées en fonction d'intégrales de la forme $\int f(t, x, v)\varphi(v)\mathrm{d}v$. En particulier (en variables adimensionnées), en un point x et à l'instant t, la densité locale ρ, la vitesse macroscopique locale u et la température locale T sont définies par

$$\rho := \int f(t, x, v)\mathrm{d}v, \quad \rho u := \int f(t, x, v)v\mathrm{d}v \quad \text{et} \quad \rho|u|^2 + d\rho T := \int f(t, x, v)|v|^2\mathrm{d}v.$$

[2] Le mot cinétique qualifie qu'on prend en considération les vitesses des particules.

De manière générale, les équations cinétiques se décomposent en deux parties, une partie *transport*, qui traduit le fait que les particules se déplacent avec une vitesse v en suivant des lignes droites, et une partie *force* qui traduit le fait que le mouvement des particules est dû à une force extérieure ou juste aux collisions des particules entre elles par exemple. Elles prennent la forme suivante :

$$\partial_t f + v \cdot \nabla_x f = Q(f). \tag{0.2.1}$$

Dans le reste de cette section, nous allons donner quelques exemples classiques et fondamentaux dans la théorie cinétique, chacun ayant son domaine de validité. Commençons par l'**équation de Boltzmann** [Bol72]. Cette équation modélise un gaz suffisamment raréfié[3], dans lequel les particules interagissent par des *collisions* élastiques, réversibles, localisées et instantanées[4] (voir [Vil98], [Cer88] et [Cer00] et les références incluses pour l'étude mathématiques et les applications de cette équation, ou de ses variantes, à des problèmes théoriques ou concrets). On note par Q_B l'opérateur de collision quadratique de Boltzmann défini par

$$Q_B(f,f)(x,v) = \iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{B}(|v_* - v|, \sigma)(f' f'_* - f f_*) \mathrm{d}v_* \mathrm{d}\sigma, \tag{0.2.2}$$

où l'on a utilisé la notation classique $f' = f(x, v')$, $f_* = f(x, v_*)$ et $f'_* = f(x, v'_*)$ et où (v', v'_*) et (v, v_*) sont les vitesses avant et après le choc respectivement, de deux particules qui subissent une collision et le vecteur unitaire $\sigma = \frac{v'-v'_*}{|v'-v'_*|} \in \mathbb{S}^{d-1}$ donne les différents angles possibles après collision. Le noyau de collision B s'appelle la *section efficace*.

L'équation (0.2.2) implique la conservation de la quantité de mouvement et de l'énergie cinétique au cours des collisions (sous l'hypothèse de chocs élastiques pour cette dernière).

Mais commençant d'abord par étudier l'équation cinétique la plus basique. C'est le cas lorsque les particules ne sont soumises à aucune force, elles se déplacent en ligne droite et leur densité $f = f(t, x, v)$ suit ce qu'on appelle l'**équation de transport libre** :

$$\partial_t f + v \cdot \nabla_x f = 0,$$

qui est la formulation cinétique du modèle (0.1.1), et dont la solution n'est rien d'autre que la translation de la distribution à l'instant initial :

$$f(t, x, v) = f_0(x - tv, v).$$

Lorsqu'en revanche les particules sont soumises à une forces extérieure F (à valeur dans $\mathbb{R}^d$), il convient d'ajouter un terme modélisant l'effet de celles-ci, on obtient alors

[3] Cette hypothèse permet de négliger les collisions faisant intervenir plus de deux particules.
[4] C'est-à-dire se déroulant sur des échelles de temps et d'espace très inférieures aux échelles macroscopique.

l'**équation de Vlasov**. Cette équation décrit l'évolution temporelle de la fonction de distribution f des particules dans un plasma ou un faisceau de particules chargées en négligeant l'effet des collisions binaires. Elle intervient par exemple dans la description des plasmas qui apparaissent en astrophysique pour décrire les étoiles, les gaz interstellaires et le vent solaire ..., comme elle intervient aussi dans de nombreuses applications industrielles. Elle a été obtenue par Anatoli Vlassov [Vla] et s'écrit :

$$\partial_t f + v \cdot \nabla_x f + \mathcal{F} \cdot \nabla_v f = 0. \tag{0.2.3}$$

Selon la force $\mathcal{F}$ exercée (ou prise), cette dernière équation peut nous conduire à plusieurs autres équations ou même systèmes couplés d'équations tels que *Vlasov-Poisson*, dans le cas où $\mathcal{F}$ est une force électrique qui dérive d'un potentiel $\Phi = \Phi(x)$ (i.e. $\mathcal{F} = \nabla_x \Phi$) et que ce dernier satisfait une équation de Poisson, *Vlasov-Maxwell* dans le cas d'une force électromagnétique, etc.

Nous terminons cette section par deux autres exemples – sur lesquels nous reviendrons dans ce manuscrit – que nous allons présenter dans les deux sous-sections qui suivent.

0.2.1 Équation de Boltzmann linéaire

Dans le cas où l'on considère un ensemble de particules diluées dans un milieu à l'*équilibre thermodynamique* (que nous allons préciser) dont on connait le profil des vitesses $F(v)$, l'équation de **Boltzmann linéaire** donne un modèle simplifié, qui prend la forme (0.2.1) avec Q donné par

$$Q(f) := \int_{\mathbb{R}^d} \mathrm{b}(x, v, v_*)(f_* F - f F_*) \mathrm{d}v_*, \tag{0.2.4}$$

où l'on a utilisé les mêmes notations que dans (0.2.2).

Par définition, une distribution $F(x, v)$ est appelée *équilibre local* pour une équation cinétique collisionnelle, si elle est invariante par l'action de l'opérateur de collision :

$$Q(F) = 0.$$

L'étude de cette équation dépend du noyau de collision b ainsi que l'équilibre F. Dans ce qui suit, nous allons donner des hypothèses standards et classiques sur le noyau b, et nous terminons la sous-section par une proposition qui résume les principales propriétés de l'opérateur Q.

Tout d'abord, posons $\sigma := \sigma(x, v, v_*) := \mathrm{b}(x, v, v_*) F(v)$. L'opérateur de collision Q peut être décomposé en un terme de “gain” et un terme de “perte” comme suit :

$$Q(f) = Q^+(f) - Q^-(f),$$

avec

$$Q^+(f) := \int_{\mathbb{R}^d} \sigma(x,v,v_*) f(v_*) \mathrm{d}v_* \qquad \text{et} \qquad Q^-(f) := \nu(x,v) f, \tag{0.2.5}$$

où ν est la *fréquence de collision* définie par

$$\nu(x,v) := \int_{\mathbb{R}^d} \mathrm{b}(x,v,v_*) F(v_*) \mathrm{d}v_* = \int_{\mathbb{R}^d} \sigma(x,v,v_*) \frac{F(v_*)}{F(v)} \mathrm{d}v_* = \frac{Q^+(F)}{F(v)}. \tag{0.2.6}$$

Notons que sous des hypothèses raisonnables sur la section efficace σ (ou sur le noyau $\mathrm{b} = \sigma F^{-1}$), l'existence d'une unique fonction d'équilibre $F(x,v) \geqslant 0$ satisfaisant

$$Q(F) = 0 \qquad \text{et} \qquad \int_{\mathbb{R}^d} F(x,v) \mathrm{d}v = 1 \quad \text{p.p.t.} \quad x \in \mathbb{R}^d,$$

est en fait une conséquence du théorème de Krein-Rutman (voir [DGP00] pour plus de détails). Un cas particulier dans lequel cette condition est satisfaite est lorsque b est tel que :

- Une fonction positive, localement intégrable dans $\mathbb{R}^{3d}$ et pour tout $x, v, v_* \in \mathbb{R}^d$,

  $$\mathrm{b}(x,v,v_*) = \mathrm{b}(x,v_*,v).$$

  Dans ce cas, on dit que la section efficace σ satisfait un *principe d'équilibre détaillé* ou un *principe de micro-réversibilité.*

Sinon, l'hypothèse que nous allons donner maintenant concerne la continuité et la coercivité de l'opérateur Q.

- Supposons qu'il existe une constante positive C tel que :

  $$\int_{\mathbb{R}^d} \frac{\nu(x,v_*)}{\mathrm{b}(x,v,v_*)} F(v_*) \mathrm{d}v_* \leqslant C, \qquad \forall x, v \in \mathbb{R}^d.$$

Dans tout le manuscrit, L^p_ω désigne l'espace L^p avec la mesure $\omega \mathrm{d}v$ (ou $\omega \mathrm{d}v\mathrm{d}x$). Par exemple, $L^2_{\nu F^{-1}}(\mathbb{R}^d)$ est l'espace défini par

$$L^2_{\nu F^{-1}}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R};\ \int_{\mathbb{R}^d} |f|^2 \frac{\nu}{F} \,\mathrm{d}v < \infty \right\}.$$

Alors, sous les deux dernières hypothèses sur b et si on suppose de plus que $F \in L^1_\nu(\mathbb{R}^d)$, on a la Proposition suivante :

Proposition 0.2.1. *Soient f et g deux fonctions dans $L^2_{\nu F^{-1}}(\mathbb{R}^d)$. Alors, nous avons les assertions suivantes :*

1. *L'opérateur $Q : L^1_\nu \longrightarrow L^1$ est borné et conservatif, ainsi l'équation (0.2.4) préserve*

la masse totale de la distribution f[5] *et on a :*

$$\int_{\mathbb{R}^d} Q(f)\ \mathrm{d}v = 0, \quad \textit{pour tout } f \in L^1_\nu(\mathbb{R}^d).$$

2. *L'opérateur* Q *est auto-adjoint par rapport à la mesure* $\frac{\mathrm{d}v}{F}$ *:*

$$\int_{\mathbb{R}^d} Q(f)\ g\ \frac{\mathrm{d}v}{F} = \int_{\mathbb{R}^d} f\ Q(g)\ \frac{\mathrm{d}v}{F}. \tag{0.2.7}$$

3. *L'opérateur* $\frac{1}{\nu}Q$ *est borné dans* $L^2_{\nu F^{-1}}$ *et est dissipatif. De plus,*

$$\int_{\mathbb{R}^d} Q(f)f\ \frac{\mathrm{d}v}{F} \leqslant -\frac{1}{2C}\int_{\mathbb{R}^d} |f-\rho F|^2\ \frac{\nu \mathrm{d}v}{F}, \quad \forall f \in L^2_{\nu F^{-1}}. \tag{0.2.8}$$

4. *Le noyau de* Q *est de dimension* 1 *et est engendré par* F *:* $\quad \mathrm{Ker}(Q) = \langle F \rangle$.

Les points de la Proposition au dessus ont été récolté dans [MMM11, Mel10, DGP00].

Des exemples où toutes les hypothèses et propriétés précédentes sont satisfaites et lorsque b est donnée par :

$$\mathrm{b}(x, v, v_*) = \nu_0(x)\langle v\rangle^\beta \langle v_*\rangle^\beta \quad \text{ou} \quad \mathrm{b}(x, v, v_*) = \nu_0(x)\langle v - v_*\rangle^\beta,$$

avec $0 < \tilde{\nu}_1 \leqslant \nu_0(x) \leqslant \tilde{\nu}_2$ pour tout $x \in \mathbb{R}^d$ et $\beta \in \mathbb{R}$ tel que, $\langle v\rangle^\beta F \in L^1(\mathbb{R}^d)$ et où $\langle v\rangle := \sqrt{1+|v|^2}$. Dans les deux cas, on obtient

$$0 < \nu_1\langle v\rangle^\beta \leqslant \nu(x,v) \leqslant \nu_2\langle v\rangle^\beta, \qquad \forall x, v \in \mathbb{R}^d. \tag{0.2.9}$$

L'inégalité (0.2.9) et les conditions qui précèdent la Proposition 0.2.1 sont également remplies pour un noyau de collision (plus physique) satisfaisant le principe d'équilibre détaillé donné par

$$\mathrm{b}(x, v, v_*) = \nu_0(x)|v - v_*|^\beta.$$

Voir Remarks & Examples 3.1 dans [MMM11] pour une discussion des hypothèses sur la section efficace, et Lemma 6.1 pour la preuve des énoncés dans certains cas.

Dans ce dernier exemple, le noyau b présente une singularité au point $v = v_*$ pour des valeurs de β assez négative. Ainsi, on l'opérateur Q devient *non local*. On revient sur ce point dans la sous-section 0.3.2.

Dans tous les exemples précédents, si β est non nul, la fréquence de collision ν n'est pas bornée par deux constantes strictement positives (ce qui est dû au fait que la section efficace n'est pas bornée et dégénère pour les grandes vitesses), et l'inégalité (0.2.8) présente deux mesures différentes dans les deux intégrales de l'inégalité. Ainsi, l'opérateur Q n'a pas de *trou spectral*.

[5]Pour $F \in L^1_\nu$ on a : $L^2_{\nu F^{-1}} \subset L^1_\nu$.

0.2.2 Équation de Fokker-Planck

Un autre exemple d'équation cinétique très utilisée en physique des plasmas est l'**équation de Fokker-Planck cinétique**. Cette équation décrit d'une manière déterministe le mouvement Brownien des particules. Elle prend aussi la forme (0.2.1) où l'opérateur collisionnel Q est un opérateur différentiel cette fois-ci, donné par

$$Q(f) := \operatorname{div}_v \left(F\nabla_v \left(\frac{f}{F} \right) \right) = \Delta_v f + \operatorname{div}_v \big(E_v(v) f \big). \tag{0.2.10}$$

En utilisant la théorie des probabilités, on peut écrire l'équation de Fokker-Planck d'un point de vue Lagrangien semblable à celui du modèle à N particules. Dans ce cas, $X(t)$ et $P(t)$ deviennent des variables aléatoires et on obtient une équation stochastique qui s'écrit

$$\begin{cases} \mathrm{d}X = P\mathrm{d}t, \\ \mathrm{d}P = -E_v(P)\mathrm{d}t + \mathrm{d}B, \end{cases}$$

où $B = B(t)$ désigne le mouvement brownien. On voit que le Laplacien, qui correspond au mouvement Brownien, peut être vu comme étant un terme aléatoire dû à l'agitation thermique alors que la force E_v peut être vue comme correspondant à une force de friction sur les vitesses. On peut retrouver des modèles de type Fokker-Planck en effectuant une limite dite de collisions rasantes à partir d'une équation de Boltzmann Linéaire (voir par exemple [LT04]). L'opérateur de Fokker-Planck présente aussi de nombreuses similitudes avec l'opérateur de Landau, voir partie II, chapitre 3 dans [Vil98] pour les liens entre ces deux derniers.

Dans le reste de cette sous-section, nous donnons un théorème d'existence de solutions pour l'équation de Fokker-Planck ainsi que les principales propriétés de l'opérateur différentiel Q.

D'abord, on complète les équations (0.2.1) et (0.2.10) par une donnée initiale f_0. Ainsi,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left(F\nabla_v \left(\frac{f}{F} \right) \right), & x \in \mathbb{R}^d, v \in \mathbb{R}^d, t > 0, \\ f(0,x,v) = f_0(x,v), & x \in \mathbb{R}^d, v \in \mathbb{R}^d. \end{cases} \tag{0.2.11}$$

Le cadre fonctionnel de l'étude de l'équation (0.2.11) a été défini dans [NP15] où nous définissons les espaces fonctionnels ad hoc

$$Y_F^p \big(\mathbb{R}^{2d} \big) := L^p \big(\mathbb{R}^d; L^p_{F^{1-p}}(\mathbb{R}^d) \big), \tag{0.2.12}$$

où

$$L^p_{F^{1-p}}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R};\ \int_{\mathbb{R}^d} \frac{|f|^p}{F^{p-1}}\, \mathrm{d}v < \infty \right\}$$

et

$$L^\infty_{F^{-1}}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R};\ f/F \in L^\infty(\mathbb{R}^d) \right\}.$$

On définit l'espace V par

$$V := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R} \ \int_{\mathbb{R}^d} \frac{|f|^2}{F} \, \mathrm{d}v < \infty \text{ et } \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{F} \right) \right|^2 F \mathrm{d}v < \infty \right\},$$

V' étant son dual. Enfin, l'espace Y est donné par

$$Y := \left\{ f \in L^2 \left([0,T] \times \mathbb{R}^d, V \right) ; \ \partial_t f + v \cdot \nabla_x f \in L^2 \left([0,T] \times \mathbb{R}^d, V' \right) \right\}. \tag{0.2.13}$$

Une fois les espaces fonctionnels sont définis, on peut énoncer la Proposition suivante :

Proposition 0.2.2 ([NP15]). *Soient f et g deux fonctions régulières dans V. Alors, nous avons les assertions suivantes :*

1. *L'opérateur Q est conservatif, ainsi l'équation (0.2.11) préserve la masse totale de la distribution f et on a :*
$$\int_{\mathbb{R}^d} Q(f) \, \mathrm{d}v = 0, \quad \text{pour tout } f \in V.$$
2. *L'opérateur Q est auto-adjoint par rapport à la mesure $\frac{\mathrm{d}v}{F}$:*
$$\int_{\mathbb{R}^d} Q(f) \ g \ \frac{\mathrm{d}v}{F} = - \int_{\mathbb{R}^d} \nabla_v \left(\frac{f}{F} \right) \cdot \nabla_v \left(\frac{g}{F} \right) F^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} f \ Q(g) \ \frac{\mathrm{d}v}{F}. \tag{0.2.14}$$
3. *L'opérateur Q est dissipatif :*
$$\int_{\mathbb{R}^d} Q(f) \ f \ \frac{\mathrm{d}v}{F} = - \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{F} \right) \right|^2 F \, \mathrm{d}v \leqslant 0. \tag{0.2.15}$$
4. *Le noyau de Q est de dimension 1 et est engendré par F :* $\quad \mathrm{Ker}(Q) = \langle F \rangle$.
5. *L'opérateur Q est continu de V dans V'.*

Remarque 0.2.3. Par analogie avec l'inégalité (0.2.8), l'opérateur de Fokker-Planck n'a pas de trou spectral, dans le cas où l'équilibre est de la forme $F(v) = \frac{C_{\beta,d}}{\langle v \rangle^\beta}$ par exemple, puisque par l'inégalité de Hardy-Poincaré [BDGV10], sur laquelle on revient dans le chapitre 3 (Lemme 3.2.5 et la Remarque qui le suit), on a :

$$\Lambda_{\beta,d} \int_{\mathbb{R}^d} \frac{\left| f - \tilde{\rho} F \right|^2}{\langle v \rangle^2} \frac{\mathrm{d}v}{F} \leqslant \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{F} \right) \right|^2 F \, \mathrm{d}v, \qquad \forall f \in V, \tag{0.2.16}$$

où $\Lambda_{\beta,d}$ est une constante strictement positive et $\tilde{\rho}$ [6] est définie par

$$\tilde{\rho} := \left(\int_{\mathbb{R}^d} \frac{F}{\langle v \rangle^2} \mathrm{d}v \right)^{-1} \int_{\mathbb{R}^d} \frac{f}{\langle v \rangle^2} \mathrm{d}v.$$

[6] Pour $\beta > d+2$, la fonction $\tilde{\rho}$ peut être remplacée par $\rho := \int_{\mathbb{R}^d} f \mathrm{d}v$ dans l'inégalité (0.2.16).

En revenant à l'inégalité (0.2.15), on obtient :

$$\int_{\mathbb{R}^d} Q(f)f \,\frac{\mathrm{d}v}{F} \leqslant -\Lambda_{\beta,d} \int_{\mathbb{R}^d} \left|f - \tilde{\rho}F\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^2 F}, \qquad \forall f \in V.$$

On termine cette section par le Théorème suivant :

Théorème 0.2.1 ([NP15]). *Supposons que $f_0 \in Y^2_F(\mathbb{R}^{2d})$. Alors l'équation (0.2.11) admet une unique solution f dans la classe des fonctions Y.*

La preuve de ce Théorème est donnée dans [NP15]. Elle a été inspiré de [Deg86] et est basée sur l'application du Théorème de Lions [Lio13].

0.3 Équations macroscopiques

La mécanique des fluides est un domaine de la physique consacré à l'étude du comportement des fluides (liquides, gaz et plasmas) et des forces internes associées. C'est une branche de la mécanique des milieux continus qui modélise la matière à l'aide de particules assez petites pour relever de l'analyse mathématique, mais assez grandes par rapport aux molécules pour être décrites par des fonctions continues. Elle comprend deux sous-domaines : la statique des fluides, qui est l'étude des fluides au repos, et la dynamique des fluides ou *hydrodynamique*, qui est l'étude des fluides en mouvement.

La solution à un problème hydrodynamique implique généralement le calcul de diverses observables du fluide, tels que la vitesse $u(t,x) \in \mathbb{R}^d$, la pression $p(t,x) \in \mathbb{R}$, la densité $\rho(t,x) \geqslant 0$ et la température $T(t,x) \geqslant 0$, en tant que fonctions de l'espace $\mathbb{R}^d_x$ et du temps $t \geqslant 0$.

Le reste de cette section est consacré à la présentation de quelques modèles hydrodynamiques fondamentaux sur lesquels nous reviendrons dans ce manuscrit. Commençons par les plus anciennes.

Équations d'Euler.
Ces équations décrivent l'écoulement des fluides (liquide ou gaz) dans l'approximation des milieux continus. Ces écoulements sont adiabatiques[7], sans échange de quantité de mouvement par viscosité ni d'énergie par conduction thermique. L'histoire de ces équations remonte à Leonhard Euler [Eul57] qui les a établies pour des écoulements incompressibles (1757) et dont la relation avec la thermodynamique est due à Pierre-Simon de Laplace (1816). Le système est donné par

$$\begin{cases} \partial_t u + (u \cdot \nabla_x)u + \nabla_x p = \mathcal{F}, \\ \nabla_x \cdot u = 0, \end{cases} \tag{0.3.1}$$

[7] C'est à dire sans qu'aucun transfert thermique n'intervienne entre le système étudié et son environnement, ou encore sans échange de chaleur entre les deux milieux.

où $u := u(t, x) \in \mathbb{R}^d$ représente la vitesse du fluide, $p(t, x) \in \mathbb{R}$ désigne les forces de pression, et $\mathcal{F}(t, x) \in \mathbb{R}^d$ est la force externe.

Équations de Navier-Stokes.
Elles décrivent le mouvement des fluides newtoniens (donc des gaz et de la majeure partie des liquides). La résolution de ces équations modélisant un fluide comme un milieu continu à une seule phase est difficile, mais elles permettent souvent, par une résolution approchée, de proposer une modélisation de nombreux phénomènes, comme les courants océaniques et des mouvements des masses d'air de l'atmosphère pour les météorologistes, le comportement des gratte-ciel ou des ponts sous l'action du vent pour les architectes et les ingénieurs, ou encore celui des avions, ainsi que l'écoulement de l'eau dans un tuyau et de nombreux autres phénomènes d'écoulement de divers fluides. Ces équations sont nommées ainsi pour honorer les travaux de deux scientifiques du XIX^e siècle : le mathématicien et ingénieur des Ponts Henri Navier, qui le premier a introduit la notion de viscosité dans les équations d'Euler en 1822 [Nav23], et le physicien George Gabriel Stokes, qui a donné sa forme définitive à l'équation de conservation de la quantité de mouvement en 1845 [Sto51, Gal10] :

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \nu \Delta_x u + \nabla_x p = \mathcal{F}, \\ \nabla_x \cdot u = 0, \end{cases} \tag{0.3.2}$$

où $\nu > 0$ est la viscosité du fluide, $u := u(t, x) \in \mathbb{R}^d$ représente sa vitesse, $p(t, x) \in \mathbb{R}$ désigne les forces de pression, et $\mathcal{F}(t, x) \in \mathbb{R}^d$ est la force externe. Le terme non-linéaire $u \cdot \nabla_x u$ – également connu sous le nom de *terme convectif* – est générateur d'instabilité et tenu pour responsable de l'écoulement turbulent du fluide dans certaines situations. Le terme spécial $\nu \Delta_x u$ est connu sous le nom de *terme visqueux*[8].

Les modèles précédents, comme les suivants, sont toujours complétés par des données initiales et des conditions aux bord dans le cas des domaines bornés, ce qui n'est pas le cas dans ce manuscrit puisque on considère $x \in \mathbb{R}^d$, à part le dernier chapitre sur Gevrey où $x \in \mathbb{T}^d$ ou $x \in \mathbb{R}^d$.

Nous allons donner un dernier exemple classique d'équation hydrodynamique, qu'on rencontre souvent en physique et en mathématique, c'est les **équations de diffusion**.

0.3.1 Équations de diffusion

De nombreux phénomènes physiques, dans des domaines scientifiques différents, se décrivent mathématiquement par les équations de diffusion. Ces équations décrivent le comportement du déplacement collectif de particules (molécules, atomes, photons. neutrons, etc.)

[8] C'est à partir de $u \cdot \nabla_x u$ et $\nu \Delta_x u$ qu'a été défini le nombre de Reynold **Re** : $Re := \frac{|u \cdot \nabla_x u|}{|\nu \Delta_x u|}$.

ou de quasi-particules[9] dans un milieu causé par le mouvement aléatoire de chaque particule lorsque les échelles de temps et d'espace macroscopiques sont grandes devant leurs homologues microscopiques. L'équation de diffusion *classique* se formule :

$$\partial_t \rho + \mathrm{div}_x\big(\kappa_\rho \nabla_x \rho\big) = 0, \tag{0.3.3}$$

où l'inconnue ρ est la grandeur qui se diffuse, par exemple la concentration particulaire, et κ_ρ le coefficient de diffusion collectif associé à ρ.
Dans le cas où κ_ρ est constant, l'équation se réduit à l'équation de la chaleur :

$$\partial_t \rho - \kappa \Delta_x \rho = 0. \tag{0.3.4}$$

Nous employons le mot « classique » dans l'équation de diffusion (0.3.3) ou (0.3.4) lorsque l'opérateur en espace est le Laplacien classique[10] ou standard. Dans le cas de la diffusion fractionnaire (définie ci-dessous), le mot « fractionnaire » revient à la puissance fractionnaire qui apparait dans le Laplacien. On parle ainsi du *Laplacien fractionnaire*.

L'**équation de diffusion fractionnaire**, pour un certain $\alpha \in]0,2[$, est donnée par :

$$\partial_t \rho + \kappa(-\Delta_x)^{\frac{\alpha}{2}} \rho = 0. \tag{0.3.5}$$

Il est à noter que les deux équations (0.3.4) et (0.3.5) bien qu'elles portent le même nom mais elles ont deux interprétations différentes, puisque la première fait intervenir un opérateur (différentiel) *local* et la seconde un opérateur *non-local* $(-\Delta)^{\frac{\alpha}{2}}$, qui fait l'objet de la sous-section suivante.

0.3.2 Laplacien fractionnaire

Le Laplacien fractionnaire $\Delta^{\frac{\alpha}{2}}$[11] est un opérateur qui donne le Laplacien standard lorsque $\alpha = 2$. On peut considérer $-(-\Delta)^{\frac{\alpha}{2}}$ comme l'opérateur intégro-différentiel linéaire elliptique le plus basique d'ordre α et peut être défini de plusieurs manières équivalentes. Nous considérons la gamme de puissances $\alpha \in]0,2[$, auquel cas pour $u \in \mathcal{S}(\mathbb{R}^d)$ nous pouvons écrire l'opérateur comme

$$\Delta^{\frac{\alpha}{2}} u(x) := c_{d,\alpha} \mathrm{VP} \int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \mathrm{d}y = c_{d,\alpha} \lim_{\varepsilon \to 0^+} \int_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \mathrm{d}y \tag{0.3.6}$$

où $c_{d,\alpha} > 0$ est une constante universelle dépendant de la dimension d et la puissance α.

Lien avec d'autres opérateurs. Dans le cas où le noyau de collision B ou b (section

[9] Les quasi-particules sont des entités conçues comme des particules et facilitant la description des systèmes de particules.
[10] dans le cas de coefficient de diffusion constant par exemple.
[11] $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$.

efficace) est singulier[12], les opérateurs de collisions (0.2.2) ou (0.2.4) ressemblent fort au Laplacien fractionnaire. Mieux que ça, l'opérateur (0.3.6) peut être écrit sous la forme d'un opérateur de Boltzmann linéaire (0.2.4) par exemple. En effet, pour $u := fF^{-1}$, on obtient (0.2.4) dès que $\mathrm{b}(x,v,v_*) = F(v)F(v_*)|v_* - v|^{-(d+\alpha)}$, puisque :

$$Q(uF) = \int_{\mathbb{R}^d} \mathrm{b}(x,v,v_*)F(v_*)F(v)\big(u(v_*) - u(v)\big)\mathrm{d}v_*.$$

Cette dernière correspondance justifie le gain de régularité pour l'opérateur Boltzmann linéaire malgré la présence des singularités, d'ailleurs l'opérateur (0.2.4) est parfois appelé « scattering operator ».

Une autre définition utile du Laplacien fractionnaire, et qui va être utilisée dans ce document, est celle donnée par la transformée de Fourier :

$$\mathcal{F}\big((-\Delta)^{\frac{\alpha}{2}}\big)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi), \tag{0.3.7}$$

où l'on a pris la définition suivante de la transformée de Fourier

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot\xi} u(x)\ \mathrm{d}x.$$

Ainsi, cette dernière définition du Laplacien fractionnaire est une extension naturelle du Laplacien classique. Une autre définition naturelle est celle qui repose sur la théorie spectrale des opérateurs autoadjoints, où les valeurs propre de l'opérateur en question sont des puissances des valeurs propre du Laplacien classique.

Notez qu'on peut passer de la première définition (opérateur intégral singulier) à la seconde définition (0.3.7), et inversement, grâce à l'égalité suivante :

$$\mathcal{F}\left(\frac{1}{\omega_a |x|^a}\right)(\xi) = \frac{1}{\omega_{d-a}|\xi|^{d-a}},$$

où $a \in]0, d[$ et $\omega_a = \frac{2\pi^{\frac{a}{2}}}{\Gamma(\frac{a}{2})}$, avec Γ désignant la fonction Gamma (voir par exemple [LL01]). Cette dernière formule implique les égalités suivantes :

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) = -\mathcal{F}^{-1}\big(|\xi|^\alpha \mathcal{F}(u)\big)(x) = -\mathcal{F}^{-1}\big(|\xi|^\alpha\big) * u(x) = \mathrm{VP}\left(\frac{1}{|x|^{d+\alpha}}\right) * u(x),$$

prises au sens des distributions, où $*$ désigne le produit de convolution sur $\mathbb{R}^d$. Dans ce cas, la constante $c_{d,\alpha}$ de la définition (0.3.6) est donnée par $c_{d,\alpha} = -\frac{\omega_{-\alpha}}{\omega_{d+\alpha}} > 0$.

Nous renvoyons le lecteur à Landkof [Lan] et Stein [Ste70] pour une discussion des propriétés des opérateurs fractionnaires et des intégrales singulières. Nous n'avons besoin de connaître que les définitions ci-dessus, (0.3.6) et (0.3.7), pour les besoins de cette thèse.

[12] On obtient de telles singularités dès que l'interaction microscopique à l'origine de la collision est à longue portée.

Le Laplacien fractionnaire peut être aussi défini comme l'opérateur infinitésimal qui génère les processus de Lévy α-stables. Ces processus stochastiques, tels que le mouvement Brownien et le processus de Poisson, peuvent présenter des discontinuités dans le temps. Les processus de Lévy associés au Laplacien fractionnaire sont α-stables, ce qui signifie que si X_t est un processus de Lévy, alors $X_{c^\alpha t}$ suit la même distribution que cX_t. Ils peuvent donc être considérés comme des généralisations du mouvement Brownien, qui est un processus 2-stable. Pour une fonction u, lisse, on a

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[u(x) - u(x+X_h)].$$

Cette dernière définition est importante pour les applications aux probabilités ou l'utilisation des approches probabilistes dans les EDP.

0.3.3 Invariance par changement d'échelle

De (0.3.6) ou (0.3.7), on peut remarquer que pour tout $\lambda > 0$, on a :

$$(-\Delta)^{\frac{\alpha}{2}}[u(\lambda x)] = \lambda^\alpha(-\Delta)^{\frac{\alpha}{2}}[u](\lambda x). \tag{0.3.8}$$

Ainsi, si ρ est solution de l'équation de diffusion fractionnaire (0.3.5) (respectivement, de l'équation de diffusion classique (0.3.4)[13]), alors $\tilde{\rho}_\lambda(t,x) := \rho(\lambda^\alpha t, \lambda x)$ est aussi solution de l'équation (0.3.5) (respectivement, $\tilde{\rho}_\lambda(t,x) := \rho(\lambda^2 t, \lambda x)$ est solution de l'équation (0.3.4)).

0.3.4 Système de Vlasov-Navier-Stokes

Dans ces deux dernières sections 0.2 et 0.3, nous avons présenté des modèles cinétiques et hydrodynamiques indépendamment. On peut avoir parfois des systèmes couplés d'équations qui viennent de ces deux derniers modèles. À titre d'exemple, on peut citer le système de Vlasov-Navier-Stokes sur lequel nous allons travailler dans le dernier chapitre de ce manuscrit. Comme son nom l'indique, il est couplé de l'équation de Vlasov (0.2.3) et des équations de Navier-Stokes (0.3.2) avec des forces particulières. Il est donné par

$$\text{(VNS)}\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[(u-v)f] = 0, & \text{dans } (0,T)\times\Omega\times\mathbb{R}^d,\\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p = j_f - \rho_f u, & \text{dans } (0,T)\times\Omega,\\ \nabla_x\cdot u = 0, & \text{dans } (0,T)\times\Omega, \end{cases} \tag{0.3.9}$$

où

$$j_f := j_f(t,x) := \int_{\mathbb{R}^d} vf(t,x,v)\mathrm{d}v \qquad \text{et} \qquad \rho_f := \rho_f(t,x) := \int_{\mathbb{R}^d} vf(t,x,v)\mathrm{d}v.$$

Ce système d'EDP non linéaires décrit le transport de particules (décrites par leur fonction de densité f) dans un fluide (décrit par sa vitesse u et sa pression p). Il appar-

[13] La formule (0.3.3) reste vraie dans le cas $\alpha = 2$.

tient à la grande famille des systèmes cinétiques des fluides ou *fluido-cinétiques*, qui ont été introduits dans les travaux pionniers de O'Rourke [O'R81] et de Williams [Wil85]. Parmi tous les couplages possibles (nous renvoyons à l'introduction de [GHKM18] pour d'autres exemples), le couplage Vlasov-Navier-Stokes a été intensivement étudié en raison de sa pertinence physique (voir [BGLM15] par exemple) et des défis mathématiques qu'il pose. Le système de Vlasov-Navier-Stokes est entièrement couplé : les deux inconnues f et u dépendent l'une de l'autre. Ceci est dû à la *force de Brinkman* (le terme source dans l'équation du fluide) et à l'*accélération de la traînée* (le terme inertiel dans l'équation cinétique). Nous renvoyons à [BGLM15] pour la justification physique de ces deux éléments, et à [DGR08, BDGR17, BDGR18, Hil18, HMS17] pour la dérivation mathématique (partielle) de la première. Les constantes physiques sont toutes normalisées dans (0.3.9).

0.4 Limites de diffusion fractionnaire

0.4.1 Approximation hydrodynamique

Une description cinétique est nettement plus complexe qu'une description hydrodynamique, principalement en raison de la dimension plus élevée de l'espace des phases. Dans les applications pratiques, les équations cinétiques nécessitent des calculs extrêmement coûteux[14]. Par conséquent, lorsque cela est possible, il est préférable de remplacer un modèle cinétique par un modèle hydrodynamique. Cette simplification repose sur l'hypothèse d'équilibre thermodynamique local. Rappelons qu'une distribution $F(x,v)$ est appelée *équilibre local* si elle est invariante par l'action de l'opérateur de collision, c'est à dire : $Q(F) = 0$. Aussi, notez que les équations au niveau macroscopique n'impliquent plus les distributions des particules dans l'espace des phases mais directement les observables en chaque position $x \in \mathbb{R}^d$, telles que la densité spatiale ρ, la vitesse macroscopique u et la température T définies dans le deuxième paragraphe de la section 0.2, et que les échelles de temps et d'espace à ce niveau sont très supérieures par rapport aux échelles mésoscopiques. Ainsi, on introduit un changement d'échelle pour ces dernières variables faisant intervenir un nouveau paramètre $\varepsilon \ll 1$. Intuitivement et d'après l'hypothèse d'équilibre thermodynamique local, on s'attend à ce que la distribution rescalée[15] converge vers l'équilibre de l'équation. Par ailleurs, les différentes équations cinétiques présentées précédemment vérifient toutes la conservation locale de la densité spatiale exprimée par l'équation

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$

puisque $\int Q(f) \mathrm{d}v = 0$. On peut alors chercher à trouver d'autres équations sur ρ et u pour obtenir un système fermé d'équations. Ce type de stratégie permet de dériver les modèles hydrodynamiques.

[14]en particulier l'équation de Boltzmann avec son intégrale de collision à haute multiplicité.
[15]C'est à dire la solution de l'équation cinétique après changement d'échelle

Exemple de l'équation de Boltzmann. L'étude des limites hydrodynamiques de l'équation de Boltzmann (équation (0.2.1) avec $Q_B(f,f)$ défini dans (0.2.2)) consiste à

1. prouver une forme d'équilibre thermodynamique local dans l'asymptotique $\varepsilon \to 0$,
2. en déduire des équations limites pour les champs hydrodynamiques associées à f.

Pour cette équation, l'équilibre local est la *Maxwellienne*, qu'on note généralement par M au lieu de F, donnée par

$$M(x,v) = \rho(x)\frac{e^{-|v-u(x)|^2/2T(x)}}{(2\pi T(x))^{\frac{d}{2}}}, \tag{0.4.1}$$

et le paramètre ε ici représente le *nombre de Knudsen*.
Différents modèles hydrodynamiques ont été dérivés à partir de cette équation comme les équations de l'acoustique, d'Euler et de Navier-Stokes incompressibles, Stokes etc, tout en considérant différents changements d'échelle, chacun ayant son interprétation physique. Voir [GSR04, Vil02] et les références incluses pour les limites hydrodynamique de l'équation de Boltzmann.

Les limites hydrodynamiques dépendent du changement d'échelle, de l'opérateur de collision Q ainsi que son équilibre.

Limite de diffusion classique ou fractionnaire

Ce n'est qu'une limite hydrodynamique particulière où l'équation limite est l'*équation de diffusion classique ou fractionnaire*. C'est généralement le cas pour les équations cinétiques avec un opérateur de collision linéaire Q. En effet, lorsque l'interaction entre les particules est le phénomène dominant et que le temps d'observation est très grand, c'est à dire que les collisions sont très nombreuses et que l'échelle de temps correspondant aux collisions est très petite par rapport au temps d'observation, nous introduisons le paramètre $\varepsilon \ll 1$, qui désigne le libre parcours moyen, temps caractéristique entre deux collisions, nous considérons ensuite le changement d'échelle suivant

$$t = \frac{t'}{\theta(\varepsilon)} \quad \text{et} \quad x = \frac{x'}{\varepsilon} \quad \text{avec} \quad \theta(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Ainsi, l'équation mise à l'échelle, satisfaite par la distribution $f^\varepsilon(t',x',v) = f(t,x,v)$ (on omet les primes), est donnée par

$$\theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v\cdot\nabla_x f^\varepsilon = Q(f^\varepsilon). \tag{0.4.2}$$

Noter que pour $\theta(\varepsilon) = \varepsilon^\alpha$, on retrouve le changement d'échelle *diffusif* (0.3.3) par lequel la solution de l'équation de diffusion (classique ou fractionnaire) est invariante.

Formellement, en passant à la limite dans l'équation (0.4.2), quand ε tend vers 0, on obtient que la limite f^0 est dans le noyau de Q qui est engendré par l'équilibre F, ce qui signifie que $f^0 = \rho(t,x)F(v)$. Ainsi, cela revient à trouver l'équation satisfaite par la densité ρ.

0.4.2 Résultats antérieurs et analyse formelle

La dérivation d'équations macroscopiques de type diffusion à partir d'équations cinétiques telles que (0.2.1) a été étudiée pour la première fois par E. Wigner [WB61], A. Bensoussan, J.L. Lions et G. Papanicolaou dans [BLP79] et E.W. Larsen et J.B. Keller [LK74], suivi d'une longue série articles, comme ceux de C. Bardos, R. Santos et R. Sentis [BSS84], P. Degond et S. Mas-Gallic dans [DMG87], P. Degond, T. Goudon et F. Poupaud [DGP00] et P. Degond, L. Pareschi et G. Russo [DPR04], en considérant différents opérateurs de collision et sous différentes hypothèses.

L'étude de la limite de diffusion a continué et le cas d'équilibre à queue lourde est un sujet qui a reçu beaucoup d'attention dans les quinze dernières années. En gros, on distingue deux cas : le cas d'équilibre *gaussien*, F donné par la formule (0.4.1) avec $(\rho, u, T) = (1, 0, 1)$, dont les résultats ont été obtenus dans la première période (c'est à dire, références du paragraphe précédent, avant 2004), et le cas d'équilibre *à queue lourde* ou *à décroissance polynomiale*, $F(v) \underset{\infty}{\sim} \frac{1}{|v|^\beta}$. De nouvelles approches – complètement différentes – ont mené à l'équation de diffusion fractionnaire, selon les puissances qui apparaissent dans l'équilibre F et le changement d'échelle, $\theta(\varepsilon)$, considéré. On peut citer quelques travaux sur l'équation de Boltzmann linéaire, par exemple ceux de J. Milton, T. Komorowski et S. Olla [JKO09] et A. Mellet, S. Mischler et C. Mouhot [MMM11] en utilisant les *chaînes de Markov* dans la première référence et une *transformation de Fourier-Laplace* dans la seconde, le travail de A. Mellet [Mel10] pour une section efficace qui dépend de la variable spatiale, N. Ben Abdallah, A. Mellet et M. Puel dans [BAMP11a, BAMP11b], toujours sur la même équation mais avec une nouvelle méthode, par *un développement de Hilbert*, ils ont obtenu une convergence forte pour une donnée initiale Sobolev. Pour l'équation de Fokker-Planck, les travaux sont très récents, voir [LP19] pour une approche spectrale en dimension 1, [FT21, FT20] pour une approche probabiliste et [BM22] par une approche spectrale aussi mais en dimension supérieure. Dans ce dernier papier, les auteurs ont considéré un problème spectral modifié, différent de celui considéré dans [LP19] afin d'éviter le problème d'absence de trou spectral dans l'opérateur de Fokker-Planck (voir Remarque 0.2.3). Dans cette thèse, une nouvelle méthode a été développé dans un travail avec M. Puel, présenté dans les chapitres 2 et 3 qui font l'objet des papiers [DP23a, DP23b].

0.4.2.1 Le cas d'équilibre gaussien

Pour les équilibres gaussiens, il est classique (voir [LK74, BLP79, BSS84, DGP00, DPR04] pour Boltzmann linéaire et [DMG87] pour Fokker-Planck) qu'en prenant l'échelle de temps *classique* $\theta(\varepsilon) = \varepsilon^2$, on obtient une *équation de diffusion*

$$\partial_t \rho - \nabla_x \cdot (D \nabla_x \rho) = 0, \tag{0.4.3}$$

où D est la matrice de diffusion donnée par

$$D = \int v Q^{-1}(-vF) \mathrm{d}v. \tag{0.4.4}$$

En effet, le développement formel $f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 ...$ donne

$$\begin{array}{rcl} Q(f^0) & = & 0, \\ Q(f^1) & = & v \cdot \nabla_x f^0, \\ Q(f^2) & = & \partial_t f^0 + v \cdot \nabla f^1. \end{array}$$

La première équation donne $f^0 = \rho(t,x)F(v)$ et la seconde, sous certaines conditions, donne $f^1 = Q^{-1}(v \cdot \nabla_x f^0)$, tandis que la dernière équation – de compatibilité – pour l'équation donnant f^2 donne

$$\partial_t \int f^0 \mathrm{d}v + \int \nabla_x \cdot (v Q^{-1}(v \cdot \nabla_x f^0)) \mathrm{d}v = 0,$$

ce qui est une autre formulation de (0.4.3) puisque $f^0 = \rho(t,x)F(v)$ et F est normalisé par $\int F \mathrm{d}v = 1$.

Remarque 0.4.1. Comme on considère un état d'équilibre gaussien, il a suffisamment de moments bornés (même autant de moments bornés qu'on veut). En particulier, la matrice de diffusion D donnée par l'intégrale (0.4.4) est tout le temps bien définie.

0.4.2.2 Le cas d'équilibre à queue lourde

Les fonctions de distribution à queue lourde apparaissent dans de nombreux contextes, nous allons citer quelques uns tirés de [MMM11]. Par exemple, on observe que la plupart des plasmas astrophysiques ont des fonctions de distribution de vitesse présentant des queues de loi de puissance (voir Summers et Thorne [ST91] ou Mendis et Rosenberg [MR94]). Les mécanismes de collision dissipatifs dans les gaz granulaires peuvent également produire des queues de loi de puissance : voir par exemple [BGT92, EB02] pour le soi-disant “modèle de Maxwell inélastique” introduit dans [BCG00]. On peut également se référer à l'article de synthèse plus général [Vil06]. Nous mentionnons également qu'un travail [BG06] a montré que même les mécanismes de collision élastiques peuvent produire des comportements de queue de loi de puissance dans le cas d'un mélange de gaz avec un noyau de collision maxwellien. Les queues des lois de puissance sont également courantes en économie où elles sont appelées distributions de Pareto (ainsi qu'en statistiques et probabilités plus généralement). Voici quelques exemples de travaux mathématiques (très différents) utilisant des modèles de physique statistique pour rendre compte de ces lois de puissance en économie : Newman [New05], Duering-Toscani [DT07] et Wright [Wri05].

Contrairement au cas d'équilibres gaussiens, les équilibres à queue lourde n'ont pas autant de moments bornés, mais ça dépend de l'ordre de décroissance de l'équilibre. Considérons des équilibres de la forme

$$F(v) = \frac{C_\beta^2}{\langle v\rangle^\beta} = \frac{C_\beta^2}{(1+|v|^2)^{\frac{\beta}{2}}}, \tag{0.4.5}$$

avec $\beta > d$ [16] et où C_β^2 est une constante de normalisation, $\displaystyle\int_{\mathbb{R}^d} F\mathrm{d}v = 1$. Alors, pour F donné par (0.4.5), la matrice de diffusion D définie par l'intégrale (0.4.4) est donnée par :

1. Dans le cas de l'opérateur de Boltzmann linéaire (0.2.4) :

$$D = \int \frac{v\otimes v}{\nu(x,v)} F(v)\mathrm{d}v, \tag{0.4.6}$$

   où ν est la fréquence de collision définie dans (0.2.6). Voir [DGP00] pour plus de détail.

2. Dans le cas de l'opérateur de Fokker-Planck (0.2.10) :

$$D = \int (v\otimes v)\frac{|v|^2+3}{3\beta-2d-4} F(v)\mathrm{d}v. \tag{0.4.7}$$

   Voir [Nas13], sous-sous-section 0.1.3.2 et Annexe A, pour le calcul de D.

Ainsi, dans les deux cas précédents, la matrice de diffusion D peut être infinie. En effet, dans le cas de Boltzmann, pour ν satisfaisant (0.2.9) : $\nu(x,v) \sim \nu_0\langle v\rangle^\gamma$, l'intégrale (0.4.6) est bien définie pour $\beta > d+2-\gamma$ et l'analyse menant à (0.4.3) peut être effectuée, toujours avec $\theta(\varepsilon) = \varepsilon^2$. De même pour l'opérateur de Fokker-Planck pour $\beta > d+4$. Pour les cas critiques $\beta = d+2-\gamma$ et $\beta = d+4$, on obtient encore une fois une équation de diffusion classique à la limite quitte à modifier l'échelle de temps, $\theta(\varepsilon) = \varepsilon^2\ln(\varepsilon^{-1})$, puisque les intégrales (0.4.6) et (0.4.7) sont infinies pour les dernières valeurs de β. On dit qu'on a une *limite de diffusion classique avec échelle de temps anormale*. Voir [MMM11] pour Boltzmann et [NP15, CNP19] pour les cas normal et anormal de Fokker-Planck respectivement.

En revanche, dans les cas $\beta < d+2-\gamma$ pour Boltzmann, avec $\gamma < \min\{\beta-d; d+2-\beta\}$, et $\beta < d+4$ pour Fokker-Planck, l'intégrale donnant D est divergente et la limite de

[16] Afin d'avoir $F \in L^1(\mathbb{R}^d)$

diffusion conduisant à (0.4.3) s'effondre, ce qui signifie que le choix de l'échelle de temps $\theta(\varepsilon) = \varepsilon^2$ était inapproprié. Cela nécessite un nouveau changement d'échelle en temps pour faire apparaître le bon opérateur limite. L'échelle de diffusion appropriée et prise par les auteurs de [MMM11, BAMP11a] pour Boltzmann et [LP19, FT20, FT21, BM22] pour Fokker-Planck est donnée par

$$\theta(\varepsilon) = \varepsilon^{\alpha} \qquad \text{avec} \qquad \alpha := \begin{cases} \frac{\beta-d-\gamma}{1-\gamma} & \text{dans le cas de Boltzmann,} \\ \\ \frac{\beta-d+2}{3} & \text{dans le cas de Fokker-Planck.} \end{cases}$$

Noter que les conditions $d < \beta < d+2-\gamma$, avec $\gamma < \min\{\beta-d; d+2-\beta\}$, pour Boltzmann et $d < \beta < d+4$ pour Fokker-Planck impliquent que $\gamma < 1$ et $\alpha \in (0,2)$ dans les deux cas, et à la limite, ρ est solution de l'*équation de diffusion fractionnaire*

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}} \rho = 0,$$

pour un certain $\kappa > 0$, défini par différentes intégrales, selon la méthode utilisée, à part dans [LP19] (avec $d = 1$ et $\beta \notin \{2,3,4\}$) où les auteurs l'ont calculé explicitement :

$$\kappa = 2C_\beta^2(\beta+1)9^{-\frac{\beta+1}{3}} \cos\left(\frac{\pi}{2}\frac{\beta+1}{3}\right) \frac{\Gamma(1-\frac{\beta+1}{3})}{\Gamma(1+\frac{\beta+1}{3})} > 0, \tag{0.4.8}$$

où Γ est la fonction d'Euler.

Dans le cas fractionnaire, il est intéressant de regarder la façon dont sont reliés l'opérateur Q, l'état d'équilibre F et la puissance du Laplacien fractionnaire qui apparaît à la limite. Pour cela, nous avons choisi de présenter la méthode des moments dans la sous-section qui suit.

0.4.3 Méthode des moments

Dans cette sous-section, nous allons présenter une méthode qui nous permet de dériver l'équation de diffusion à partir des équations de Boltzmann linéaire et Fokker-Planck.

0.4.3.1 Formulation faible et estimations à priori

En intégrant l'équation (0.4.2) multipliée par une fonction test $\chi \in \mathcal{D}([0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$, on obtient :

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f^\varepsilon \chi \, \mathrm{d}x\mathrm{d}v = \int \left[\partial_t \chi + \theta(\varepsilon)^{-1} Q^*(\chi) + \varepsilon\theta(\varepsilon)^{-1}(v \cdot \nabla_x \chi)\right] f^\varepsilon \, \mathrm{d}x\mathrm{d}v. \tag{0.4.9}$$

Puisque à la limite, la solution est décomposée en $\rho(t,x)$ et $F(v)$, et on cherche à trouver l'équation satisfaite par ρ, on peut prendre $\chi = \varphi(t,x)$ avec $\varphi \in \mathcal{D}([0,+\infty) \times \mathbb{R}^d)$. Ainsi, (0.4.9) devient :

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f^\varepsilon \varphi F \,\mathrm{d}x\mathrm{d}v = \int \big[\partial_t\varphi + \varepsilon\theta(\varepsilon)^{-1}(v\cdot\nabla_x\varphi)\big] f^\varepsilon F \,\mathrm{d}x\mathrm{d}v. \tag{0.4.10}$$

On remarque que le second terme du côté droite dans la formulation faible ci-dessus, (0.4.10), est d'ordre $\varepsilon^{1-\alpha}$ pour $\theta(\varepsilon)=\varepsilon^\alpha$, et le passage à la limite quand $\varepsilon\to 0$ n'est pas évident. Pour cela, il faut introduire une correction en ε, ce qui est l'objet de la *méthode des moments*. C'est une méthode qui consiste à faire une correction sur la fonction teste, en introduisant une nouvelle fonction χ^ε, solution d'un *problème auxiliaire* qui nous facilite le passage à la limite, tout en récupérant la fonction test initiale χ à la limite.

Le cas de l'équation de Boltzmann linéaire. Considérons l'équation rescalée :

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v\cdot\nabla_x f^\varepsilon = Q^+(f^\varepsilon) - \nu(x,v) f^\varepsilon, & x\in\mathbb{R}^d, v\in\mathbb{R}^d, t>0, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x\in\mathbb{R}^d, v\in\mathbb{R}^d, \end{cases} \tag{0.4.11}$$

où Q^+ est l'opérateur défini dans (0.2.5). Sous les deux hypothèses de la Proposition 0.2.1 et grâce à l'inégalité (0.2.8), on obtient le Lemme de compacité suivant :

Lemme 0.4.2 ([MMM11])**.** *La solution f^ε de (0.4.11) est bornée dans $L^\infty(0,T;L^2_{F^{-1}})$ uniformément par rapport à ε. De plus, les estimations suivantes sont satisfaites :*

$$\|\rho^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \|f_0\|_{L^2_{F^{-1}}} \tag{0.4.12}$$

et

$$\|f^\varepsilon - \rho^\varepsilon F\|^2_{L^\infty(0,T;L^2_{\nu F^{-1}})} \leqslant C\|f_0\|^2_{L^2_{F^{-1}}}\theta(\varepsilon), \tag{0.4.13}$$

où $\rho^\varepsilon := \int_{\mathbb{R}^d} f^\varepsilon \mathrm{d}v$. En particulier, ρ^ε converge faiblement $*$ dans $L^\infty(0,T;L^2)$ vers ρ et f^ε converge faiblement $*$ dans $L^\infty(0,T;L^2_{\nu F^{-1}})$ vers $f=\rho(x,t)F(v)$.

Le cas de l'équation de Fokker-Planck. Considérons l'équation rescalée suivante :

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v\cdot\nabla_x f^\varepsilon = \nabla_v\cdot\left(F\nabla_v\left(\frac{f^\varepsilon}{F}\right)\right), & x\in\mathbb{R}^d, v\in\mathbb{R}^d, t>0, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x\in\mathbb{R}^d, v\in\mathbb{R}^d. \end{cases} \tag{0.4.14}$$

Nous avons le Lemme de compacité suivant :

Lemme 0.4.3 ([LP19, NP15])**.** *Pour une donnée initiale $f_0\in Y^p_F(\mathbb{R}^{2d})$, l'espace défini dans (0.2.12), où $p\geqslant 2$ et pour un temps $T>0$.*

1. *La solution f^ε de (0.4.14) est bornée $L^\infty\big([0,T];Y^p_F(\mathbb{R}^{2d})\big)$ uniformément par rapport à ε. De plus, elle satisfait*

$$\|f^\varepsilon(T)\|^p_{Y^p_F} + \frac{p(p-1)}{\theta(\varepsilon)}\int_0^T\int_{\mathbb{R}^{2d}}\left|\nabla_v\left(\frac{f^\varepsilon}{F}\right)\right|^2\left|\frac{f^\varepsilon}{F}\right|^{p-2} F\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \leqslant \|f_0\|^p_{Y^p_F}. \tag{0.4.15}$$

2. *La densité* $\rho^\varepsilon(t,x) := \int_{\mathbb{R}^d} f^\varepsilon \mathrm{d}v$ *satisfait l'estimation suivante :*

$$\|\rho^\varepsilon(t)\|^p_{L^p(\mathbb{R}^d)} \leqslant C_\beta^{-2(p-1)} \|f_0\|^p_{Y^p_F(\mathbb{R}^{2d})} \quad \textit{pour tout} \quad t \in [0,T]. \tag{0.4.16}$$

3. *À une sous-suite près, la densité* ρ^ε *converge faiblement* $*$ *dans* $L^\infty\big([0,T]; L^p(\mathbb{R}^d)\big)$ *vers* ρ.

4. *À une sous-suite près, la suite* f^ε *converge faiblement* $*$ *dans* $L^\infty\big([0,T]; Y^p_F(\mathbb{R}^{2d})\big)$ *vers la function* $f = \rho(t,x)F(v)$.

Comme une conséquence du Lemme précédent, nous avons l'estimation suivante :

Corollaire 0.4.4 ([LP19])**.** *Soient* F *donné par* (0.4.5), $F(v) = \frac{C_\beta^2}{\langle v\rangle^\beta}$, *et* $\beta \in (d, d+4)$. *Soit* f^ε *solution de* (0.4.14) *avec* $\theta(\varepsilon) = \varepsilon^{\frac{\beta-d+2}{3}}$. *Supposons que* $\|f_0/F\|_\infty \leqslant C$. *Alors,* f^ε *satisfait l'estimation suivante :*

$$\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left| f^\varepsilon - \rho^\varepsilon F \right|^2 \frac{\mathrm{d}v}{F} \right)^{\frac{\beta-d+2}{\beta-d}} \mathrm{d}x\mathrm{d}s \leqslant C \varepsilon^{\frac{\beta-d+2}{3}}. \tag{0.4.17}$$

Les estimations des deux Lemmes de compacités 0.4.2 et 0.4.3 assurent l'existence de f et ρ, limites de f^ε et ρ^ε respectivement – à une sous-suite près – dans les espaces L^p (ou L^p à poids) donnés dans ces deux derniers Lemmes. Ainsi, si on arrive à construire une fonction test χ^ε qui converge vers χ dans l'espace fonctionnel qui convient alors, le passage à la limite dans (0.4.10) est facilement obtenu et l'opérateur appliqué à la limite χ donne l'équation de ρ.

0.4.3.2 Construction de la fonction test

La construction de la fonction χ^ε dépend de l'opérateur considéré. Nous allons commencer par présenter le cas de l'équation de Boltzmann pour une section efficace bornée, qui est plus simple par rapport au cas de l'équation de Fokker-Planck pour différentes raisons dont les plus importantes reviennent à :

- l'écriture explicite de χ^ε en fonction de sa limite χ, ce qui nous permet d'avoir des estimations précises sur χ^ε et montrer la convergence vers χ dans l'espace adéquat ;
- la possibilité de passer à la limite dans chaque terme, séparément, dans le côté droite de (0.4.10).

En effet, A. Mellet a introduit dans [Mel10] un problème auxiliaire associé à l'équation (0.4.11) : pour une fonction $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)$, la fonction $\chi^\varepsilon = \chi^\varepsilon(t,x,v)$ est définie comme solution de

$$\nu(x,v)\chi^\varepsilon - \varepsilon v \cdot \nabla_x \chi^\varepsilon = \nu(x,v)\varphi(t,x). \tag{0.4.18}$$

Notez que la solution χ^ε est régulière et est donnée par :

$$\chi^\varepsilon(t,x,v) = \int_0^\infty e^{-\int_0^z \nu(x+\varepsilon vs,v)\mathrm{d}s}\nu(x+\varepsilon vz,v)\varphi(x+\varepsilon vz,t)\mathrm{d}z. \tag{0.4.19}$$

Sous l'hypothèse suivante sur la section efficace σ :

$$\nu_1 F(v) \leqslant \sigma(x,v,v') = \mathrm{b}(x,v,v')F(v) \leqslant \nu_2 F(v) \quad \text{pour tout} \;\; x,v,v' \in \mathbb{R}^d, \tag{0.4.20}$$

où ν_1 et ν_2 sont deux constantes strictement positives, qui implique en particulier que :

$$\nu_1 \leqslant \nu(x,v) \leqslant \nu_2 \qquad \text{pour tout} \;\; x,v \in \mathbb{R}^d, \tag{0.4.21}$$

et grâce à l'identité

$$\int_0^\infty \nu(x+\varepsilon vz,v)\; e^{-\int_0^z \nu(x+\varepsilon vs,v)\mathrm{d}s}\;\mathrm{d}z = \int_0^\infty e^{-u}\;\mathrm{d}u = 1,$$

on obtient l'estimation suivante :

$$\begin{aligned} |\chi^\varepsilon - \varphi| &= \left| \int_0^\infty e^{-\int_0^z \nu(x+\varepsilon vs,v)\mathrm{d}s}\nu(x+\varepsilon vz,v)\big[\varphi(x+\varepsilon vz,t)-\varphi(t,x)\big]\mathrm{d}z \right| \\ &\leqslant \nu_2 \int_0^\infty e^{-\nu_1 z}\varepsilon z \|\mathrm{D}\varphi\|_{L^\infty}\mathrm{d}z = \frac{\nu_2}{\nu_1}\varepsilon\|\mathrm{D}\varphi\|_{L^\infty}. \end{aligned}$$

D'où, χ^ε est bornée dans L^∞ et converge vers φ uniformément par rapport à x et t. De plus, nous avons les limites et estimations suivantes :

Lemme 0.4.5. *Pour tout $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^d)$, si χ^ε est la fonction définie par (0.4.19), alors*

$$\int_{\mathbb{R}^d} F(v)\big[\chi^\varepsilon(t,x,v)-\varphi(t,x)\big]\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{0.4.22}$$

et

$$\int_{\mathbb{R}^d} F(v)\big[\partial_t\chi^\varepsilon(t,x,v)-\partial_t\varphi(t,x)\big]\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{0.4.23}$$

uniformément par rapport à x et t. De plus, il existe une constante $C>0$ tel que

$$\|\chi^\varepsilon\|_{L^2_F} \leqslant C\|\varphi\|_{L^2_F} \quad \textit{et} \quad \|\partial_t\chi^\varepsilon\|_{L^2_F} \leqslant C\|\partial_t\varphi\|_{L^2_F}. \tag{0.4.24}$$

En revenant à la formulation (0.4.10), en l'intégrant en temps entre 0 et $+\infty$ et avec la nouvelle fonction test satisfaisant (0.4.18), on écrit :

$$\begin{aligned} &-\iint f^\varepsilon(t,x,v)\partial_t\chi^\varepsilon(t,x,v)\;\mathrm{d}x\mathrm{d}v\mathrm{d}t - \int f_0(x,v)\chi^\varepsilon(0,x,v)\;\mathrm{d}x\mathrm{d}v \\ &= \varepsilon^{-\alpha}\iint \big[f^\varepsilon(Q^+)^*(\chi^\varepsilon) - f^\varepsilon\nu(x,v)\chi^\varepsilon + \varepsilon v\cdot\nabla_x\chi^\varepsilon\big]\;\mathrm{d}x\mathrm{d}v\mathrm{d}t \\ &= \varepsilon^{-\alpha}\iint f^\varepsilon\big[(Q^+)^*(\chi^\varepsilon) - \nu(x,v)\varphi(t,x)\big]\;\mathrm{d}x\mathrm{d}v\mathrm{d}t, \end{aligned}$$

ou encore,

$$
\begin{aligned}
&-\iint f^\varepsilon(t,x,v)\partial_t\chi^\varepsilon(t,x,v)\ \mathrm{d}x\mathrm{d}v\mathrm{d}t-\int f_0(x,v)\chi^\varepsilon(0,x,v)\ \mathrm{d}x\mathrm{d}v\\
&=\varepsilon^{-\alpha}\iint f^\varepsilon\big[(Q^+)^*(\chi^\varepsilon)-(Q^+)^*(\varphi)\big]\ \mathrm{d}x\mathrm{d}v\mathrm{d}t.
\end{aligned}
$$

Maintenant, en décomposant f^ε comme $\rho^\varepsilon F+(f^\varepsilon-\rho^\varepsilon F)$ et en utilisant le fait que $Q^+(F)=\nu F$, on obtient :

$$
\begin{aligned}
&-\iint f^\varepsilon(t,x,v)\partial_t\chi^\varepsilon(t,x,v)\ \mathrm{d}x\mathrm{d}v\mathrm{d}t-\int f_0(x,v)\chi^\varepsilon(0,x,v)\ \mathrm{d}x\mathrm{d}v\\
&=\varepsilon^{-\alpha}\iint Q^+(\rho^\varepsilon F)\big[\chi^\varepsilon-\varphi\big]\ \mathrm{d}x\mathrm{d}v\mathrm{d}t+\varepsilon^{-\alpha}\iint Q^+(f^\varepsilon-\rho^\varepsilon F)\big[\chi^\varepsilon-\varphi\big]\ \mathrm{d}x\mathrm{d}v\mathrm{d}t\\
&=\varepsilon^{-\alpha}\iint \rho^\varepsilon\nu F\big[\chi^\varepsilon-\varphi\big]\ \mathrm{d}x\mathrm{d}v\mathrm{d}t+\varepsilon^{-\alpha}\iint Q^+(f^\varepsilon-\rho^\varepsilon F)\big[\chi^\varepsilon-\varphi\big]\ \mathrm{d}x\mathrm{d}v\mathrm{d}t. \qquad (0.4.25)
\end{aligned}
$$

Le Lemme 0.4.5 permet de passer à la limite dans la première ligne dans les égalités ci-dessus afin de récupérer :

$$
-\iint \rho(t,x)\partial_t\varphi(t,x)\ \mathrm{d}x\mathrm{d}t-\int \rho_0(x)\varphi(0,x)\ \mathrm{d}x,
$$

où $\rho_0:=\int f_0\ \mathrm{d}v$. Pour la dernière ligne dans l'égalité (0.4.25), il nous faut d'autres hypothèses sur la fréquence de collision ν pour pouvoir passer à la limite plus facilement. Supposons que :

1. Pour tout $x,v\in\mathbb{R}^d$, on a : $\nu(x,-v)=\nu(x,v)$.

2. Il existe une fonction $\nu_0(x)$ (satisfaisant $\nu_1\leqslant\nu_0(x)\leqslant\nu_2$) tel que :

$$
\nu(x,v)\underset{|v|\to\infty}{\longrightarrow}\nu_0(x)\qquad \text{uniformément par rapport à } x.
$$

3. Enfin, supposons que ν est de classe C^1 par rapport à x et que :

$$
\|\mathrm{D}_x\nu\|_{L^\infty(\mathbb{R}^{2d})}\leqslant C. \qquad (0.4.26)
$$

Alors, nous avons les deux résultats suivants :

Lemme 0.4.6 ([Mel10]). *Pour toute fonction test $\varphi\in\mathcal{D}([0,\infty)\times\mathbb{R}^d)$, soit χ^ε la fonction définie par (0.4.19). Alors,*

$$
\lim_{\varepsilon\to0}\ \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbb{R}^{2d}}Q^+(f^\varepsilon-\rho^\varepsilon F)\big[\chi^\varepsilon(t,x,v)-\varphi(t,x)\big]\mathrm{d}x\mathrm{d}v\mathrm{d}t=0,
$$

où $\alpha\in(0,2)$ tel que $|v|^{\alpha+d}F(v)\to\kappa_0$ quand $|v|\to\infty$.

Remarque 0.4.7. Notez que dans le cas où ν satisfait l'inégalité (0.4.21), l'échelle de temps est donnée par $\theta(\varepsilon) = \varepsilon^{-\alpha}$, où α correspond à la puissance qui apparait dans le comportement de l'équilibre F.

Proposition 0.4.8 ([Mel10]). *Pour toute fonction test $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^d)$, soit χ^ε définie par* (0.4.19). *Alors,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int_{\mathbb{R}^d} \nu(x,v)F(v)\big[\chi^\varepsilon(x,v,t) - \varphi(x,t)\big]\mathrm{d}v = -\kappa_0 \mathcal{L}^*(\varphi),$$

où $\mathcal{L}^*$ est l'adjoint de l'opérateur elliptique $\mathcal{L}$ défini par :

$$\mathcal{L}(\varphi) = \mathrm{P.V.} \int_{\mathbb{R}^d} \eta(x,y)\frac{\varphi(x)-\varphi(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y,$$

avec

$$\eta(x,y) = \nu_0(x)\nu_0(y)\int_0^\infty z^\alpha e^{-z\int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s}\,\mathrm{d}z.$$

Remarques 0.4.9.

1. Notez que la limite de la Proposition 0.4.8 est uniforme par rapport à x et t.
2. D'après la seconde hypothèse sur ν, il existe η_1 et η_2 tels que :

   $$0 < \eta_1 \leqslant \eta(x,y) \leqslant \eta_2 < \infty.$$

   En particulier, l'opérateur $\mathcal{L}$ a le même ordre que l'opérateur Laplacien fractionnaire $(-\Delta)^{\alpha/2}$.
3. Notez aussi que l'opérateur $\mathcal{L}$ est auto-adjoint puisque $\eta(x,y) = \eta(y,x)$.

Commentaire sur le Lemme 0.4.6 et la Proposition 0.4.8.
• La limite du Lemme 0.4.6 est basée sur un développement de Taylor par rapport à x appliqué à la différence $\chi^\varepsilon - \varphi$, sur l'inégalité (1.2.8) pour χ^ε, et sur l'inégalité (0.4.13) :

$$\int_0^\infty \int_{\mathbb{R}^{2d}} |f^\varepsilon - \rho^\varepsilon F|^2 \,\frac{\mathrm{d}v}{F}\mathrm{d}x\mathrm{d}t \leqslant C\|f_0\|^2_{L^2_{F^{-1}}}\theta(\varepsilon), \quad \forall f \in L^2_{F^{-1}},$$

et les hypothèses (0.4.20) et (0.4.21) simplifient beaucoup les calculs. Le cas d'une section efficace qui dégénère pour les grandes vitesses est traité en détail dans le chapitre 1.
• La limite de la Proposition 0.4.8 est obtenue en deux étapes. La première consiste à montrer que les petites vitesses ($|v| \leqslant C$) ne participent pas à la limite, dont la preuve est basée sur des estimations des quantités $\chi^\varepsilon - \varphi$ et $\nu(x+\varepsilon zv, x) - \nu(x,v)$. Dans la seconde étape, on rescale la variable vitesse et on passe à la limite quand ε tend vers 0, quitte à découper l'intégrale en plusieurs morceaux. Notez que la symétrie de F et ν par rapport à v est cruciale dans la preuve de cette Proposition.

Difficultés dans le cas de Fokker-Planck.
Contrairement à l'équation de Boltzmann linéaire (0.4.11), le passage à la limite dans la formulation faible pour l'équation de Fokker-Planck n'est pas évident, car tous les termes participent à la limite et l'absence de trou spectral complique plus l'étude, puisque l'inégalité (0.4.13) n'est plus valide et son analogue (0.4.17) ne suffit pas pour montrer la limite du Lemme 0.4.6. Ainsi, le passage à la limites dans les termes séparément (comme dans (0.4.25)) n'est pas possible et la construction précédente ne tient pas. À cet effet, G. Lebeau et M. Puel [LP19] ont développé une *méthode spectrale* pour la construction de la fonction teste en dimension 1. En effet, ils ont proposé de regarder tout l'opérateur de Fokker-Planck (avec la partie *advection*) comme un seul bloc et de considérer la fonction propre associée comme une fonction test.

0.4.3.3 Construction de Lebeau-Puel et difficulté en dimension supérieure

Dans [LP19], les auteurs ont proposé de prendre la solution de l'équation ci-dessous (écrite en variable Fourier, ξ, pour la position x, et après le changement d'inconnue $\frac{f}{\sqrt{F}}$ afin de travailler avec un opérateur auto-adjoint dans L^2) comme une fonction test adéquate :

$$(Q + \mathrm{i}\varepsilon\xi v)M_{\mu,\varepsilon}(v) = \mu M_{\mu,\varepsilon}(v), \quad v \in \mathbb{R}, \tag{0.4.27}$$

où Q est l'opérateur de Fokker-Planck. Leur construction consiste à reconnecter deux branches construites comme suit : Ils ont d'abord construit pour chaque μ, ε (fixés) une branche dans le demi-espace $\mathbb{R}_+$, en introduisant une équation approchée pour les grandes vitesses dans un premier temps, et en prolongeant ces solutions aux petites vitesses dans un second temps, par un argument de point fixe dans chaque étape. Ensuite, par symétrie de l'équation (qui découle de la symétrie de l'équilibre), ils ont obtenu une deuxième branche dans l'autre demi-espace $\mathbb{R}_-$, et afin de recoller les deux fonctions pour obtenir une solution $C^1(\mathbb{R})$, il y avait une contrainte à satisfaire. Cette contrainte vient du recollement des dérivées des deux branches, ce qui a conduit à une relation sur μ que les auteurs ont résolu par le théorème des fonctions implicites en obtenant μ en fonction de ε.

Cette construction basée sur le recollement de branches semble très compliquée à adapter en dimension supérieure, car elle signifie que le théorème des fonctions implicites utilisé pour étudier la contrainte doit être appliqué à l'ensemble de l'opérateur dérivé, ce qui nous a poussé à chercher une nouvelle stratégie pour résoudre le problème spectral associé à l'opérateur de Fokker-Planck et de récupérer la fonction test adéquate.

0.4.3.4 Méthode spectrale : avantage et généralisation

Avec cette méthode, on capture l'équation de diffusion dans tous les cas, c'est à dire pour toutes les phases du paramètre β (qui apparait dans l'équilibre F donné par (0.4.5)), et pour obtenir que ce soit de la diffusion classique ou fractionnaire. Notez aussi, que dans cette construction, la valeur propre $\mu(\varepsilon)$ associée à la fonction propre $M_{\mu,\varepsilon}$ joue un rôle très important et c'est son comportement pour $\varepsilon \sim 0$ qui donne la bonne échelle en temps $\theta(\varepsilon)$ et le coefficient de diffusion κ. Nous reviendrons sur ces deux points dans la section d'après.

L'objectif principal de cette thèse est de proposer une nouvelle méthode[17] alternative, inspirée des travaux d'Herbert Koch sur l'équation KdV non linéaire [Koc15], où l'on résout le problème spectral associé à l'opérateur de Fokker-Planck, toujours en tenant compte de la partie d'advection. Nous avons établi le résultat d'abord en dimension 1, en travaillant sur tout l'espace $\mathbb{R}$ et en évitant les problèmes de recollement, avant de le généraliser en dimension supérieure par des méthodes d'EDP. Cette méthode est probablement intéressante pour des potentiels plus généraux (de type convolution par exemple) ou pour des problèmes non linéaires.

0.5 Principaux résultats obtenus

Cette thèse est composée de trois Parties, les deux premières portent sur les limites de diffusion fractionnaire pour des équations cinétiques, tandis que la dernière et qui est complètement indépendante, porte sur la régularité Gevrey pour les solutions du système de Vlasov-Naver-Stokes (VNS). Les Parties I et III sont constituées d'un seul chapitre, contrairement à la Partie II qui contient trois. Chaque Partie est résumée dans une section dans ce qui suit.

0.5.1 Diffusion fractionnaire pour l'équation de Boltzmann linéaire

Cette section comprend un résumé du chapitre 1 sur la limite de diffusion pour l'équation de Boltzmann linéaire avec équilibre à queue lourde et section efficace qui dégénère pour les grandes vitesses et qui dépend de la variable de position au même temps. Ce résultat a été obtenu au début de la première année de thèse, tout en adaptant la preuve donnée par A. Mellet dans [Mel10], en la combinant avec les hypothèses de [MMM11].

Rappelons que l'équation considérée est donnée par :

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q^+(f^\varepsilon) - \nu(x,v) f^\varepsilon, & x \in \mathbb{R}^d, v \in \mathbb{R}^d, t > 0, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x \in \mathbb{R}^d, v \in \mathbb{R}^d, \end{cases} \tag{0.5.1}$$

avec

$$Q^+(f) := \int_{\mathbb{R}^d} \sigma(x,v,v') f(v') \mathrm{d}v' \quad \text{et} \quad \nu(x,v) := \int_{\mathbb{R}^d} \sigma(x,v',v)\, \mathrm{d}v' = \frac{Q^+(F)}{F(v)}.$$

Avant d'énoncer le résultat, nous donnons les hypothèses sur σ et F.

[17] nouvelle au point de vue construction, puisque la méthode spectrale a été développé avant, par G. Lebeau et M. Puel.

Hypothèses 0.5.1. Les deux premières sont très classiques :

(A1) La section efficace σ est positive et localement intégrable sur $\mathbb{R}^{2d}$ pour tout x, et la fréquence de collision ν satisfait :

$$\nu(x,-v)=\nu(x,v)>0, \qquad \forall x,v\in\mathbb{R}^d.$$

(A2) Il existe une fonction $F(v)\in L^1_\nu(\mathbb{R}^d)$ indépendante de x telle que :

$$Q(F)=0.$$

De plus, la fonction F est symétrique, positive et normalisée à 1 :

$$F(-v)=F(v)>0 \quad \text{pour tout } v\in\mathbb{R}^d \quad \text{et} \quad \int_{\mathbb{R}^d}F(v)\mathrm{d}v=1.$$

Les hypothèses suivantes concernent le comportement de F et ν pour de grandes vitesses.

(B1) Il existe $\alpha>0$ et une constante $\kappa_0>0$ tel que

$$|v|^{\alpha+d}F(v)\longrightarrow\kappa \quad \text{quand} \quad |v|\to\infty.$$

(B2) Il existe ν_1 et ν_2 deux constantes strictement positives, une fonction $\nu_0(x)$ et une constante $\beta\in\mathbb{R}$ tel que

$$\nu_1\langle v\rangle^\beta\leqslant\nu(x,v)\leqslant\nu_2\langle v\rangle^\beta, \qquad \forall x,v\in\mathbb{R}^d \tag{0.5.2}$$

et

$$|v|^{-\beta}\nu(x,v)\longrightarrow\nu_0(x) \quad \text{quand} \quad |v|\longrightarrow\infty, \quad \text{uniformément par rapport à } x.$$

Supposons de plus que ν est de classe C^1 par rapport à x et que

$$\left\|\langle v\rangle^{-\beta}\partial_x\nu(x,v)\right\|_{L^\infty(\mathbb{R}^{2d})}\leqslant C.$$

(B3) Enfin, nous supposons qu'il existe une constante strictement positive M telle que

$$\int_{\mathbb{R}^d}F'\frac{\nu}{\mathrm{b}}\,\mathrm{d}v'+\left(\int_{\mathbb{R}^d}\frac{F'}{\nu'}\frac{\mathrm{b}^2}{\nu^2}\,\mathrm{d}v'\right)^{\frac12}\leqslant M, \qquad \forall x,v\in\mathbb{R}^d,$$

avec les notations $\mathrm{b}:=\mathrm{b}(x,v,v'):=\sigma(x,v,v')F^{-1}(v)$, $F':=F(v')$ et $\nu':=\nu(x,v')$.

Théorème 0.5.1 (Diffusion fractionnaire pour l'équation Boltzmann linéaire)**.** *Supposons (A1-A2) et (B1-B2-B3) avec $\alpha>0$ et $\beta<\min\{\alpha;2-\alpha\}$. On définit γ par :*

$$\gamma:=\frac{\alpha-\beta}{1-\beta}.$$

Soit $f^\varepsilon(t,x,v)$ la solution de (0.5.1) de donnée initiale $f_0\in L^2_{F^{-1}}(\mathbb{R}^{2d})$ avec $f_0\geqslant0$ et $\theta(\varepsilon)=\varepsilon^\gamma$. Alors, f^ε converge faiblement-$*$ dans $L^\infty(0,T,L^2_{\nu F^{-1}}(\mathbb{R}^{2d}))$ vers $\rho(x,t)F(v)$ où

ρ est solution de l'équation

$$\begin{cases} \partial_t \rho + \kappa \mathcal{L}(\rho) = 0, \\ \rho(0,x) = \rho_0(x) = \displaystyle\int_{\mathbb{R}^d} f_0 \, \mathrm{d}v, \end{cases} \tag{0.5.3}$$

où $\mathcal{L}$ est un opérateur elliptique d'ordre γ défini par :

$$\mathcal{L}(\rho) := \frac{1}{1-\beta} \mathrm{VP} \int_{\mathbb{R}^d} \eta(x,y) \frac{\rho(x)-\rho(y)}{|x-y|^{d+\gamma}} \, \mathrm{d}y$$

avec

$$\eta(x,y) = \nu_0(x)\nu_0(y) \int_0^\infty z^\gamma e^{-z\int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s} \, \mathrm{d}z.$$

Remarque 0.5.2.

1. Noter que $\beta < 1$ et $\gamma \in (0,2)$ pour $\alpha > 0$ et $\beta < \min\{\alpha; 2-\alpha\}$.

2. D'après (B2), il existe $\eta_1, \eta_2 \in \mathbb{R}_+^*$ tel que :
$$\eta_1 \leqslant \eta(x,y) \leqslant \eta_2, \qquad \forall x, y \in \mathbb{R}^d.$$

3. L'opérateur $\mathcal{L}$ est auto-adjoint puisque $\eta(x,y) = \eta(y,x)$, et a le même ordre que l'opérateur Laplacien fractionnaire $(-\Delta)^{\gamma/2}$.

Les démarches de la preuve sont les mêmes présentées dans la sous-sous-section 0.4.3.2 et l'idée clé repose sur la "normalisation" de la fréquence de collision ν par $|v|^{-\beta}$ (pour les grandes vitesses) dans le problème auxiliaire. Donc on considère l'équation suivante :

$$|v|^{-\beta}\nu(x,v)\chi^\varepsilon - \varepsilon|v|^{-\beta} v \cdot \nabla_x \chi^\varepsilon = |v|^{-\beta}\nu(x,v)\varphi(t,x),$$

pour $|v| \geqslant 1$ au lieu de (0.4.18), et on garde (0.4.18) pour $|v| \leqslant 1$. La solution de l'équation ci-dessus est donnée par :

$$\chi^\varepsilon(t,x,v) = \int_0^\infty e^{-\int_0^z |v|^{-\beta}\nu(x+\varepsilon|v|^{-\beta}vs,v)\mathrm{d}s} |v|^{-\beta}\nu(x+\varepsilon|v|^{-\beta}vz,v)\varphi(x+\varepsilon|v|^{-\beta}vz,t)\mathrm{d}z.$$

Rappelons que la formulation faible de l'équation (0.5.1) a été établi dans la sous-sous-section 0.4.3.2 et est donnée par (0.4.25). Ainsi, ça revient à redémontrer le Lemme 0.4.5 (en version plus faible), le Lemme 0.4.6 et la Proposition 0.4.8 sous les hypothèses 0.5.1.

0.5.2 Problème spectral et diffusion fractionnaire pour l'équation de Fokker-Planck

Cette section concerne les résultats de la Partie II sur la limite de diffusion fractionnaire pour l'équation de Fokker-Planck avec équilibre à queue lourde. On s'intéresse plutôt à la construction d'une solution pour le problème spectral associé à l'opérateur de Fokker-Planck qu'à la limite de diffusion elle-même, puisque le résultat a été déjà établi en dimension 1 [LP19] par une approche spectrale difficile à généraliser, comme nous l'avons expliqué dans la section précédente, puis par une approche probabiliste en dimension 1 puis en dimension supérieure [FT21, FT20] et dernièrement [BM22] par une approche spectrale en toute dimension, différente à la notre. Nous reviendrons sur la différence entre notre construction et celle de [BM22].

Rappelons que l'équation étudiée dans cette partie est donnée par :

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot \left(F \nabla_v \left(\frac{f^\varepsilon}{F}\right)\right), & x \in \mathbb{R}^d, v \in \mathbb{R}^d, t > 0, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x \in \mathbb{R}^d, v \in \mathbb{R}^d. \end{cases} \tag{0.5.4}$$

Dans les chapitres 2 et 3, on considère un équilibre F donné par :

$$F(v) = \frac{C_\beta^2}{(1+|v|^2)^{\frac{\beta}{2}}}, \tag{0.5.5}$$

où on rappelle que C_β est une constante de normalisation, $\displaystyle\int_{\mathbb{R}^d} F(v)\mathrm{d}v = 1$, et $\beta > d$. Nous discutons le cas d'un équilibre quelconque dans le chapitre 4.

Cette section est structurée comme suit : nous commençons par introduire les notations utilisées dans la Partie II, puis nous présentons les résultats de la construction du couple propre et nous terminons par donner le lien avec la limite de diffusion.

0.5.2.1 Notations

Afin de simplifier le calcul et de travailler avec un opérateur auto-adjoint dans L^2, nous procédons à un changement d'inconnue en écrivant

$$f = F^{\frac{1}{2}} g = C_\beta M g,$$

avec

$$M = C_\beta^{-1} F^{\frac{1}{2}} = \frac{1}{(1+|v|^2)^{\frac{\gamma}{2}}} \qquad \text{et} \qquad \gamma = \frac{\beta}{2}.$$

L'équation (0.5.4) devient

$$\theta(\varepsilon)\partial_t g^\varepsilon + \varepsilon v \cdot \nabla_x g^\varepsilon = \frac{1}{M}\nabla_v\left(M^2 \nabla_v\left(\frac{g^\varepsilon}{M}\right)\right) = \Delta_v g^\varepsilon - W(v) g^\varepsilon,$$

avec

$$W(v) = \frac{\Delta_v M}{M} = \frac{\gamma(\gamma - d + 2)|v|^2 - \gamma d}{(1+|v|^2)^2}. \tag{0.5.6}$$

L'équation peut être vu comme

$$\theta(\varepsilon)\partial_t g^\varepsilon = -\mathcal{L}_\varepsilon g^\varepsilon,$$

où

$$\mathcal{L}_\varepsilon := -(Q + \varepsilon v \cdot \nabla_x) \quad \text{avec} \quad Q := -\Delta_v + W(v).$$

Nous opérons une transformée de Fourier en x et comme l'opérateur Q a des coefficients qui ne dépendent pas de x, nous obtenons :

$$\theta(\varepsilon)\partial_t \hat{g}^\varepsilon = -\mathcal{L}_\eta \hat{g}^\varepsilon, \tag{0.5.7}$$

où

$$\mathcal{L}_\eta := -\Delta_v + W(v) + \mathrm{i}\eta v_1 \quad \text{avec} \quad v_1 = v \cdot \frac{\xi}{|\xi|} \quad \text{et} \quad \eta = \varepsilon|\xi|,$$

et où ξ étant la variable Fourier en espace.

En dimension 1 on note : $\eta = \varepsilon\xi$ et $\mathcal{L}_\eta := -\partial_v^2 + W(v) + \mathrm{i}\eta v$.

0.5.2.2 Construction d'une solution propre pour l'opérateur de Fokker-Planck en dimension 1

Cette sous-section concerne le chapitre 2 où on s'intéresse à la construction d'un couple propre $(\mu(\eta), M_{\mu,\eta})$ solution du problème spectral

$$\mathcal{L}_\eta(M_{\mu,\eta}) = \big[-\partial_v^2 + W(v) + \mathrm{i}\eta v\big] M_{\mu,\eta} = \mu M_{\mu,\eta}, \tag{0.5.8}$$

considéré en dimension 1, par une méthode qui s'étend plus facilement aux dimensions supérieures.

Théorème 0.5.2 (Solution propre pour l'opérateur Fokker-Planck en dimension 1 [DP23a]). *Supposons que $\beta \in]1,5[\backslash\{2\}$. Soit $\eta_0, \lambda_0 > 0$ assez petits. Alors, pour tout $\eta \in [0,\eta_0]$, il existe un unique couple propre $\big(\mu(\eta), M_\eta\big)$ dans $\{\mu \in \mathbb{C}, |\mu| \leqslant \eta^{\frac{2}{3}}\lambda_0\} \times L^2(\mathbb{R},\mathbb{C})$, solution du problème spectral (0.5.8). De plus, nous avons*

1. *La convergence suivante dans $L^2(\mathbb{R},\mathbb{C})$,*

$$\|M_\eta - M\|_{L^2(\mathbb{R})} \underset{\eta\to 0}{\longrightarrow} 0. \tag{0.5.9}$$

2. *La relation entre la valeur propre $\mu(\eta)$, l'échelle de la variable temporelle $\theta(\varepsilon)$ et le coefficient κ est donnée par le développement suivant :*

$$\mu(\eta) = \kappa\eta^{\frac{\beta+1}{3}}\big(1 + O(\eta^{\frac{\beta+1}{3}})\big), \tag{0.5.10}$$

où κ est une constante strictement positive donnée par

$$\kappa = -2C_\beta^2 \int_0^\infty s^{1-\gamma}\mathrm{Im}H_0(s)\mathrm{d}s, \tag{0.5.11}$$

et où H_0 est l'unique solution de l'équation

$$\Big[-\partial_s^2+\frac{\gamma(\gamma+1)}{s^2}+\mathrm{i}s\Big]H_0(s)=0,\quad \forall s\in\mathbb{R}^*, \tag{0.5.12}$$

satisfaisant

$$\int_{\{|s|\geqslant 1\}}|H_0(s)|^2\mathrm{d}s<\infty \quad \textit{et} \quad H_0(s)\underset{0}{\sim}|s|^{-\gamma}. \tag{0.5.13}$$

Pour $\eta\in[-\eta_0,0]$, par conjugaison (complexe) de l'équation, on obtient

$$\mu(\eta)=\bar{\mu}(-\eta)=\kappa|\eta|^{\frac{\beta+1}{3}}(1+O(|\eta|^{\frac{\beta+1}{3}})).$$

Remarques 0.5.3.

1. L'hypothèse $\beta\neq 2$ est technique, cela évite d'introduire des termes logarithmiques dans l'expression de $\mu(\eta)$.

2. Pour $\beta<5$, nous avons $\alpha:=\frac{\beta+1}{3}<2$.

3. Nous avons beaucoup mieux que la convergence L^2 donnée par (0.5.9), puisque nous démontrons que M_η est dominée par la fonction d'Airy qui décroît exponentiellement,

$$|M_\eta(v)|\leqslant\frac{C\eta^{\frac{\gamma}{3}}}{(\eta^{\frac{1}{3}}|v|)^{\frac{1}{4}}}\exp\left[-\frac{\sqrt{2}}{3}(\eta^{\frac{1}{3}}|v|)^{\frac{2}{3}}\right] \quad \text{pour} \quad |v|\geqslant\eta^{-\frac{1}{3}}.$$

4. Toutes les estimations obtenues dans [LP19] ont été récupérées et la formule (0.4.8) du coefficient de diffusion κ peut être retrouver.

Idées de la preuve.
La construction du couple $\big(\mu(\eta),M_\eta\big)$ se fait en deux étapes principales, toutes deux basées sur le théorème des fonctions implicites, et dont l'application de ce dernier théorème dans la première étape a été inspiré du papier [Koc15]. En effet, dans un premier temps, nous considérons ce que nous appelons une *équation pénalisée*. Nous introduisons un terme supplémentaire dans l'équation (0.5.8) afin de tuer la direction M dans le noyau de l'opérateur linéaire calculé en $\eta=0$. Cela donne l'équation suivante

$$\begin{cases}\big[-\partial_v^2+W(v)+\mathrm{i}\eta v\big]M_{\mu,\eta}(v)=\mu M_{\mu,\eta}(v)-\langle M_{\mu,\eta}-M,\Phi\rangle\Phi,\ v\in\mathbb{R},\\[2mm] M_{\mu,\eta}\in L^2(\mathbb{R}).\end{cases} \tag{0.5.14}$$

où Φ est une fonction qui satisfait certaine conditions, données dans le chapitre 2. Cette équation pénalisée a une solution pour tout μ et η sur l'espace tout entier, ce qui nous permet d'éviter le problème de recollement et de travailler directement sur $\mathbb{R}$. C'est l'un des points clés qui nous permet de généraliser cette construction en toute dimension.

L'objectif de la première étape est de montrer l'existence d'une unique solution pour l'équation (0.5.14) pour tout η et μ (fixés). Pour cela, nous décomposons l'opérateur "$-\partial_v^2 + W(v) + \mathrm{i}\eta v - \mu$" en deux parties. La première partie est choisie de telle sorte qu'elle admette "un inverse à droite" qui est continu en tant qu'opérateur linéaire entre deux espaces fonctionnels appropriés, continu par rapport aux paramètres η et μ et compact en $\eta = \mu = 0$. La seconde partie de l'opérateur est laissée dans le côté droit de l'équation, c'est-à-dire qu'elle est considérée comme un terme source. Trouver une solution à (0.5.14) revient à un processus de point fixe.

Dans la deuxième étape, pour s'assurer que le terme supplémentaire disparaît, nous devons choisir $\mu(\eta)$ obtenu par le théorème des fonctions implicites autour du point $(\mu, \eta) = (0, 0)$ appliqué à l'équation $\langle M_{\mu,\eta} - M, \Phi \rangle = 0$.

0.5.2.3 Construction d'une solution propre pour l'opérateur de Fokker-Planck en dimension quelconque

Cette sous-section est consacrée au résultat principal du chapitre 3. L'objectif est d'étendre le résultat obtenu en dimension 1 pour toute dimension $d \geqslant 1$. La stratégie est la même mais les outils sont différents, puisque la dimension 1 était basée sur des méthodes d'EDO pour inverser l'opérateur par exemple et établir certaines estimations L^∞ qui donnent d'autres estimations permettant de passer à la limite plus facilement, tandis qu'en dimension supérieure, nous n'avons pas ce type d'estimations et nous sommes obligés d'établir des estimations, en travaillant sur l'EDP, à chaque fois que nous les avons besoin.

Théorème 0.5.3 (Solution propre pour l'opérateur de Fokker-Planck pour $d \geqslant 1$ [DP23b]). *Supposons que $\beta \in]d, d+4[\setminus \{d+1\}$. Soit $\eta_0, \lambda_0 > 0$ assez petits. Alors, pour tout $\eta \in [0, \eta_0]$, il existe un unique couple propre $\big(\mu(\eta), M_\eta\big)$ dans $\{\mu \in \mathbb{C}, |\mu| \leqslant \eta^{\frac{2}{3}} \lambda_0\} \times L^2(\mathbb{R}^d, \mathbb{C})$, solution du problème spectral*

$$\mathcal{L}_\eta(M_{\mu,\eta})(v) = \big[- \Delta_v + W(v) + \mathrm{i}\eta v_1 \big] M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v), \;\; v \in \mathbb{R}^d. \tag{0.5.15}$$

De plus, nous avons

1. La convergence suivante dans l'espace de Sobolev $H^1(\mathbb{R}^d)$:

$$\big\| M_\eta - M \big\|_{H^1(\mathbb{R}^d)} \underset{\eta \to 0}{\longrightarrow} 0. \tag{0.5.16}$$

2. La valeur propre $\mu(\eta)$ satisfait

$$\mu(\eta) = \kappa \eta^{\frac{\beta-d+2}{3}} \big(1 + O(\eta^{\frac{\beta-d+2}{3}})\big), \tag{0.5.17}$$

où κ est une constante strictement positive donnée par :

$$\kappa = -2C_\beta^2 \int_{\{s_1>0\}} s_1 |s|^{-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s, \tag{0.5.18}$$

et où H_0 est l'unique solution de

$$\Big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|^2} + \mathrm{i}s_1 \Big] H_0(s) = 0, \quad \forall s \in \mathbb{R}^d \setminus \{0\}, \tag{0.5.19}$$

satisfaisant

$$\int_{\{|s_1|\geqslant 1\}} |H_0(s)|^2 \mathrm{d}s < \infty \quad \textit{et} \quad H_0(s) \underset{0}{\sim} |s|^{-\gamma}. \tag{0.5.20}$$

Remarque 0.5.4. Comme en dimension 1, l'hypothèse $\beta \neq d+1$ est technique, cela évite d'introduire des termes logarithmiques dans l'expression de $\mu(\eta)$.

Idées de la preuve.
Comme en dimension 1, la preuve du théorème 0.5.3 se fait en deux étapes principales. L'équation pénalisée est la même, avec $v \in \mathbb{R}^d$ cette fois-ci :

$$\begin{cases} \big[-\Delta_v + W(v) + \mathrm{i}\eta v_1 \big] M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v) - \langle M_{\mu,\eta} - M, \Phi \rangle \Phi(v), \ v \in \mathbb{R}^d, \\ M_{\mu,\eta} \in L^2(\mathbb{R}^d). \end{cases} \tag{0.5.21}$$

où Φ est une fonction qui satisfait certaines hypothèses. Dans le chapitre 3 nous avons choisi $\Phi := c_{\gamma,d}\langle v \rangle^{-\gamma-2}$, où $c_{\gamma,d}$ est une constante telle que $\langle M, \Phi \rangle = 1$. Comme nous l'avons dit précédemment, le terme supplémentaire $\langle M_{\mu,\eta} - M, \Phi \rangle \Phi$ nous permet d'éviter le problème de recollement et garantie l'existence d'une solution à l'équation (0.5.21) sur l'espace $\mathbb{R}^d$ tout entier. Notez également que le signe précédant le produit scalaire $\langle M_{\mu,\eta} - M, \Phi \rangle$ est important pour établir certaines estimations.

Le but de la première étape est de montrer l'existence d'une unique solution pour l'équation (0.5.21) à η et μ fixés. Une décomposition de l'opérateur "$-\Delta_v + W(v) + \mathrm{i}\eta v_1 - \mu$" en deux parties est cruciale, pour les mêmes raisons qu'en dimension 1. L'inversibilité de la première partie ainsi que l'étude de l'opérateur inverse et de ses propriétés sont basées sur une version élaborée du théorème de Lax-Milgram.

Dans la deuxième étape, pour s'assurer que le terme supplémentaire disparaît, il faut choisir $\mu(\eta)$ obtenu par le théorème des fonctions implicites autour du point $(\mu, \eta) = (0, 0)$. L'étude de cette contrainte et l'approximation de la valeur propre en dimension supérieure sont beaucoup plus compliquées que dans le cas de la dimension 1. En effet, en dimension 1, le théorèmes des fonctions implicites nous fournit une solution, ainsi qu'une convergence vers l'équilibre M, dans un espace L^∞ à poids, dont le poids est suffisamment bon (1/poids $\in L^\infty \cap L^2$), tandis qu'en dimension supérieure, la solution obtenue est dans l'espace L^2 muni de la mesure $\frac{\mathrm{d}v}{\langle v \rangle^2}$, ce qui n'est pas suffisant pour le reste de l'étude. Cependant, une étape de plus est à réaliser dans ce cas, elle consiste à améliorer l'espace auquel appartient la solution trouvée par Lax-Milgram. Elle est basée sur des estimations L^2 délicates, qui nécessitent une décomposition de l'espace $\mathbb{R}^d$ en trois parties, $|v|$ petit, $|v_1|$ petit mais $|v|$ grand, et enfin $|v_1|$ grand, dont les estimations sur chaque partie s'établirent différemment.

0.5.2.4 Lien avec la limite de diffusion

Dans cette sous-section, nous revenons sur les deux dernières remarques de la sous-sous-section 0.4.3.4 concernant le lien entre la solution propre construite et la limite de diffusion.

Théorème 0.5.4 (Limite de diffusion fractionnaire pour l'équation de Fokker-Planck)**.** *Supposons que $d < \beta < d+4$ avec $\beta \neq d+1$. Supposons que $f_0 \in L^2_{F^{-1}}(\mathbb{R}^{2d}) \cap L^\infty_{F^{-1}}(\mathbb{R}^{2d})$ est une fonction positive. Soit f^ε la solution de* (0.5.4) *dans Y, défini dans* (0.2.13)*, de donnée initiale f_0, avec $\theta(\varepsilon) = \varepsilon^\alpha$ et $\alpha := \frac{\beta-d+2}{3}$. Soit κ la constante donnée par* (0.5.18)*. Alors, f^ε converge faiblement étoile dans $L^\infty\big([0,T],\ L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ vers $\rho(t,x)F(v)$ où ρ est solution de l'équation*

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}}\rho = 0, \quad \rho(0,x) = \int_{\mathbb{R}^d} f_0 \,\mathrm{d}v. \tag{0.5.22}$$

En effet, soit $\hat{g}^\varepsilon$ la solution de l'équation (0.5.7). On a :

$$\hat{g}^\varepsilon(t,\xi,v) = e^{-t\theta(\varepsilon)\mathcal{L}_\eta}\hat{g}(0,\xi,v),$$

ce qui donne, en revenant à la variable d'espace x, ce qui suit

$$g^\varepsilon(t,x,v) = \frac{1}{(2\pi)^d}\int e^{\mathrm{i}x\cdot\xi}\ \hat{g}^\varepsilon(t,\xi,v)\ \mathrm{d}\xi.$$

En variable Fourier, ça revient à vérifier que $\hat{\rho}(t,\xi) := \displaystyle\int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot\xi}\ \rho(t,x)\ \mathrm{d}x$ satisfait

$$\partial_t\hat{\rho} + \kappa|\xi|^\alpha\hat{\rho} = 0. \tag{0.5.23}$$

Soit M_η l'unique solution dans L^2 de l'équation $\mathcal{L}_\eta(M_\eta) = \mu(\eta)M_\eta$, donnée par le Théorème 0.5.3. On a :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \hat{g}^\varepsilon(t,\xi,v)M_\eta(v)\ \mathrm{d}v &= \int \partial_t\hat{g}^\varepsilon M_\eta\ \mathrm{d}v = -\varepsilon^{-\alpha}\int \mathcal{L}_\varepsilon(\hat{g}^\varepsilon)M_\eta\ \mathrm{d}v \\ &= -\varepsilon^{-\alpha}\int \hat{g}^\varepsilon\mathcal{L}_\varepsilon(M_\eta)\ \mathrm{d}v = -\varepsilon^{-\alpha}\mu(\eta)\int \hat{g}^\varepsilon(t,\xi,v)M_\eta(v)\ \mathrm{d}v.\end{aligned}$$

Rappelons que $\hat{\rho}^\varepsilon := \displaystyle\int \hat{f}^\varepsilon\ \mathrm{d}v = \int \hat{g}^\varepsilon M\ \mathrm{d}v$ et que $\eta = \varepsilon|\xi|$. Par le Théorème 0.5.3, $\varepsilon^{-\alpha}\mu(\eta) \to \kappa|\xi|^\alpha$ et $M_\eta \to M$ dans L^2, quand $\varepsilon \to 0$. Par conséquent, en passant à la limite dans l'égalité précédente, nous obtenons l'équation suivante

$$\partial_t\hat{\rho} = -\kappa|\xi|^\alpha\hat{\rho}.$$

Ce qui signifie que $\hat{\rho}$ satisfait (0.5.23). Ainsi, la solution f^ε de (0.5.4) converge, dans un certain sens, vers $\rho(t,x)F(v)$, où ρ est la solution de l'équation de diffusion fractionnaire

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}} \rho = 0, \quad \rho(0,x) = \int f_0 \, \mathrm{d}v. \tag{0.5.24}$$

Heuristique sur le calcul de la valeur propre. A l'aide d'un calcul formel présenté en dimension 1 pour simplifier, et justifié dans les chapitres 2 et 3, nous présenterons comment l'échelle de temps $\theta(\varepsilon)$ est choisie et comment elle apparaît dans le problème spectral. Supposons que le couple $(\mu(\eta), M_{\mu,\eta})$ soit solution du problème

$$\mathcal{L}_\eta(M_{\mu,\eta}) = [-\partial_v^2 + W(v) + \mathrm{i}\eta v]M_{\mu,\eta} = \mu(\eta)M_{\mu,\eta}, \quad v \in \mathbb{R}.$$

En intégrant l'équation précédente multipliée par M et en utilisant le fait que $Q(M) = 0$, on obtient :

$$\mathrm{i}\eta \int_{\mathbb{R}} v M_{\mu,\eta} M \mathrm{d}v = \mu(\eta) \int_{\mathbb{R}} M_{\mu,\eta} M \mathrm{d}v.$$

Par conséquent,

$$\mu(\eta) = -\mathrm{i}\eta\Big(\int_{\mathbb{R}} v M_{\mu,\eta} M \mathrm{d}v\Big)\Big(\int_{\mathbb{R}} M_{\mu,\eta} M \mathrm{d}v\Big)^{-1}.$$

La convergence $M_{\mu,\eta} \to M$ quand $\eta \to 0$ implique que $\Big(\int_{\mathbb{R}} M_{\mu,\eta} M \mathrm{d}v\Big)^{-1} \to \|M\|_2^{-2} = C_\beta^2$. Formellement, par un développement de Hilbert

$$M_{\mu,\eta} = M + \eta N_\eta + o(\eta), \text{ où } [-\partial_v^2 + W(v)]N_\eta = -\mathrm{i}\eta v M,$$

on obtient

$$|M_{\mu,\eta}(v) - M(v)| \lesssim \eta \langle v \rangle^{3-\gamma} \quad \text{pour} \quad |v| \leqslant s_0 \eta^{-\frac{1}{3}},$$

et en rescalant l'intégrale sur les grandes vitesses par $v = \eta^{-\frac{1}{3}} s$ on obtient enfin :

$$\mu(\eta) \sim \eta^2 \int_{\{|v| \leqslant \eta^{-\frac{1}{3}} s_0\}} |v| \langle v \rangle^{3-2\gamma} \mathrm{d}v + 2\eta^{\frac{2\gamma+1}{3}} \int_{\{s \geqslant s_0\}} s^{1-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s,$$

où H_0 est la limite de $M_{\mu,\eta}$ rescalée, solution de l'équation réscalée limite

$$\Big[-\partial_s^2 + \frac{\gamma(\gamma+1)}{s^2} + \mathrm{i}s\Big]H_0(s) = 0, \quad \forall s \in \mathbb{R}^*.$$

Rappelons que $\mu(\eta) = \mu(\varepsilon|\xi|)$. Ainsi, si $\frac{2\gamma+1}{3} > 2$ alors on récupère de la diffusion classique avec le scaling usuel $\theta(\varepsilon) = \varepsilon^2$, et ça sera les petites vitesses qui donne le coefficient de diffusion. En revanche, si $\frac{2\gamma+1}{3} < 2$ alors, nous obtenons une diffusion fractionnaire avec une puissance de $\frac{2\gamma+1}{6}$ pour le laplacien, l'échelle de temps est donnée par $\theta(\varepsilon) = \varepsilon^{\frac{2\gamma+1}{3}}$, et le coefficient de diffusion est déterminé par l'intégrale pour les grandes vitesses dans ce cas.

0.5.2.5 Diffusion fractionnaire pour l'équation de Fokker-Planck avec drift

Dans les chapitres 2 et 3 résumés dans les deux sous-sections précédentes, nous avions travaillé avec un équilibre F donné par une formule explicite (0.5.5), en particulier, centré et symétrique (i.e. paire), et ce qui donne une formule explicite aussi pour le potentiel W. Dans cette sous-section, qui résume le chapitre 4, nous discutons le cas d'équilibres plus généraux[18]. Sous certaines hypothèses sur le comportement de F à l'infini ainsi que le potentiel W, nous obtenons de la diffusion fractionnaire pour l'équation de Fokker-Planck avec ou sans drift, selon la décroissance de F.

Notations. Soit $\beta > d+1$. On note par j_F le vecteur

$$j_F := \int_{\mathbb{R}^d} vF(v)\,\mathrm{d}v,$$

et par j_1 le scalaire

$$j_1 := \int_{\mathbb{R}^d} v_1 F(v)\,\mathrm{d}v.$$

Soit f^ε la solution de l'équation (0.5.4). On note par $\tilde{f}_\varepsilon$ la fonction définie par

$$\tilde{f}_\varepsilon(t,x,v) := f^\varepsilon(t, x - \varepsilon^{1-\alpha} j_F t, v),$$

solution de l'équation suivante :

$$\theta(\varepsilon)\partial_t \tilde{f}_\varepsilon + \varepsilon(v - j_F)\cdot\nabla_x \tilde{f}_\varepsilon = \nabla_v \cdot \left(F\nabla_v\left(\frac{\tilde{f}_\varepsilon}{F}\right)\right). \tag{0.5.25}$$

Enfin, on note par $\tilde{\rho}_\varepsilon$ la fonction $\tilde{\rho}_\varepsilon(t,x) := \rho^\varepsilon(t, x - \varepsilon^{1-\alpha} j_F t)$. Notez qu'on garde aussi les mêmes notations introduites au début de cette section ($M := F^{\frac{1}{2}}$, ...).

Hypothèses 0.5.5.

(H1) Supposons que $F(v) > 0$ pour tout $v \in \mathbb{R}^d$ et qu'il existe $\beta > d$ et une constante C tels que :

$$F(v) \underset{|v|\to\infty}{\sim} \frac{C}{|v|^\beta} \quad \text{et} \quad \int_{\mathbb{R}^d} F(v)\mathrm{d}v = \int_{\mathbb{R}^d} M^2(v)\mathrm{d}v = 1. \tag{0.5.26}$$

(H2) Concernant le potentiel W, on suppose qu'il existe une constante $\sigma \in (0, 2\gamma - d + 2)$ telle que

$$0 < W(v) - \frac{\gamma(\gamma - d + 2)}{|v|^2} \underset{|v|\to\infty}{=} O\left(\frac{1}{|v|^{2+\sigma}}\right), \tag{0.5.27}$$

où $\gamma := \frac{\beta}{2}$.

Notre construction, donnant les Théorèmes 0.5.3 et 0.5.4, permet d'aborder le cas d'un équilibre plus général satisfaisant les hypothèses 0.5.5, puisque nous n'avons pas utilisé

[18] Le cas de la dimension 1 a été fait à la fin de la première année de thèse, en adaptant la méthode donnée dans [LP19].

le fait que F (ou M) était donnée explicitement, mais plutôt utilisé la décomposition du potentiel W en deux parties dans la première étape (sa queue pour les grandes vitesses et le reste qui est d'ordre inférieur) afin de montrer l'existence de solutions pour l'équation pénalisée, et la symétrie de l'équilibre dans la deuxième étape, afin d'utiliser l'inégalité de Hardy-Poincaré pour établir certaines estimations. L'hypothèse sur le comportement de W est récupérée par l'hypothèse (H2) ci-dessus. Concernant la non symétrie de l'équilibre, elle est traitée comme suit : afin d'utiliser l'inégalité de Hardy, on rajoute et on retranche un terme qui fait apparaitre la projection de la fonction $M_{\mu,\eta}$ sur M dans l'espace L^2 muni de la mesure $\frac{\mathrm{d}v}{\langle v\rangle^2}$. Le terme qu'on rajoute est utilisé dans l'inégalité et apparait dans l'approximation de la valeur propre, tandis que l'autre terme est mis dans l'équation. Donc ça revient à faire une sorte de correction sur l'équation, ce qui donne exactement le terme de drift j_F pour $\beta > d+1$. Ce dernier terme est dominé lorsque $\beta \in (d, d+1)$ et n'apparait pas dans l'équation. Observons que $\beta = d+1$ est le seuil de la convergence absolue de l'intégrale $j_1 = \displaystyle\int_{\mathbb{R}^d} v_1 M^2(v)\mathrm{d}v$ définissant le drift macroscopique.

Proposition 0.5.6 (Approximation de la valeur propre)**.** *Soit $\beta = 2\gamma \in (d, d+4)$ et soit $\alpha := \frac{2\gamma-d+2}{3}$. Alors, la valeur propre $\mu(\eta)$ associée à la solution du problème (0.5.15) satisfait*

1. *Pour $\beta \in (d, d+1)$,*
$$\mu(\eta) = \kappa\eta^\alpha\big(1 + O(\eta^\alpha)\big), \tag{0.5.28}$$

2. *Pour $\beta \in (d+1, d+4)$,*
$$\mu(\eta) - \mathrm{i}\eta\, j_1 = \kappa\eta^\alpha\big(1 + O(\eta^\alpha)\big), \tag{0.5.29}$$

où κ est une constante strictement positive donnée par

$$\kappa := \begin{cases} \mathrm{i}\displaystyle\int_{\{|s_1|>0\}} s_1|s|^{-\gamma} H_0(s)\ \mathrm{d}s, & \textit{pour}\ \ \gamma \in (\frac{d}{2}, \frac{d+1}{2}), \\ \mathrm{i}\displaystyle\int_{\{|s_1|>0\}} s_1|s|^{-\gamma}[H_0(s) - |s|^{-\gamma}]\ \mathrm{d}s, & \textit{pour}\ \ \gamma \in (\frac{d+1}{2}, \frac{d+4}{2}), \end{cases} \tag{0.5.30}$$

et où H_0 est l'unique solution de l'équation (0.5.19) satisfaisant (0.5.20).

Pour la limite de diffusion, la convergence de $\tilde{f}_\varepsilon$ et $\tilde{\rho}_\varepsilon$, à des sous-suites près, est assurée grâce aux estimations complémentaires suivantes :

Lemme 0.5.7 (Lemme complémentaire)**.** *Soit $\beta > d+1$. Pour une donnée initiale $f_0 \in Y_F^p(\mathbb{R}^{2d})$ où $p \geqslant 2$, et pour un temps $T > 0$, nous avons les estimations suivantes :*

1. *La solution* $\tilde{f}_\varepsilon$ *de* (0.5.25) *est bornée dans* $L^\infty\big([0,T]; Y_F^p(\mathbb{R}^{2d})\big)$ *uniformément par rapport à* ε*. De plus,*

$$\|\tilde{f}_\varepsilon(T)\|_{Y_F^p}^p + \frac{p(p-1)}{\theta(\varepsilon)} \int_0^T \int_{\mathbb{R}^{2d}} \left|\nabla_v \left(\frac{\tilde{f}_\varepsilon}{F}\right)\right|^2 \left|\frac{\tilde{f}_\varepsilon}{F}\right|^{p-2} F \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \leqslant \|f_0\|_{Y_F^p}^p.$$

2. *La densité* $\tilde{\rho}^\varepsilon$ *satisfait :*

$$\|\tilde{\rho}_\varepsilon(t)\|_{L^p(\mathbb{R}^d)}^p = \|\rho^\varepsilon(t)\|_{L^p(\mathbb{R}^d)}^p \leqslant C\|f_0\|_{Y_F^p(\mathbb{R}^{2d})}^p \quad \textit{pour tout} \quad t \in [0,T].$$

3. *Supposons que* $\|f_0/F\|_\infty \leqslant C$*. Alors,*

$$\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\tilde{f}_\varepsilon - \tilde{\rho}_\varepsilon F\right|^2 \frac{\mathrm{d}v}{F}\right)^{\frac{\beta-d+2}{\beta-d}} \mathrm{d}x\mathrm{d}t \leqslant C\varepsilon^{\frac{\beta-d+2}{3}}.$$

De manière analogue à celle qui mène au Théorème 0.5.4, on obtient :

Théorème 0.5.5 (Diffusion fractionnaire pour l'équation de Fokker-Planck avec drift)**.** *Supposons que* $d < \beta < d+4$ *avec* $\beta \neq d+1$*. Supposons que* $f_0 \in L^2_{F^{-1}}(\mathbb{R}^{2d}) \cap L^\infty_{F^{-1}}(\mathbb{R}^{2d})$ *est une fonction positive. Soit* f^ε *la solution de* (0.5.4) *dans* Y*, de donnée initiale* f_0 *avec* $\theta(\varepsilon) = \varepsilon^\alpha$ *et* $\alpha := \frac{\beta-d+2}{3}$*. Soit* κ *la constante définie dans* (0.5.30)*. Alors,*

1. *Pour* $\beta \in (d, d+1)$*, la suite* f^ε *converge faiblement étoile dans* $L^\infty\big([0,T]; L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ *vers* $\rho(t,x)F(v)$ *où* ρ *est solution de l'équation*

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}}\rho = 0, \quad \rho(0,x) = \int_{\mathbb{R}^d} f_0 \,\mathrm{d}v. \tag{0.5.31}$$

2. *Pour* $\beta \in (d+1, d+4)$*, la suite* $\tilde{f}_\varepsilon$ *converge faiblement étoile dans* $L^\infty\big([0,T]; L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ *vers* $\rho(t,x)F(v)$ *où* ρ *est solution de l'équation* (0.5.31)*.*

Remarque 0.5.8. Dans le cas d'un équilibre symétrique, on récupère bien les résultats des deux sous-sections précédentes, puisque $j_1 = 0$, $j_F = 0_{\mathbb{R}^d}$ et $\overline{H_0}(-s_1, s') = H_0(s_1, s')$.

0.5.2.6 Différence avec l'approche de Bouin-Mouhot

Les travaux de cette Partie ont été faits indépendamment des résultats de E. Bouin et C. Mouhot [BM22] et la méthode est complètement différente, d'ailleurs on ne regarde pas le même problème spectral. En effet, dans cette thèse on s'est intéressé à l'amélioration et la généralisation de la construction de Lebeau-Puel pour résoudre le problème

$$\big[Q + \mathrm{i}\eta v_1\big] M_{\mu,\eta} = \mu M_{\mu,\eta}.$$

Tandis que dans [BM22], les auteurs ont considéré un problème spectral modifié, plus précisément, ils ont regardé une fonction propre avec un facteur de $\frac{1}{\langle v\rangle^2}$ à côté de la valeur propre :

$$\big[Q + \mathrm{i}\eta v_1\big]\phi_\eta = \mu \frac{\phi_\eta}{\langle v\rangle^2}, \tag{0.5.32}$$

avec $\phi_\eta \in L^2\big(\mathbb{R}^d; \frac{\mathrm{d}v}{\langle v\rangle^2}\big)$ satisfaisant $\displaystyle\int_{\mathbb{R}^d} \phi_\eta(v) M(v)\, \frac{\mathrm{d}v}{\langle v\rangle^2} = 1$. Cette idée pertinente permet "d'éviter le problème de trou spectral pour ce dernier opérateur" et d'utiliser l'inégalité

$$\int |f - rM|^2\, \frac{\mathrm{d}v}{\langle v\rangle^2} \leqslant C \int f Q(f)\, \mathrm{d}v,$$

où r est une densité à poids définie par

$$r(t,x) := \int f\, \frac{\mathrm{d}v}{\langle v\rangle^2}. \tag{0.5.33}$$

Ainsi, par une méthode totalement différente, les deux derniers auteurs ont montré l'existence de ce qu'ils ont appelé par "fluide mode", un couple $\big(\mu(\eta), \phi_\eta\big)$ solution du problème (0.5.32) ([BM22, Lemma 1.1]).

Cette dernière construction a donné comme conséquence la convergence de f^ε (solution de (0.5.4)) vers $r(t,x)F(v)$ lorsque ε tend vers 0, avec r (définie par (3.1.7)) solution de l'équation de diffusion fractionnaire (0.5.23) ([BM22, Theorem 1.4]). Enfin, la limite de diffusion avec la bonne densité $\rho := \int f\, \mathrm{d}v$ est récupérée grâce à la convergence $f^\varepsilon \to r(t,x)F(v)$, l'inégalité de Hölder et l'hypothèse $\|f_0/F\|_\infty \leqslant C$.

0.5.3 Propagation de régularité Gevrey pour les solutions du système de Vlasov-Navier-Stokes

Cette sous-section concerne la dernière Partie (III) de ce document, où nous abordons le problème de propagation de la régularité Gevrey (et d'analyticité) pour les solutions du système de Vlasov-Navier-Stokes (VNS) sur $\mathbb{T}^d \times \mathbb{R}^d$ (ou $\mathbb{R}^d \times \mathbb{R}^d$) tant qu'il existe une solution Sobolev pour ce système :

$$\left\{\begin{aligned} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[(u-v)f] &= 0, && \text{dans } (0,T)\times\mathbb{T}^d\times\mathbb{R}^d,\\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p &= j_f - \rho_f u, && \text{dans } (0,T)\times\mathbb{T}^d,\\ \nabla_x\cdot u &= 0, && \text{dans } (0,T)\times\mathbb{T}^d,\\ u(0,\cdot) = u_0, \quad f(0,\cdot,\cdot) &= f_0, \end{aligned}\right. \tag{0.5.34}$$

avec

$$\rho := \rho_f(t,x) := \int_{\mathbb{R}^d} f\, \mathrm{d}v \quad \text{et} \quad j := j_f(t,x) := \int_{\mathbb{R}^d} v f\, \mathrm{d}v,$$

où $f(t,x,v) \geqslant 0$ est la densité des particules et $u(t,x) \in \mathbb{R}^d$ est la vitesse du fluide.

Plus précisément, nous donnons des estimations quantitatives de la croissance de la norme de Gevrey et de la décroissance du rayon de régularité pour la solution de (0.5.34) en fonction de la norme de Sobolev elle-même estimée en fonction du champ de force

$\|u\|_{W^{1,\infty}}$, de la densité locale $\|\rho_f\|_\infty$ et du volume du support dans la variable de vitesse de la distribution f. Comme application, nous montrons l'existence globale de solutions Gevrey pour le système VNS dans $\mathbb{T}^3\times\mathbb{R}^3$ pour une *énergie modulée* initiale[19] suffisamment petite, grâce au résultat prouvé par Han-Kwan, Moussa et Moyano dans [HKMM20]. De plus, la propagation de la régularité Gevrey reste vraie sur $\mathbb{R}^d\times\mathbb{R}^d$ quitte à remplacer les séries de Fourier (dues au travail sur le tore) par des intégrales.

Avant d'énoncer les principaux résultats, donnons les notations et définitions nécessaires.

0.5.3.1 Notations and définitions

Nous utilisons :
• les notations multi-indices

$$v^\alpha := v_1^{\alpha_1}\dots(v_d)^{\alpha_d} \quad \text{et} \quad \mathrm{D}_\eta^\alpha := (\mathrm{i}\partial_1)^{\alpha_1}\dots(\mathrm{i}\partial_d)^{\alpha_d},$$

où $\alpha\in\mathbb{N}^d, v\in\mathbb{R}^d$, $\eta\in\mathbb{R}^d$ et $\mathrm{i}^2=-1$;
• la définition du coefficient (transformation) de Fourier de $f\in L^2(\mathbb{T}^d\times\mathbb{R}^d)$ suivante :

$$\hat{f}_k(\eta) := \frac{1}{(2\pi)^d}\iint_{\mathbb{T}^d\times\mathbb{R}^d} e^{-\mathrm{i}x\cdot k-\mathrm{i}v\cdot\eta} f(x,v)\mathrm{d}x\mathrm{d}v$$

et de $u\in L^2(\mathbb{T}^d)$:

$$\hat{u}_k := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-\mathrm{i}x\cdot k}u(x)\mathrm{d}x.$$

Nous définissons les crochets japonais : $\langle k\rangle := (1+|k|^2)^{\frac12}$ et $\langle k,\eta\rangle := (1+|k|^2+|\eta|^2)^{\frac12}$ pour tout $k,\eta\in\mathbb{R}^d$. Enfin, on note la norme de Sobolev standard de f dans $H^\sigma_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)$ par $\|f\|_\sigma$ et on note $H^\sigma_{x,v,M}(\mathbb{T}^d\times\mathbb{R}^d)$ l'espace de Sobolev à poids de norme

$$\|f\|^2_{\sigma,M} := \sum_{|\alpha|\leqslant M}\|v^\alpha f\|^2_\sigma,$$

qui peut s'écrire, en variables de Fourier, comme

$$\|f\|^2_{\sigma,M} := \sum_{|\alpha|\leqslant M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\right|^2\langle k,\eta\rangle^{2\sigma}\mathrm{d}\eta.$$

Definition 0.5.9 (Classe $\frac1s$-Gevrey dans $\mathbb{T}^d\times\mathbb{R}^d$)**.** Une fonction à valeurs réelles $f\in C^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ est dite de classe $\frac1s$-Gevrey de rayon de régularité $\lambda>0$, correction de

[19]Cette énergie a été introduit dans [CK15] et a joué un rôle important dans [HKMM20]. Elle est définie par :

$$\mathcal{E}(0) := \frac12\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)|v-\langle j_{f_0}\rangle|^2\mathrm{d}v\mathrm{d}x + \frac12\int_{\mathbb{T}^3}|u_0(x)-\langle u_0\rangle|^2\,\mathrm{d}x + \frac14\left|\langle j_{f_0}\rangle-\langle u_0\rangle\right|^2,$$

où $\langle j_{f_0}\rangle := \int_{\mathbb{R}^3} j_{f_0}(x)\mathrm{d}x$ et $\langle u_0\rangle := \int_{\mathbb{R}^3} u_0(x)\mathrm{d}x$.

Sobolev $\sigma > 0$ et poids $M \in \mathbb{N}$, si pour un certain $s \in (0,1]$, on a $f \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ et

$$\|f\|^2_{\lambda,\sigma,M,s} := \sum_{|\alpha|\leqslant M} \|v^\alpha f\|^2_{\lambda,\sigma,s} < +\infty,$$

avec

$$\|v^\alpha f\|^2_{\lambda,\sigma,s} := \|A^\sigma_k(\eta) \mathrm{D}^\alpha_\eta \hat{f}\|^2_{L^2_{k,\eta}} := \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{2\sigma} e^{2\lambda\langle k,\eta\rangle^s} |\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)|^2 \mathrm{d}\eta,$$

et où

$$A^\sigma_k(\eta) := \langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}$$

est le multiplicateur de Fourier. On note par $\mathcal{G}^{\lambda,\sigma,M,\frac{1}{s}}(\mathbb{T}^d \times \mathbb{R}^d)$ l'espace des fonctions de cette classe.

Definition 0.5.10 (Classe $\frac{1}{s}$-Gevrey dans $\mathbb{T}^d$)**.** Une fonction vectorielle réelle $u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ est dite de classe $\frac{1}{s}$-Gevrey de rayon de régularité $\lambda > 0$ et correction de Sobolev $\sigma > 0$ si pour un certain $s \in (0,1]$, on a $u \in L^2(\mathbb{T}^d)$ et

$$\|u\|^2_{\lambda,\sigma,s} := \|e^{\lambda\Lambda^s} u\|^2_\sigma := \|\Lambda^\sigma e^{\lambda\Lambda^s} u\|^2_2 := \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma} e^{2\lambda\langle k\rangle^s} |\hat{u}_k|^2 < +\infty,$$

où

$$\Lambda := (\mathrm{Id} - \Delta_x)^{\frac{1}{2}}.$$

En Fourier, $\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s} =: A^\sigma_k(0)$ est le multiplicateur de Fourier et $\hat{u}_k$ sont les coefficients de Fourier de u sur $\mathbb{T}^d$. On note $\mathcal{G}^{\lambda,\sigma,\frac{1}{s}}(\mathbb{T}^d)$ l'espace des fonctions de cette classe.

Remarque 0.5.11. Le cas $s = 1$ dans les définitions précédentes correspond aux fonctions analytiques réelles.

Enfin, on note par $V_\infty(t)$ le support en vitesse de la solution de l'équation de transport (Vlasov) f :

$$V_\infty(t) := \sup\left\{|v| : \exists\ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d \text{ tel que } f(t,x,v) > 0\right\},$$

et par $V^M(t)$ la quantité : $\quad V^M(t) := V^{2M}_\infty(t)\|f_0\|^2_\infty e^{2dt}$, où d ici c'est la dimension.

0.5.3.2 Principaux résultats

Théorème 0.5.6 (Propagation of Gevrey regularity)**.** *Soit (f_0, u_0) la donnée initiale du système VNS (0.5.34) dans $\mathbb{T}^d \times \mathbb{R}^d$ telle que, f_0 a un support compact en vitesse et $\|f_0\|_{\lambda_0,\sigma,M,s} + \|u_0\|_{\lambda_0,\sigma,s}$ est finie pour un certain $s \in (0,1]$, $\lambda_0 > 0$, $\sigma > \frac{d}{2} + \frac{s}{2} + 2$ et $M > \frac{d}{2}+1$. Alors, l'unique solution classique $(f,u) \in C(0,T_{max}; H^\sigma_{x,v;M}) \times C(0,T_{max}; H^\sigma_x)$ satisfait pour tout $t \in [0,T_{max})$ les bornes supérieures*

$$\|f\|_{\lambda,\sigma,M,s} \leqslant C_1(1+t)g(t) \tag{0.5.35}$$

et

$$\|u\|_{\lambda,\sigma,s} \leqslant \left(\|u_0\|_{\lambda_0,\sigma,s} + C_2 \int_0^t (1+\zeta) g(\zeta) e^{-C_2 \int_0^\zeta g(\tau) \mathrm{d}\tau} \mathrm{d}\zeta \right) e^{C_2 \int_0^t g(\tau)\mathrm{d}\tau}, \tag{0.5.36}$$

et pour tout $t \in [0, T_{max})$ *la borne inférieure*

$$\lambda(t) \geqslant C_3 (t+1)^{-1} \exp\Big[- C_3 \int_0^t \left(1 + \|u\|_\sigma + \|f\|_{\sigma,M}\right) \mathrm{d}\tau \Big] > 0, \tag{0.5.37}$$

où

$$g(t) := \exp\big[C_0 \int_0^t \big(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(\tau) + 1\big)\, \mathrm{d}\tau\big],$$

et où T_{max} *est le temps maximal d'existence et les constantes* C_0*,* C_1*,* C_2 *et* C_3 *dépendent de la donnée initiale* (f_0, u_0)*, du rayon de régularité* λ_0*, de la correction de Sobolev* σ*, du poids* M *et de la dimension* d*.*

Remarque 0.5.12. Si pour un certain $T \in [0, T_{\max}]$, $\sigma > \frac{d}{2} + 1$ et $M > \frac{d}{2} + 1$, on a

$$\limsup_{t \to T} \left(\|f(t)\|_{\sigma,M} + \|u(t)\|_\sigma \right) < +\infty,$$

alors, $T < T_{\max}$. Autrement dit, la régularité Gevrey continue à se propager sur $[0, T]$ tant que $\|f\|_{\sigma,M} + \|u\|_\sigma$ est borné sur $[0, T]$.

Remarque 0.5.13. Nous pourrions supprimer l'hypothèse de support compact sur les données initiales, mais les estimations de la croissance de la norme Gevrey et du rayon de régularité seraient bornées en termes de $\|f\|_{\sigma,M}$ et $\|u\|_\sigma$, au lieu de $\|u\|_{W^{1,\infty}} + \|\rho_f\|_\infty + V^M(t) + 1$, et pour un *temps court*. Si nous voulons avoir une propagation pour des solutions globales (avec données petites), nous devons avoir plus de moments finis (en vitesses) et montrer une propagation des moments dans ce cas. Ceci est analogue aux résultats de Pfaffelmoser [Pfa92] et Lions-Perthame [LP91] pour le système Vlasov-Poisson. Cette dernière contrainte vient de l'estimation du commutateur dont on a besoin pour contrôler le terme de force qui vient de l'équation de Vlasov.

Theorem 0.5.14 (Propagation d'Analyticité)**.** *Soit* (f_0, u_0) *une donnée initiale pour le système* (0.5.34) *dans* $\mathbb{T}^d \times \mathbb{R}^d$ *telle que,* $\|f_0\|_{\lambda_0,\sigma,M,1} + \|u_0\|_{\lambda_0,\sigma,1}$ *est finie pour un certain* $\lambda_0 > 0$*,* $\sigma > \frac{d+1}{2} + 2$ *et* $M > \frac{d}{2} + 1$*. Alors, la solution classique* $(f, u) \in C(0, T_{max}; H^\sigma_{x,v;M}) \times C(0, T_{max}; H^\sigma_x)$ *satisfait pour tout* $t \in [0, T_{max})$ *les bornes supérieures*

$$\|f(t)\|_{\lambda,\sigma,M,s} \leqslant \|f_0\|_{\lambda_0,\sigma,M,s} \exp\big[C_4 \int_0^t (1 + \|u(\tau)\|_\sigma) \mathrm{d}\tau\big] \tag{0.5.38}$$

et

$$\|u\|_{\lambda,\sigma,1} \leqslant \left(\|u_0\|_{\lambda_0,\sigma,s} + C_5 \int_0^t \|f(\zeta)\|_{\lambda,\sigma,M,s} e^{-C_5 \int_0^\zeta Y(\tau)\mathrm{d}\tau} \mathrm{d}\zeta \right) e^{C_5 \int_0^t Y(\tau)\mathrm{d}\tau}, \tag{0.5.39}$$

et la borne inférieure (0.5.37)*, où* $Y(\tau) := \|u(t)\|_\sigma^2 + \|f(t)\|_{\sigma,M}^2$ *et* T_{max} *est le temps maximal d'existence. Les constantes* C_4 *et* C_5 *dépendent de la donnée initiale* (f_0, u_0)*, du rayon de régularité* λ_0*, de la correction de Sobolev* σ*, du poids* M *et de la dimension* d*.*

Remarque 0.5.15 (Temps d'existence et donnée initiale)**.**

Pour tout $t < \frac{1}{C}\ln\left(\frac{1+Y_0}{Y_0}\right)$ avec $Y_0 := Y(0)$, on a l'estimation suivante :

$$\|u(t)\|_\sigma^2 + \|f(t)\|_{\sigma,M}^2 \leqslant \left(1 - \frac{Y_0}{1+Y_0}e^{Ct}\right)^{-1}.$$

Comme application du Théorème 0.5.6, nous obtenons l'existence globale des solutions Gevrey pour le système VNS (0.5.34) dans $\mathbb{T}^3 \times \mathbb{R}^3$. Ce résultat découle directement des résultats de Han-Kwan, Moussa et Moyano dans [HKMM20].

Théorème 0.5.7 (Existence globale de solutions Gevrey pour $s \in (0,1)$)**.** *Soit* (f_0, u_0) *une donnée initiale pour le système* (0.5.34) *dans* $\mathbb{T}^3 \times \mathbb{R}^3$ *telle que,* f_0 *a un support compact en vitesse et* $\|f_0\|_{\lambda_0,\sigma,M,s} + \|u_0\|_{\lambda_0,\sigma,s}$ *est finie pour certain* $s \in (0,1)$*,* $\lambda_0 > 0$*,* $\sigma > \frac{7}{2} + \frac{s}{2}$ *et* $M > \frac{5}{2}$*. Soit* $\mathcal{E}(0)$ *assez petite au sens du Théorème 2.1 dans [HKMM20]. Alors, il existe une unique solution globale classique* $(f, u) \in C(0,\infty; H^\sigma(\mathbb{T}^3 \times \mathbb{R}^3)) \times C(0,\infty; H^\sigma(\mathbb{T}^3))$ *pour le système VNS* (0.5.34) *satisfaisant pour tout* $t \in [0,\infty)$ *les bornes supérieures* (0.5.35) *et* (0.5.36)*, et le rayon de régularité* $\lambda(t)$ *satisfait la borne inférieure* (0.5.37)*.*

0.5.3.3 Idées de la preuve

Les preuves des résultats précédents reposent sur des estimations d'énergie basées sur une méthode d'espace de Fourier motivée par l'approche utilisée dans [KV09, LO97] pour étudier la propagation de la régularité Gevrey et analytique pour le système $2d$-Euler et [VR21] pour la régularité de Gevrey pour le système de Vlasov-Poisson. En effet, on établit des estimations sur les normes Gevrey des solutions des équations de Vlasov et Navier-Stokes séparément, en écrivant ces dernières équations en Fourier et en utilisant des inégalités triangulaires sur les crochets japonais pour les multiplicateurs de Fourier, des inégalités de Young (pour la convolution) pour les produits et des inégalités sur le commutateur pour les termes de force. Ensuite, afin de se libérer du couplage uf, nous combinons les deux estimations et on conclut par Gronwall. Plus précisément, les estimations sur Vlasov donnent :

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\lambda,\sigma,M,s}^2 \lesssim{}& (\|u\|_{W^{1,\infty}} + 1)\,\|f\|_{\lambda,\sigma,M,s}^2 + \|u\|_\sigma\|f\|_{\sigma,M}\|f\|_{\lambda,\sigma,M,s} \\ &+ \left(\dot{\lambda} + \lambda(1+\|u\|_\sigma) + \lambda^2\|u\|_{\lambda,\sigma,s}\right)\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}^2 \\ &+ \left(\lambda\|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|_{\lambda,\sigma+\frac{s}{2},s}\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}, \qquad (0.5.40)\end{aligned}$$

et pour Navier-Stokes :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_{\lambda,\sigma+1,s} \lesssim \left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_\sigma\|u\|_{\lambda,\sigma,s}$$
$$+ \|f\|_{\lambda,\sigma,M,s}\|u\|_{\lambda,\sigma,s}$$
$$+ \left(\dot\lambda + \lambda\|u\|_\sigma + \lambda^2(\|u\|_\sigma + \|u\|_{\lambda,\sigma,s})\right)\|u\|^2_{\lambda,\sigma+\frac{s}{2},s}. \qquad (0.5.41)$$

Ceci étant vrai pour $s \in (0,1]$, $M > \frac{d}{2}+1$ et $\sigma > \frac{d}{2}+\frac{s}{2}+2$.

Les paramètres s, σ et M sont fixes, tandis que λ est une fonction en t. Le poids dans lequel M intervient a servi pour contrôler les moments ρ_f et j_f en terme de la distribution de densité f, et la fonction λ sera choisie de façon à absorber les normes dont la régularité de Sobolev déborde (à cause des dérivées en temps dans la méthode de l'énergie : $\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}$ et $\|u\|_{\lambda,\sigma+\frac{s}{2},s}$). On peut voir que si on choisit λ de telle sorte que la norme $\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}$ soit éliminée dans l'inégalité (0.5.40) par exemple, c'est à dire λ satisfait l'inéquation différentielle

$$\dot\lambda + \lambda(1 + \|f\|_{\sigma,M} + \|u\|_\sigma) + \lambda^2(\|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}) \leqslant 0, \qquad (0.5.42)$$

on aura :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} \lesssim (\|u\|_{W^{1,\infty}} + 1)\,\|f\|^2_{\lambda,\sigma,M,s} + \|u\|_\sigma\|f\|_{\sigma,M}\|f\|_{\lambda,\sigma,M,s}.$$

Ainsi, si les normes $\|u\|_{W^{1,\infty}}$, $\|f\|_{\sigma,M}$ et $\|u\|_\sigma$ sont finis alors, le lemme de Gronwall nous permettra de conclure. Noter que pour $\sigma > \frac{d}{2}+1$, on a $\|u\|_{W^{1,\infty}} \lesssim \|u\|_\sigma$. Il revient à estimer les normes Sobolev $\|f\|_{\sigma,M}$ et $\|u\|_\sigma$. Ceci se fait de la même façon que pour Gevrey. Noter aussi que le choix de λ ci-dessus est bon pour les estimations sur u et l'inégalité (0.5.41) devient :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_{\lambda,\sigma+1,s} \lesssim (\|\nabla_x u\|_\infty + \|f\|_{\sigma,M})\,\|u\|^2_{\lambda,\sigma,s} + (\|u\|^2_\sigma + \|f\|_{\lambda,\sigma,M,s})\|u\|_{\lambda,\sigma,s}.$$

Enfin, pour trouver une fonction λ satisfaisant (0.5.42), il suffit de prendre la fonction

$$\lambda(t) := e^{-\int_0^t a(\tau)\mathrm{d}\tau}\left(\lambda_0^{-1} + \int_0^t b(\tau)e^{-\int_0^\tau a(\zeta)\mathrm{d}\zeta}\,\mathrm{d}\tau\right)^{-1}$$

solution de l'équation différentielle

$$\dot\lambda + a(t)\lambda + b(t)\lambda^2 = 0,$$

avec $a(t) := 1 + \|f(t)\|_{\sigma,M} + \|u(t)\|_\sigma$ et $b(t) := \|f(t)\|_{\lambda,\sigma,M,s} + \|u(t)\|_{\lambda,\sigma,s}$.

Fractional diffusion for the linear Boltzmann equation

CHAPTER 1

Fractional diffusion for the linear Boltzmann equation with heavy-tail equilibrium and general cross-section

Résumé

Ce chapitre est consacré à la limite hydrodynamique de l'équation linéaire de Boltzmann, dans le cas d'un équilibre à queue lourde et d'une section efficace qui dépend de la variable d'espace et qui dégénère pour les grandes vitesses. Pour une échelle de temps appropriée, une équation de diffusion fractionnaire est obtenue. Ce problème a été traité dans [MMM11] pour une section efficace indépendante de l'espace, en utilisant une transformée de Fourier-Laplace, et traité dans [Mel10] pour une section efficace dépendant de l'espace mais bornée, en utilisant la méthode des Moments en introduisant un problème auxiliaire. Dans ce travail, nous adapterons la dernière méthode, tout en combinant les hypothèses des deux références citées.

Abstract

This chapter is devoted to the hydrodynamic limit for the linear Boltzmann equation, in the case of a heavy tail equilibrium and a cross section which depends on the space variable and which degenerates for large velocities. For an appropriate time scale, a fractional diffusion equation is obtained. This problem has been addressed in [MMM11] for a space-independent cross section, using a Fourier-Laplace transform, and addressed in [Mel10] for a space-dependent but bounded cross section, using the Moments method by introducing an auxiliary problem. In this work, we will adapt the last method, while combining the assumptions of the two cited references.

Contents

1.1 Introduction

1.1.1 Setting of the problem

In this chapter, we are dealing with the following parameterized linear kinetic equation:

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v\cdot\nabla_x f^\varepsilon = Q(f^\varepsilon), & x\in\mathbb{R}^d, v\in\mathbb{R}^d, t>0, \\ \\ f^\varepsilon(0,x,v) = f_0(x,v), & x\in\mathbb{R}^d, v\in\mathbb{R}^d. \end{cases} \tag{1.1.1}$$

where ε is a positive parameter and $\theta(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ to be chosen later, and where Q is the collisional linear Boltzmann operator given by

$$Q(f) := \int_{\mathbb{R}^d} [\sigma(x,v,v')f(v') - \sigma(x,v',v)f(v)]\,\mathrm{d}v' \tag{1.1.2}$$

with a non-negative *collision kernel* $\sigma = \sigma(x,v,v') \geqslant 0$. For non-negative initial data f_0, the unknown $f^\varepsilon(t,x,v) \geqslant 0$ can be interpreted as the density of particles occupying at time $t \geqslant 0$, the position $x\in\mathbb{R}^d$ with velocity $v\in\mathbb{R}^d$, since for the following rescaling

$$t' = \frac{t}{\theta(\varepsilon)} \quad \text{and} \quad x' = \frac{x}{\varepsilon},$$

the function f^ε can be seen as solution of the family of equations (without primes)

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = Q_\varepsilon(f), & t\geqslant 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \\ \\ f(0,x,v) = f_0(x,v), & x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \end{cases} \tag{1.1.3}$$

where

$$Q_\varepsilon(f) := \int_{\mathbb{R}^d} [\sigma(\varepsilon x,v,v')f(v') - \sigma(\varepsilon x,v',v)f(v)]\,\mathrm{d}v'.$$

Note that for ε small enough, the time and position variables considered in equation (1.1.1) are much larger than those in equation (1.1.3). Thus, the study of the behavior of

f^ε when ε tends to 0 can be seen as a kind of hydrodynamic limit for the kinetic equation (1.1.3), whose parameter ε designates the mean free path, and plays the role of Knudsen's number here.

Some classical notations: The collision operator Q can be decomposed into a "gain" term and a "loss" term as follows:

$$Q(f) = Q^+(f) - Q^-(f),$$

with

$$Q^+(f) := \int_{\mathbb{R}^d} \sigma(x,v,v')f(v')\mathrm{d}v' \qquad \text{and} \qquad Q^-(f) := \nu(x,v)f,$$

where ν is the *collision frequency* defined by

$$\nu(x,v) := \int_{\mathbb{R}^d} \sigma(x,v',v)\mathrm{d}v' = \frac{Q^+(F)}{F(v)}, \tag{1.1.4}$$

and F is the equilibrium of Q, a positive function satisfying:

$$Q(F) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} F(v)\ \mathrm{d}v = 1. \tag{1.1.5}$$

Remark 1.1.1.

1. This splitting, $Q = Q^+ - Q^-$, is not possible in general because the integrals defining each piece diverge, but this is not the case in this paper.

2. Such an equilibrium F satisfying (1.1.5) exists under certain assumptions on the collision kernel σ, thanks to the Krein-Rutman theorem (see [DGP00] for more details). A particular case in which this condition is satisfied is when σ is such that

$$\forall x,v,v' \in \mathbb{R}^d, \quad \sigma(x,v,v') = \mathrm{b}(x,v,v')F(v) \quad \text{with} \quad \mathrm{b}(x,v',v) = \mathrm{b}(x,v,v'), \tag{1.1.6}$$

for some non-negative $\mathrm{b} \in L^1_{loc}(\mathbb{R}^{3d})$.

3. We return to the Assumptions on σ and F with more details in the next subsection. See Remarks & Examples 3.1 in [MMM11] for a discussion of Assumptions about the cross section, and Lemma 6.1 in the same reference for the proof of statements in some cases.

Formally, passing to the limit when $\varepsilon \to 0$ in equation (1.1.1), we obtain that the limit f^0 is in the kernel of Q which is spanned by the equilibrium F, which means that $f^0 = \rho(t,x)F(v)$. Thus, it amounts to find the equation satisfied by the density ρ. This question of approximation of kinetic equations by macroscopic equations has a long history, dating back to the pioneering works of E. Wigner [WB61], A. Bensoussan, J.L. Lions, and G. Papanicolaou [BLP79], as well as E.W. Larsen and J.B. Keller [LK74]. Since then, numerous papers have addressed this topic (for further references, see the papers by C. Bardos, R. Santos, and R. Sentis [BSS84], and P. Degond, T. Goudon, and F. Poupaud [DGP00]). The resulting equations fall into two categories, depending on the rate of decrease in velocity of the equilibrium F. It is well-known (see [DGP00] for instance) that when F decreases *rapidly* for large velocities (such as when F follows a Maxwellian distribution function), they proved, in particular, that for the classical scaling $\theta(\varepsilon) = \varepsilon^2$, the distribution function f^ε converges to $\rho(t,x)F(v)$ as ε goes to 0, with ρ being the solution of the diffusion equation

$$\partial_t \rho - \nabla_x \cdot (D \nabla_x \rho) = 0, \tag{1.1.7}$$

where

$$D = \int_{\mathbb{R}^d} (v \otimes v) \frac{F(v)}{\nu(v)} \mathrm{d}v.$$

When F decreases *slowly* and it is a heavy tail distribution function, satisfying

$$F(v) \sim \frac{\kappa}{|v|^{d+\alpha}} \quad \text{as} \quad |v| \to \infty \tag{1.1.8}$$

for some $\alpha > 0$, the previous diffusion matrix D, given by (1.1.7) might be infinite. In that case, the diffusion limit leading to (1.1.7) breaks down, which means that the choice of time scale $\theta(\varepsilon) = \varepsilon^2$ was inappropriate. It has been shown in [MMM11] and [Mel10], using different methods and under various assumptions, that in such cases, the appropriate time scale involves the parameter α, which appears in the equilibrium. More specifically, for $\theta(\varepsilon) = \varepsilon^\gamma$ with γ depends on α, the following fractional diffusion equation is obtained:

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\gamma}{2}} \rho = 0, \tag{1.1.9}$$

where the fractional Laplacian appearing in the previous equation can be defined by the following singular integral:

$$(-\Delta)^{\frac{\gamma}{2}} u(x) := c_{\alpha,d} \ \mathrm{PV} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+\gamma}} \ \mathrm{d}y.$$

Let's now discuss in more detail the contexts of the last two references cited: In [MMM11], the authors addressed the problem in the space homogeneous case (that is with σ independent of x). They prouved that when F satisfies (1.1.8) and the collision frequency ν satisfies

$$\nu(v) \sim \nu_0 |v|^\beta \quad \text{as} \quad |v| \to \infty,$$

then for $\alpha > 0$ and $\beta < \min(\alpha; 2-\alpha)$, and by taking $\gamma := \frac{\alpha-\beta}{1-\beta}$, the scaling $\theta(\varepsilon) = \varepsilon^\gamma$ leads to the previous fractional diffusion equation. While in [Mel10], the same problem has been addressed but for a cross section which depends on the position variable x but assuming that the collision frequency is bounded, more precisely

$$0 < \nu_1 F(v) \leqslant \sigma(x, v, v') \leqslant \nu_2 F(v), \tag{1.1.10}$$

i.e. for $\beta = 0$ compared to the case of [MMM11], which gives in particular,

$$\nu_1 \leqslant \nu(x, v) \leqslant \nu_2. \tag{1.1.11}$$

With the additional assumption

$$\nu(x, v) \sim \nu_0(x) \quad \text{as} \quad |v| \to \infty \quad \text{with} \quad \nu_1 < \nu_0(x) < \nu_2, \tag{1.1.12}$$

the same type of macroscopic equation as (1.1.9) was found, with an elliptic operator $\mathcal{L}$ of the same order as $(-\Delta)^{\frac{\alpha}{2}}$:

$$\mathcal{L}(\rho) := \mathrm{PV} \int_{\mathbb{R}^d} \eta(x, y) \frac{\rho(x) - \rho(y)}{|x - y|^{d+\alpha}} \, \mathrm{d}y,$$

with

$$\eta(x, y) = \nu_0(x)\nu_0(y) \int_0^\infty z^\alpha e^{-z \int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s} \, \mathrm{d}z.$$

Towards the end of this section, we mention some other references to previous results in the context of anomalous and fractional diffusion. The earliest work dates back to [BGT92], where the authors showed the diffusion limit of free molecular flow in thin plane channels. There are also the works of Dogbé and Golse on Anomalous Diffusion for the Knudsen gas, see for instance [Gol98, Dog98, Dog00]. The term "anomalous" here stems from the scaling $\theta(\varepsilon) = \varepsilon^2 \ln(\varepsilon^{-1})$ considered in the critical case (i.e., the limiting case for the integrability of $\int (v \otimes v) \frac{F(v)}{\nu(v)} \mathrm{d}v$). Regarding fractional diffusion, the first result was obtained simultaneously by A. Mellet, S. Mischler, and C. Mouhot [MMM11], and M. Jara, T. Komorowski, and S. Olla [JKO09], using two completely different approaches (using a probabilistic approach for the latter refrence). This was followed by the work of A. Mellet [Mel10], N. BenAbdellah, A. Mellet, and M. Puel for a collision frequency that degenerates for small velocities with Maxwellian equilibrium in [BAMP11a], and for heavy-tail equilibrium with cross-section satisfying the assumptions (1.1.10), (1.1.11), and (1.1.12) in [BAMP11b] for a Sobolev initial data to obtain strong convergence of f^ε to $\rho(t, x)F(v)$ (via a completely different method from the previous ones, based on a Hilbert expansion).

The goal of this chapter is to show that a similar phenomenon can arise and a fractional diffusion equation is obtained from (1.1.1) in situations in which the equilibrium is heavy-tailed (satisfies (1.1.8) for some $\alpha > 0$), the cross-section σ depends on the space variable x and the collision frequency ν degenerates for large velocities

$$\nu(x, v) \sim |v|^\beta \nu_0(x) \quad \text{as} \quad |v| \to \infty \quad \text{with} \quad \nu_1 < \nu_0(x) < \nu_2,$$

for some $\beta \in \mathbb{R}$ (with $\beta < \min(\alpha; 2 - \alpha)$). For this, we will adapt the method used in [Mel10], slightly modifying the auxiliary problem in order to handle the situation when ν

degenerates for $|v|$ large enough and considering for φ smooth, the equation

$$|v|^{-\beta}\nu(x,v)\chi^{\varepsilon} - \varepsilon|v|^{-\beta}v\cdot\nabla_x\chi^{\varepsilon} = |v|^{-\beta}\nu(x,v)\varphi(t,x)$$

for $|v| \geqslant 1$ and keeping the same problem for $|v| \leqslant 1$. The function $\chi^{\varepsilon}(t,x,v)$, which will converge to φ when ε goes to 0, will be considered as the test function in the weak formulation of the equation (1.1.1). Also, due to the degeneracy of ν, the operator Q has no spectral gap and the passing to the limit in most of terms in the weak formulation is treated differently to [Mel10]. We only get weak limits.

1.1.2 Main result

First, we denote by L^p_ω the space L^p endowed with the measure $\omega\mathrm{d}v$ (or $\omega\mathrm{d}v\mathrm{d}x$). For example, $L^2_{\nu F^{-1}}(\mathbb{R}^d)$ is the space defined by

$$L^2_{\nu F^{-1}}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R};\ \int_{\mathbb{R}^d} |f|^2 \frac{\nu}{F}\,\mathrm{d}v < \infty \right\}.$$

Before stating the result, we need to make the conditions on σ and F precise.

Assumptions 1.1.2. The first two assumptions are very standard:

(A1) The cross section $\sigma(x,v,v')$ is non negative and locally integrable on $\mathbb{R}^{2d}$ for all x. The collision frequency $\nu(x,v) = \int_{\mathbb{R}^d}\sigma(x,v',v)\mathrm{d}v'$ satisfies:

$$\nu(x,-v) = \nu(x,v) > 0, \qquad \forall x, v \in \mathbb{R}^d.$$

(A2) There is a function $F(v) \in L^1_\nu(\mathbb{R}^d)$ independent of x such that:

$$Q(F) = 0.$$

Then, $Q^+(F) = \nu F$. Furthermore, the function F is symmetric, positive and normalized to 1:

$$F(-v) = F(v) > 0 \quad \text{for all } v \in \mathbb{R}^d \quad \text{and} \quad \int_{\mathbb{R}^d} F(v)\mathrm{d}v = 1.$$

The next assumptions concern the behavior of F and ν for large $|v|$. We combined the assumptions of [MMM11] and [Mel10] to work with a more general collision frequency.

(B1) There exists $\alpha > 0$ and a constant $\kappa > 0$ such that

$$|v|^{\alpha+d}F(v) \longrightarrow \kappa \quad \text{as} \quad |v| \to \infty.$$

(B2) There exists ν_1 and ν_2 two positive constants, a function $\nu_0(x)$ and a constant $\beta \in \mathbb{R}$ such that

$$\nu_1\langle v\rangle^{\beta} \leqslant \nu(x,v) \leqslant \nu_2\langle v\rangle^{\beta}, \qquad \forall x, v \in \mathbb{R}^d$$

and

$$|v|^{-\beta}\nu(x,v) \longrightarrow \nu_0(x) \quad \text{as} \quad |v| \longrightarrow \infty, \qquad \text{uniformly with respect to } x.$$

We assume that ν is C^1 with respect to x and

$$\left\| \langle v\rangle^{-\beta}\partial_x \nu(x,v) \right\|_{L^\infty(\mathbb{R}^{2d})} \leqslant C.$$

(B3) Finally, we assume that there exists a positive constant M such that

$$\int_{\mathbb{R}^d} F' \frac{\nu}{\mathrm{b}} \, \mathrm{d}v' + \left(\int_{\mathbb{R}^d} \frac{F'}{\nu'} \frac{\mathrm{b}^2}{\nu^2} \, \mathrm{d}v' \right)^{\frac{1}{2}} \leqslant M, \qquad \forall x, v \in \mathbb{R}^d,$$

with $\mathrm{b} := \mathrm{b}(x,v,v') := \sigma(x,v,v')F^{-1}(v)$, $F' = F(v')$ and $\nu' := \nu(x,v')$.

Remark 1.1.3.

1. This last assumption (B3) concerns the collision operator Q, so that it is bounded and has the property of weighted coercivity.

2. Assumptions (A1-A2) and (B1-B2-B3) are exactly the assumptions imposed in [MMM11] and [Mel10], with ν independent of x in [MMM11] and $\beta = 0$ in [Mel10].

Theorem 1.1.4 (Fractional diffusion limit for the linear Boltzmann equation)**.**
Assume (A1-A2) and (B1-B2-B3) with $\alpha > 0$ and $\beta < \min\{\alpha; 2-\alpha\}$. We define

$$\gamma := \frac{\alpha - \beta}{1-\beta}.$$

Let $f^\varepsilon(t,x,v)$ be the solution of (1.1.1) with $\theta(\varepsilon) = \varepsilon^\gamma$ and non-negative $f_0 \in L^2_{F^{-1}}(\mathbb{R}^{2d})$. Then, f^ε converges weakly star in $L^\infty(0,T; L^2_{\nu F^{-1}}(\mathbb{R}^{2d}))$ to $\rho(x,t)F(v)$ with ρ solution to

$$\begin{cases} \partial_t \rho + \kappa \mathcal{L}(\rho) = 0, \\[2ex] \rho(x,0) = \rho_0(x) := \displaystyle\int_{\mathbb{R}^d} f_0(x,v) \, \mathrm{d}v, \end{cases} \tag{1.1.13}$$

where $\mathcal{L}$ is an elliptic operator of order γ defined by

$$\mathcal{L}(\rho) := \frac{1}{1-\beta} \, \mathrm{PV} \int_{\mathbb{R}^d} \eta(x,y) \frac{\rho(x)-\rho(y)}{|x-y|^{d+\gamma}} \, \mathrm{d}y, \tag{1.1.14}$$

with

$$\eta(x,y) := \nu_0(x)\nu_0(y) \int_0^\infty z^\gamma e^{-z\int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s} \, \mathrm{d}z.$$

Remark 1.1.5.

1. Note that $\beta < 1$ and $\gamma < 2$ for $\alpha > 0$ and $\beta < \min\{\alpha; 2-\alpha\}$.

2. From $(B2)$, there exists η_1 and η_2 such that

$$0 < \eta_1 \leqslant \eta(x,y) \leqslant \eta_2 < \infty.$$

   In particular, the operator $\mathcal{L}$ has the same order as the fractional Laplacian operator $(-\Delta)^{\gamma/2}$. Moreover, for σ independent of x, the function η is constant and the formula (1.1.14) gives exactly $(-\Delta)^{\gamma/2}$ up to a multiplicative positive constant.

3. The operator $\mathcal{L}$ is self-adjoint since $\eta(x,y) = \eta(y,x)$.

4. For a cross section σ independent of x, we recover Theorem 3.2 in [MMM11] and for $\beta = 0$ we recover the results of [Mel10].

1.2 Preliminaries results

We start by recalling some classical properties of the collision operator Q.

1.2.1 Properties of the operator Q

Under Assumptions (A1-A2) and (B3), we have the following Proposition:

Proposition 1.2.1. *Let f and g two functions in $L^2_{\nu F^{-1}}(\mathbb{R}^d)$. Then we have the following assertions:*

1. *The operator $Q : L^1_\nu \longrightarrow L^1$ is bounded and conservative, so the equation (1.1.3) preserves the total mass of the distributionf and one has*

$$\int_{\mathbb{R}^d} Q(f)\ \mathrm{d}v = 0, \quad \textit{pour tout } f \in L^1_\nu(\mathbb{R}^d).$$

2. *The operator $\frac{1}{\nu}Q$ is bounded in $L^2_{\nu F^{-1}}$ and Q is dissipative. Moreover,*

$$\int_{\mathbb{R}^d} Q(f) f\ \frac{\mathrm{d}v}{F} \leqslant -\frac{1}{2M}\int_{\mathbb{R}^d} |f - \rho F|^2\ \frac{\nu \mathrm{d}v}{F}, \quad \forall f \in L^2_{\nu F^{-1}}. \tag{1.2.1}$$

   where $\rho := \displaystyle\int_{\mathbb{R}^d} f\ \mathrm{d}v$ and M is the constant given in (B3).

Remark 1.2.2. 1. It should be noted that for an unbounded collision frequency, which degenerates for large velocities, for example, the integrals in inequality (1.2.1) are conducted with two different measures, namely $\frac{\mathrm{d}v}{F}$ and $\frac{\nu \mathrm{d}v}{F}$. Thus, the operator Q does not have a spectral gap for the type of cross-sections or collision frequencies considered in this work.

2. Note that for $F \in L^2_{\nu F^{-1}}(\mathbb{R}^d)$, we get the following inclusion $L^2_{\nu F^{-1}}(\mathbb{R}^d) \subset L^1_\nu(\mathbb{R}^d)$.

Proof of Proposition 1.2.1. We adapt the proof of Lemma 4.1 in [MMM11], whose authors had adapted the proof of Proposition 1 & 2 from [DGP00]. Recall that $Q(f) = Q^+(f) - \nu f$. Thus, to show that $\frac{1}{\nu}Q$ is bounded in $L^2_{\nu F^{-1}}$, it amounts to showing that $\frac{1}{\nu}Q^+$ is bounded. Let $f \in L^2_{\nu F^{-1}}$. By using the fact that $Q^+(F) = \nu F$ and $\left(\int \sigma f \mathrm{d}v\right)^2 \leqslant \left(\int \sigma \frac{f^2}{F}\mathrm{d}v\right)\left(\int \sigma F \mathrm{d}v\right)$, we write

$$\begin{aligned}
\left\|\frac{1}{\nu}Q^+(f)\right\|^2_{L^2(\nu F^{-1})} &= \int_{\mathbb{R}^d} \frac{\left|Q^+(f)\right|^2}{\nu F}\mathrm{d}v \\
&\leqslant \int_{\mathbb{R}^d} \frac{1}{\nu F}\left(\int_{\mathbb{R}^d} \sigma(v,v')F(v')\mathrm{d}v'\right)\left(\int_{\mathbb{R}^d} \sigma(v,v')\frac{|f(v')|^2}{F(v')}\mathrm{d}v'\right)\mathrm{d}v \\
&= \int_{\mathbb{R}^d} \frac{1}{\nu F}Q^+(F)\int_{\mathbb{R}^d} \sigma(v,v')\frac{|f(v')|^2}{F(v')}\mathrm{d}v'\mathrm{d}v \\
&= \int_{\mathbb{R}^{2d}} \sigma(v,v')\frac{|f(v')|^2}{F(v')}\mathrm{d}v'\mathrm{d}v \\
&= \iint_{\mathbb{R}^{2d}} \nu(v')\frac{|f(v')|^2}{F(v')}\mathrm{d}v' \\
&= \|f\|^2_{L^2_{\nu F^{-1}}}.
\end{aligned}$$

Now, let's prouve the inequality (1.2.1). We have

$$\int_{\mathbb{R}^d} Q(f)\frac{f}{F}\mathrm{d}v = \iint_{\mathbb{R}^{2d}} \sigma(v,v')f'\frac{f}{F}\mathrm{d}v'\mathrm{d}v - \int_{\mathbb{R}^d} \nu\frac{f^2}{F}\mathrm{d}v.$$

On the one hand,

$$\int_{\mathbb{R}^d} \nu(v)\frac{f^2}{F}\mathrm{d}v = \iint_{\mathbb{R}^{2d}} \sigma(v',v)F\frac{f^2}{F^2}\mathrm{d}v\mathrm{d}v' = \iint_{\mathbb{R}^{2d}} \sigma(v,v')F'\frac{f'^2}{F'^2}\mathrm{d}v\mathrm{d}v'.$$

On the other hand,

$$\int_{\mathbb{R}^d} \nu(v)\frac{f^2}{F}\mathrm{d}v = \int_{\mathbb{R}^d} Q^+(F)\frac{f^2}{F^2}\mathrm{d}v = \iint_{\mathbb{R}^{2d}} \sigma(v,v')F'\frac{f^2}{F^2}.$$

Hence,

$$\int_{\mathbb{R}^d} Q(f)\frac{f}{F}\mathrm{d}v = -\frac{1}{2}\iint_{\mathbb{R}^{2d}} \sigma(v,v')F'\left[\frac{f}{F} - \frac{f'}{F'}\right]^2\mathrm{d}v'\mathrm{d}v.$$

Moreover, we have

$$fF' - f'F = \left(\frac{f}{F} - \frac{f'}{F'}\right)FF'.$$

Then,

$$g := f - \rho F = \int_{\mathbb{R}^d} (fF' - f'F)\mathrm{d}v' = \int_{\mathbb{R}^d} \left(\frac{f}{F} - \frac{f'}{F'}\right)FF'\mathrm{d}v',$$

which implies that

$$g^2 \leqslant \left(\int_{\mathbb{R}^d} \sigma(v,v') F' \left(\frac{f}{F} - \frac{f'}{F'} \right)^2 \mathrm{d}v' \right) \left(\int_{\mathbb{R}^d} \frac{F^2}{\sigma(v,v')} F' \mathrm{d}v' \right).$$

Thus, by integrating this last inequality against $\frac{F}{\nu}$ and using the hypothesis (B3), we obtain

$$\int_{\mathbb{R}^d} g^2 \frac{\nu}{F} \,\mathrm{d}v \leqslant \left(\sup_{v\in\mathbb{R}^d} \left(\nu \int_{\mathbb{R}^d} \frac{F}{\sigma} F' \mathrm{d}v' \right) \right) \left(\iint_{\mathbb{R}^{2d}} \sigma(v,v') F' \left(\frac{f}{F} - \frac{f'}{F'} \right)^2 \mathrm{d}v'\mathrm{d}v \right).$$

Therefore,

$$\int_{\mathbb{R}^d} g^2 \frac{\nu}{F} \mathrm{d}v \leqslant -2M \int_{\mathbb{R}^d} Q(f) \frac{f}{F} \mathrm{d}v.$$

Hence the inequality of the second point of Proposition 1.2.1. □

Remark 1.2.3. If we had considered the integral $\int_{\mathbb{R}^d} g^2 \frac{\mathrm{d}v}{F}$ instead of $\int_{\mathbb{R}^d} g^2 \frac{\nu}{F}\,\mathrm{d}v$, we would have obtained $\sup_{v\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{FF'}{\sigma} \mathrm{d}v' \right)$ in the right-hand side of the penultimate inequality, which may not be bounded for typical physical examples where $\nu(v) \sim |v|^\beta$.

1.2.2 Compactness lemma

As a consequence of the previous Proposition, we get the following compactness lemma:

Lemma 1.2.4. *For initial datum $f_0 \in L^2_{F^{-1}}(\mathbb{R}^{2d})$ and a positive time T,*

1. *The solution f^ε of (1.1.1) is bounded in $L^\infty\big(0,T; L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ uniformly with respet to ε. Moreover,*
$$\|f^\varepsilon - \rho^\varepsilon F\|^2_{L^\infty(0,T,L^2_{\nu F^{-1}})} \leqslant C\|f_0\|^2_{L^2_{F^{-1}}} \theta(\varepsilon). \tag{1.2.2}$$
2. *The density $\rho^\varepsilon := \displaystyle\int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}v$ is bounded in $L^\infty\big(0,T; L^2(\mathbb{R}^d)\big)$ uniformly with respet to ε and one has*
$$\|\rho^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \|f_0\|_{L^2_{F^{-1}}}. \tag{1.2.3}$$

In particular, ρ^ε converges weakly star in $L^\infty(0,T;L^2(\mathbb{R}^d))$ to ρ and f^ε converges weakly star in $L^\infty(0,T;L^2_{\nu F^{-1}}(\mathbb{R}^{2d}))$ to $f = \rho(x,t)F(v)$.

Proof. 1. By integrating the equation (1.1.1) multiplied by f^ε/F and using the inequality (1.2.1), we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} \frac{|f^\varepsilon|^2}{F} \mathrm{d}x\mathrm{d}v &= \frac{1}{\theta(\varepsilon)} \int_{\mathbb{R}^{2d}} Q(f^\varepsilon) \frac{f^\varepsilon}{F} \mathrm{d}x\mathrm{d}v \\ &\leqslant -\frac{1}{2M\theta(\varepsilon)} \int_{\mathbb{R}^{2d}} \left| f^\varepsilon - \rho^\varepsilon F \right|^2 \frac{\nu}{F} \mathrm{d}x\mathrm{d}v.\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{R}^{2d}} |f^\varepsilon(t,x,v)|^2 \frac{\mathrm{d}x\mathrm{d}v}{F} + \frac{1}{2M\theta(\varepsilon)}\int_0^t \int_{\mathbb{R}^{2d}} \left|f^\varepsilon - \rho^\varepsilon F\right|^2 \frac{\nu}{F}\mathrm{d}x\mathrm{d}v \leqslant \frac{1}{2}\int_{\mathbb{R}^{2d}} |f_0(x,v)|^2 \frac{\mathrm{d}x\mathrm{d}v}{F}.$$

This inequality shows that f^ε is uniformly bounded in $L^\infty(0,T,L^2_{F^{-1}})$ and gives also the inequality (1.2.2).

2. For the second point, we write

$$\int_{\mathbb{R}^d} |\rho^\varepsilon|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f^\varepsilon \mathrm{d}v \right|^2 \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^d} F\mathrm{d}v\right)\left(\int_{\mathbb{R}^{2d}} |f^\varepsilon|^2 \frac{\mathrm{d}v\mathrm{d}x}{F}\right) = \|f^\varepsilon\|^2_{L^2_{F^{-1}}}.$$

□

1.2.3 The auxiliary problem

We adapt the method used in [Mel10] and this step is the key of the proof: it consists in correcting the error on the test function in the weak formulation. Indeed, for a test function $\varphi \in \mathcal{D}([0,+\infty)\times\mathbb{R}^d)$, we introduce $\chi^\varepsilon(t,x,v)$ solution of

$$\lceil v\rfloor^{-\beta}\nu(x,v)\chi^\varepsilon - \varepsilon\lceil v\rfloor^{-\beta} v\cdot\nabla_x\chi^\varepsilon = \lceil v\rfloor^{-\beta}\nu(x,v)\varphi(t,x), \tag{1.2.4}$$

where

$$\lceil v\rfloor^{-\beta} := \begin{cases} 1 & \text{for} \quad |v|\leqslant 1, \\ |v|^{-\beta} & \text{for} \quad |v|\geqslant 1. \end{cases}$$

Note that for $\sigma>0$,

$$\lceil v\rfloor^{\sigma} \leqslant \langle v\rangle^{\sigma} \leqslant \sqrt{2}\lceil v\rfloor^{\sigma}.$$

The trick which makes it possible to generalize the work given in [Mel10] to the cross section which degenerates for large velocities is in the term $\lceil v\rfloor^{-\beta}$ introduced in the previous equation. It allows to renormalize the collision frequency ν in some sense, and it is as if we consider particles moving with a velocity $\lceil v\rfloor^{-\beta}v$.

We can integrate the equation (1.2.4) and its solution χ^ε is given explicitly, according to the function φ, by the following formula:

$$\chi^\varepsilon(t,x,v) = \int_0^\infty \lceil v\rfloor^{-\beta}\nu(x+\varepsilon\lceil v\rfloor^{-\beta}vz,v)\; e^{-\int_0^z \lceil v\rfloor^{-\beta}\nu(x+\varepsilon\lceil v\rfloor^{-\beta}vs,v)\mathrm{d}s}\varphi(t,x+\varepsilon\lceil v\rfloor^{-\beta}vz)\;\mathrm{d}z.$$

In order to simplify the writing and to avoid long expressions, we denote by $\tilde\nu$ and $\tilde\varphi$ the two following functions:

$$\tilde\nu(x,v,z) := \nu(x+\varepsilon\lceil v\rfloor^{-\beta}vz,v) \qquad \text{and} \qquad \tilde\varphi(t,x,z) := \varphi(t,x+\varepsilon\lceil v\rfloor^{-\beta}vz).$$

Then,

$$\chi^\varepsilon(t,x,v) = \int_0^\infty \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,s)\mathrm{d}s} \tilde{\varphi}(t,x,z)\ \mathrm{d}z. \tag{1.2.5}$$

The function χ^ε is smooth. Furthermore, thanks to the identity

$$\int_0^\infty \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,s)\mathrm{d}s}\ \mathrm{d}z = \int_0^\infty e^{-u}\ \mathrm{d}u = 1,$$

and the assumption (B2), we write

$$\begin{aligned} |\chi^\varepsilon - \varphi| &= \left| \int_0^\infty \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v \rfloor^{-\beta} \tilde{\nu}(x,v,s)\mathrm{d}s} [\tilde{\varphi}(t,x,z) - \varphi(t,x)]\mathrm{d}z \right| \\ &\leqslant \nu_2 \int_0^\infty e^{-\nu_1 z} \varepsilon |v| \lceil v \rfloor^{-\beta} z \|\mathrm{D}\varphi\|_{L^\infty} \mathrm{d}z \\ &= \frac{\nu_2}{\nu_1} \varepsilon |v| \lceil v \rfloor^{-\beta} \|\mathrm{D}\varphi\|_{L^\infty}. \end{aligned}$$

Thus, χ^ε is bounded in $L^\infty_{t,x}$ and converges to φ, uniformly with respect to x and t. However, this convergence is not uniform with respect to v. Moreover, we have the following Lemma:

Lemma 1.2.5. *Let $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)$ and let χ^ε the function given by (1.2.4). Then,*

$$\int_0^\infty \int_{\mathbb{R}^{2d}} F(v) \big[\chi^\varepsilon(t,x,v) - \varphi(t,x)\big]^2 \mathrm{d}x\mathrm{d}v\mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} 0 \tag{1.2.6}$$

and

$$\int_0^\infty \int_{\mathbb{R}^{2d}} F(v) \big[\partial_t \chi^\varepsilon(t,x,v) - \partial_t \varphi(t,x)\big]^2 \mathrm{d}x\mathrm{d}v\mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{1.2.7}$$

Moreover, there is a constant $C > 0$ such that

$$\|\chi^\varepsilon\|_{L^2_F((0,\infty)\times\mathbb{R}^{2d})} \leqslant C \|\varphi\|_{L^2((0,\infty)\times\mathbb{R}^d)} \tag{1.2.8}$$

and

$$\|\partial_t \chi^\varepsilon\|_{L^2_F((0,\infty)\times\mathbb{R}^{2d})} \leqslant C \|\partial_t \varphi\|_{L^2((0,\infty)\times\mathbb{R}^d)}. \tag{1.2.9}$$

This Lemma is in a way the equivalent of Lemmas 3.1 and 4.2 in [Mel10] but not quite the same thing, since in [Mel10] we have

$$\int_{\mathbb{R}^d} F(v) \big[\chi^\varepsilon(t,x,v) - \varphi(t,x)\big] \mathrm{d}v \underset{\varepsilon \to 0}{\longrightarrow} 0$$

and the same for the derivative. The proof of Lemma 1.2.5 is different from the one given in [Mel10], since in the formula of χ^ε involves $\lceil v \rfloor^{-\beta}$, and even more so in its derivative. We obtain only weak limits instead of the bounds by ε.

Proof of Lemma 1.2.5. We are going to prove the limit (1.2.7) and (1.2.6) is done exactly the same way. We have:

$$\|\partial_t\chi^\varepsilon - \partial_t\varphi\|^2_{L^2_F} = \int_0^\infty \int_{\mathbb{R}^{2d}} F\big[\partial_t\chi^\varepsilon - \partial_t\varphi\big]^2 \mathrm{d}x\mathrm{d}v\mathrm{d}t = \int_{\mathbb{R}^d} F \int_0^\infty \int_{\mathbb{R}^d} \big[\partial_t\chi^\varepsilon - \partial_t\varphi\big]^2 \mathrm{d}x\mathrm{d}t\mathrm{d}v.$$

On the one hand, using Taylor expansion and assumption (B2), we write

$$\begin{aligned}\big|\partial_t\chi^\varepsilon - \partial_t\varphi\big| &= \left|\int_0^\infty \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,s)\mathrm{d}s}\big[\partial_t\tilde{\varphi}(t,x,z) - \partial_t\varphi(x,t)\big]\mathrm{d}z\right| \\ &\leqslant \int_0^\infty \nu_2 e^{-\nu_1 z}\varepsilon|v|\lceil v\rfloor^{-\beta} z \int_0^1 \left|\partial_t \mathrm{D}_x\varphi(t, x+\varepsilon v\lceil v\rfloor^{-\beta} zs)\right|\,\mathrm{d}s\mathrm{d}z.\end{aligned}$$

Then,

$$\big|\partial_t\chi^\varepsilon - \partial_t\varphi\big|^2 \leqslant \nu_2^2\varepsilon^2\lceil v\rfloor^{2(1-\beta)}\Big(\int_0^\infty z^2 e^{-\nu_1 z}\Big)\Big(\int_0^\infty e^{-\nu_1 z}\int_0^1 \big|\partial_t\mathrm{D}_x\tilde{\varphi}(t,x,zs)\big|^2\mathrm{d}s\mathrm{d}z\Big).$$

Therefore,

$$\int_0^\infty \int_{\mathbb{R}^d} \big|\partial_t\chi^\varepsilon - \partial_t\varphi\big|^2 \mathrm{d}x\mathrm{d}t \leqslant C\varepsilon^2\lceil v\rfloor^{2(1-\beta)}\|\partial_t\mathrm{D}_x\varphi\|^2_{L^2((0,\infty)\times\mathbb{R}^d)},$$

where

$$C := \nu_2^2\int_0^\infty e^{-\nu_1 z}\mathrm{d}z \int_0^\infty z^2 e^{-\nu_1 z}\mathrm{d}z.$$

Hence, for all $v\in\mathbb{R}^d$

$$F(v)\int_0^\infty \int_{\mathbb{R}^d} \big|\partial_t\chi^\varepsilon - \partial_t\varphi\big|^2 \mathrm{d}x\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{1.2.10}$$

On the other hand,

$$\begin{aligned}\big|\partial_t\chi^\varepsilon\big|^2 &\leqslant \int_0^\infty \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,s)\mathrm{d}s}|\partial_t\tilde{\varphi}(t,x,z)|^2\mathrm{d}z \\ &\leqslant \int_0^\infty \nu_2 e^{-\nu_1 z}|\partial_t\tilde{\varphi}(t,x,z)|^2\mathrm{d}z.\end{aligned}$$

Therefore,

$$\int_0^\infty \int_{\mathbb{R}^d} \big|\partial_t\chi^\varepsilon\big|^2\mathrm{d}x\mathrm{d}t \leqslant \frac{\nu_2}{\nu_1}\|\partial_t\varphi\|^2_{L^2((0,\infty)\times\mathbb{R}^d)}. \tag{1.2.11}$$

Hence,

$$F\int_0^\infty \int_{\mathbb{R}^d} \big|\partial_t\chi^\varepsilon - \partial_t\varphi\big|^2\mathrm{d}x\mathrm{d}t \leqslant 2\frac{\nu_2}{\nu_1}\|\partial_t\varphi\|^2_{L^2} F \in L^1_v(\mathbb{R}^d). \tag{1.2.12}$$

Thus, from (1.2.10) and (1.2.12), and by Lebesgue's theorem, the limit (1.2.7) holds true. The inequality (1.2.9) follows from (1.2.11) and $\int_{\mathbb{R}^d} F\ \mathrm{d}v = 1$. □

1.3 Proof of the main Theorem

The aim of this section is to prove the main theorem whose proof is based on the moment method. For that, we start by giving the weak formulation, taking the solution of the auxiliary problem as a test function.

1.3.1 First part of the limting process

Weak formulation: We recall that f^ε solves (1.1.1) with $\theta(\varepsilon) = \varepsilon^\gamma$. By multiplying the equation (1.1.1) by χ^ε and integrating it with respect to x, v and t, we obtain:

$$\begin{aligned} &-\varepsilon^\gamma \int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \partial_t \chi^\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}t - \varepsilon^\gamma \int_0^\infty \int_{\mathbb{R}^{2d}} f_0(x,v)\chi^\varepsilon(0,x,v)\mathrm{d}x\mathrm{d}v \\ &= \int_0^\infty \int_{\mathbb{R}^{2d}} \left[Q^+(f^\varepsilon)\chi^\varepsilon - \nu f^\varepsilon \chi^\varepsilon + \varepsilon v \cdot \nabla_x \chi^\varepsilon f^\varepsilon\right] \mathrm{d}x\mathrm{d}v\mathrm{d}t. \end{aligned}$$

Now, using (1.2.4) and the fact that $Q^+(F) = \nu F$, we write

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \partial_t \chi^\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^{2d}} f_0(x,v)\chi^\varepsilon(0,x,v)\mathrm{d}x\mathrm{d}v \\ &= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} \left[f^\varepsilon (Q^+)^*(\chi^\varepsilon) - f^\varepsilon \nu \varphi\right] \mathrm{d}x\mathrm{d}v\mathrm{d}t \\ &= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \left[(Q^+)^*(\chi^\varepsilon) - (Q^+)^*(\varphi)\right] \mathrm{d}x\mathrm{d}v\mathrm{d}t \\ &= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(f^\varepsilon)\big(\chi^\varepsilon - \varphi\big) \mathrm{d}x\mathrm{d}v\mathrm{d}t, \end{aligned}$$

and using the decomposition $f^\varepsilon = \rho^\varepsilon F + g^\varepsilon$ with $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F$, we get

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \partial_t \chi^\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^{2d}} f_0(x,v)\chi^\varepsilon(0,x,v)\mathrm{d}x\mathrm{d}v \\ &= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(\rho^\varepsilon F)\big(\chi^\varepsilon - \varphi\big) \mathrm{d}x\mathrm{d}v\mathrm{d}t + \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big) \mathrm{d}x\mathrm{d}v\mathrm{d}t. \end{aligned}$$

The second step of the proof of the main theorem consists in passing to the limit in the previous weak formulation. First, we rewrite it as follows:

$$-\int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \partial_t \chi^\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^{2d}} f_0(x,v)\chi^\varepsilon(0,x,v)\mathrm{d}x\mathrm{d}v \tag{1.3.1}$$

$$= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big) \mathrm{d}x\mathrm{d}v\mathrm{d}t \tag{1.3.2}$$

$$+ \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^d} \rho^\varepsilon \int_{\mathbb{R}^d} \nu F\big(\chi^\varepsilon - \varphi\big) \mathrm{d}v\mathrm{d}x\mathrm{d}t. \tag{1.3.3}$$

Limit of the time derivative:

Lemma 1.3.1. *We have the following limits*

$$-\int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \partial_t \chi^\varepsilon \, \mathrm{d}x\mathrm{d}v\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} -\int_0^\infty \int_{\mathbb{R}^d} \rho \partial_t \varphi \, \mathrm{d}x\mathrm{d}t$$

and

$$-\int_{\mathbb{R}^d} f_0 \chi^\varepsilon(0,x,v) \, \mathrm{d}x\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} -\int_{\mathbb{R}^d} \rho_0 \varphi(0,x) \, \mathrm{d}x.$$

The proof follows from Lemmas 1.2.5 and 1.2.4 and Lebesgue's theorem. Indeed, we have

$$\begin{aligned}\left|\int_0^\infty \int_{\mathbb{R}^{2d}} f^\varepsilon \big(\partial_t \chi^\varepsilon - \partial_t \varphi\big) \mathrm{d}x\mathrm{d}v\mathrm{d}t\right| &\leqslant \|f^\varepsilon\|_{L^\infty(0,\infty,L^2_{F^{-1}}(\mathbb{R}^{2d}))} \|\partial_t \chi^\varepsilon - \partial_t\varphi\|_{L^2(0,\infty,L^2_F(\mathbb{R}^{2d}))} \\ &\leqslant \|f_0\|_{L^2_{F^{-1}}(\mathbb{R}^{2d})} \|\partial_t \chi^\varepsilon - \partial_t\varphi\|_{L^2(0,\infty,L^2_F(\mathbb{R}^{2d}))}.\end{aligned}$$

By (1.2.7), $\|\partial_t \chi^\varepsilon - \partial_t\varphi\|_{L^2(0,\infty,L^2_F(\mathbb{R}^{2d}))} \underset{\varepsilon\to 0}{\longrightarrow} 0$. Hence, the limits of the previous Lemma hold.

Limit of a corrector term:

Lemma 1.3.2. *For any test function $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)$, let χ^ε the function defined by* (1.2.5)*. Then,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big[\chi^\varepsilon(t,x,v) - \varphi(t,x)\big] \mathrm{d}x\mathrm{d}v\mathrm{d}t = 0.$$

Proof of Lemma 1.3.2. First, using assumption (B3), for all t and x, we write:

$$\begin{aligned}\big|Q^+(g^\varepsilon)\big| &\leqslant \left(\int_{\mathbb{R}^d} \sigma^2(x,v,v') \frac{F(v')}{\nu(v')} \mathrm{d}v'\right)^{\frac12} \left(\int_{\mathbb{R}^d} \big|g^\varepsilon(v')\big|^2 \frac{\nu(v')}{F(v')} \mathrm{d}v'\right)^{\frac12} \\ &= \nu F \left(\int_{\mathbb{R}^d} \frac{\sigma^2}{F^2} \frac{F'}{\nu'} \frac{1}{\nu^2} \mathrm{d}v'\right)^{\frac12} \|g^\varepsilon\|_{L^2_{\nu F^{-1}}(\mathbb{R}^d)} \\ &\leqslant M \nu F \|g\|_{L^2_{\nu F^{-1}}(\mathbb{R}^d)},\end{aligned}$$

where we used the notations $F' := F(v')$ and $\nu' := \nu(x,v')$. Also, we have thanks to assumption (B2), $\nu(x,v) \leqslant \nu_2 \langle v \rangle^\beta \leqslant 2\nu_2 \lceil v \rfloor^\beta$. Therefore,

$$\begin{aligned}\left|\int Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big)\mathrm{d}x\mathrm{d}v\mathrm{d}t\right| &\leqslant M \int_0^\infty \int_{\mathbb{R}^{2d}} \nu F \big|\chi^\varepsilon - \varphi\big| \|g^\varepsilon\|_{L^2_{\nu F^{-1}}(\mathbb{R}^d)} \mathrm{d}x\mathrm{d}v\mathrm{d}t \\ &\leqslant 2\nu_2 M \int_0^\infty \int_{\mathbb{R}^d} \|g^\varepsilon\|_{L^2_{\nu F^{-1}}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \lceil v \rfloor^\beta F \big|\chi^\varepsilon - \varphi\big| \mathrm{d}v\right) \mathrm{d}x\mathrm{d}t \\ &\leqslant 2\nu_2 M \|g^\varepsilon\|_{L^2_{\nu F^{-1}}((0,\infty)\times\mathbb{R}^{2d})} \left[\int_0^\infty \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \lceil v \rfloor^\beta F \big|\chi^\varepsilon - \varphi\big| \mathrm{d}v\right)^2 \mathrm{d}x\mathrm{d}t\right]^{\frac12}.\end{aligned}$$

Now, we have from (1.2.2): $\|g^\varepsilon\|_{L^2_{\nu F^{-1}}((0,\infty)\times\mathbb{R}^{2d})} \leqslant C\varepsilon^{\frac{\gamma}{2}}$. Then

$$\left|\int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big)\ \mathrm{d}x\mathrm{d}v\mathrm{d}t\right| \leqslant C'\varepsilon^{\frac{\gamma}{2}}\big(J_1^\varepsilon + J_2^\varepsilon\big)^{\frac{1}{2}}, \tag{1.3.4}$$

where

$$J_1^\varepsilon = \int_0^\infty \int_{\mathbb{R}^d}\left(\int_{|v|\leqslant \varepsilon^{-\frac{1}{1-\beta}}} \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2 \mathrm{d}x\mathrm{d}t$$

and

$$J_2^\varepsilon = \int_0^\infty \int_{\mathbb{R}^d}\left(\int_{|v|\geqslant \varepsilon^{-\frac{1}{1-\beta}}} \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2 \mathrm{d}x\mathrm{d}t.$$

Estimation of J_1^ε. By decomposing the set $\{|v| \leqslant \varepsilon^{-\frac{1}{1-\beta}}\}$ into two parts $\{|v| \leqslant 1\}$ and $\{1 \leqslant |v| \leqslant \varepsilon^{-\frac{1}{1-\beta}}\}$, we write

$$J_1^\varepsilon \leqslant 2\int_0^\infty \int_{\mathbb{R}^d}\left[\left(\int_{|v|\leqslant 1} \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2 + \left(\int_{1\leqslant|v|\leqslant \varepsilon^{-\frac{1}{1-\beta}}} \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2\right]\mathrm{d}x\mathrm{d}t.$$

Now let's deal with the integral

$$\int_0^\infty \int_{\mathbb{R}^d}\left(\int_\Omega \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2 \mathrm{d}x\mathrm{d}t,$$

with $\Omega = \{|v| \leqslant 1\}$ or $\Omega = \{1 \leqslant |v| \leqslant \varepsilon^{-\frac{1}{1-\beta}}\}$. As in the proof of Lemma 1.2.5, using Taylor's formula, assumption (B2) and the Cauchy-Schwarz inequality, we write

$$\begin{aligned}
\big|\chi^\varepsilon - \varphi\big| &= \left|\int_0^\infty \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,z)\ e^{-\int_0^z \lceil v\rfloor^{-\beta}\tilde{\nu}(x,v,s)\mathrm{d}s}\big[\tilde{\varphi}(t,x,z) - \varphi(x,t)\big]\mathrm{d}z\right| \\
&\leqslant \int_0^\infty \nu_2 e^{-\nu_1 z}\varepsilon|v|\lceil v\rfloor^{-\beta} z\int_0^1 |\mathrm{D}_x\varphi(t, x+\varepsilon v\lceil v\rfloor^{-\beta}zs)|\ \mathrm{d}s\mathrm{d}z \\
&\leqslant \nu_2\varepsilon|v|\lceil v\rfloor^{-\beta}\left(\int_0^\infty z^2 e^{-\nu_1 z}\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-\nu_1 z}\int_0^1 \big|\mathrm{D}_x\varphi(x+\varepsilon|v|^{-\beta}vzs,t)\big|^2\mathrm{d}s\mathrm{d}z\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d}\left(\int_\Omega \lceil v\rfloor^\beta F\big|\chi^\varepsilon - \varphi\big|\mathrm{d}v\right)^2 \mathrm{d}x\mathrm{d}t \\
&\leqslant C\int_0^\infty \int_{\mathbb{R}^d}\left[\int_\Omega \varepsilon|v|F\left(\int_0^\infty e^{-\nu_1 z}\int_0^1 \big|\mathrm{D}_x\varphi(x+\varepsilon|v|^{-\beta}vzs,t)\big|^2\mathrm{d}s\mathrm{d}z\right)^{\frac{1}{2}}\mathrm{d}v\right]^2 \mathrm{d}x\mathrm{d}t \\
&\leqslant C\varepsilon^2\left(\int_\Omega |v|F\mathrm{d}v\right)\left(\int_0^\infty\int_{\mathbb{R}^d}\int_\Omega |v|F\int_0^\infty e^{-\nu_1 z}\int_0^1 \big|\mathrm{D}_x\varphi(x+\varepsilon|v|^{-\beta}vzs,t)\big|^2\mathrm{d}s\mathrm{d}z\mathrm{d}v\right)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $C := \nu_2\int_0^\infty z^2 e^{-\nu_1 z}\mathrm{d}z$.

Now, taking $\Omega = \{|v| \leqslant 1\}$ in the previous inequality, we obtain:

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \Big(\int_{|v|\leqslant 1} \lceil v \rfloor^\beta F \big| \chi^\varepsilon - \varphi \big| \mathrm{d}v \Big)^2 \mathrm{d}x \mathrm{d}t \\
&\leqslant C\varepsilon^2 \Big(\int_{|v|\leqslant 1} |v| F \mathrm{d}v \Big)^2 \int_0^\infty e^{-\nu_1 z} \int_0^1 \Big(\int_0^\infty \int_{\mathbb{R}^d} \big| \mathrm{D}_x \varphi(x + \varepsilon |v|^{-\beta} v z s, t) \big|^2 \mathrm{d}x \mathrm{d}t \Big) \mathrm{d}s \mathrm{d}z \\
&\leqslant C\varepsilon^2 \| \mathrm{D}_x \varphi \|^2_{L^2((0,\infty)\times\mathbb{R}^d)}.
\end{aligned}$$

Similarly, for $\Omega = \{1 \leqslant |v| \leqslant \varepsilon^{-\frac{1}{1-\beta}}\}$:

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \Big(\int_{1\leqslant|v|\leqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \big| \chi^\varepsilon - \varphi \big| \mathrm{d}v \Big)^2 \mathrm{d}x \mathrm{d}t \\
&\leqslant C\varepsilon^2 \Big(\int_{1\leqslant|v|\leqslant\varepsilon^{-\frac{1}{1-\beta}}} |v| F \mathrm{d}v \Big)^2 \int_0^\infty e^{-\nu_1 z} \int_0^1 \Big(\int_0^\infty \int_{\mathbb{R}^d} \big| \mathrm{D}_x \varphi(x + \varepsilon |v|^{-\beta} v z s, t) \big|^2 \mathrm{d}x \mathrm{d}t \Big) \mathrm{d}s \mathrm{d}z \\
&\leqslant C\varepsilon^2 \| \mathrm{D}_x \varphi \|^2_{L^2((0,\infty)\times\mathbb{R}^d)} \Big(\int_{1\leqslant|v|\leqslant\varepsilon^{-\frac{1}{1-\beta}}} |v| F \mathrm{d}v \Big)^2 \\
&\leqslant C\varepsilon^2 \| \mathrm{D}_x \varphi \|^2_{L^2((0,\infty)\times\mathbb{R}^d)} \Big(\frac{\varepsilon^{\gamma-1} - 1}{1-\gamma} \Big)^2,
\end{aligned}$$

where we used the inequality $F(v) \leqslant C|v|^{-\alpha-d}$ for $|v| \geqslant 1$, thanks to assumption (B1), and where we recall that $\gamma := \frac{\alpha-\beta}{1-\beta}$. Hence,

$$J_1^\varepsilon \leqslant C \| \mathrm{D}_x \varphi \|^2_{L^2((0,\infty)\times\mathbb{R}^d)} \big(\varepsilon^\gamma + \varepsilon \big)^2. \tag{1.3.5}$$

Estimation of J_2^ε. For the range of velocities $\{|v| \geqslant \varepsilon^{-\frac{1}{1-\beta}}\}$, we write

$$\begin{aligned}
J_2^\varepsilon &:= \int_0^\infty \int_{\mathbb{R}^d} \Big(\int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \big| \chi^\varepsilon - \varphi \big| \mathrm{d}v \Big)^2 \mathrm{d}x \mathrm{d}t \\
&\leqslant \Big(\int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \mathrm{d}v \Big) \Big(\int_0^\infty \int_{\mathbb{R}^d} \int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \big| \chi^\varepsilon - \varphi \big|^2 \mathrm{d}v \mathrm{d}x \mathrm{d}t \Big) \\
&\leqslant 2 \Big(\int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \mathrm{d}v \Big) \int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \Big(\int_0^\infty \int_{\mathbb{R}^d} \big(|\chi^\varepsilon|^2 + |\varphi|^2 \big) \mathrm{d}x \mathrm{d}t \Big) \mathrm{d}v.
\end{aligned}$$

Since $\|\chi^\varepsilon\|_{L^2_{x,t}} \leqslant \frac{\nu_2}{\nu_1} \|\varphi\|_{L^2}$ (see proof of inequality (1.2.11)), and since we have $F(v) \leqslant C|v|^{-\alpha-d}$ for $|v| \geqslant \varepsilon^{-\frac{1}{1-\beta}}$ then,

$$J_2^\varepsilon \leqslant C \|\varphi\|^2_{L^2((0,\infty)\times\mathbb{R}^d)} \Big(\int_{|v|\geqslant\varepsilon^{-\frac{1}{1-\beta}}} |v|^\beta F \mathrm{d}v \Big)^2 \leqslant C \|\varphi\|^2_{L^2((0,\infty)\times\mathbb{R}^d)} \varepsilon^{2\gamma}. \tag{1.3.6}$$

Finally, going back to (1.3.4) and using the estimates (1.3.5) and (1.3.6), we obtain

$$\left| \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big)\mathrm{d}x\mathrm{d}v\mathrm{d}t \right| \leqslant C\varepsilon^{\frac{\gamma}{2}}\big(\varepsilon + \varepsilon^\gamma\big).$$

Which implies that,

$$\left| \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} Q^+(g^\varepsilon)\big(\chi^\varepsilon - \varphi\big)\mathrm{d}x\mathrm{d}v\mathrm{d}t \right| \leqslant C\big(\varepsilon^{\frac{2-\gamma}{2}} + \varepsilon^{\frac{\gamma}{2}}\big).$$

Hence the limit of Lemma 1.3.2 holds, since $\gamma < 2$. □

To conclude the proof of the main theorem, it remains to pass to the limit in the last line of the weak formulation, the integrals (1.3.3), which is the subject of the following subsection.

1.3.2 Obtention of the limiting operator

We have the following Proposition:

Proposition 1.3.3. *For any test function $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^d)$, let χ^ε the function defined by (1.2.5). Then,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_{\mathbb{R}^d} \nu(x,v)F(v)\big[\chi^\varepsilon(x,v,t) - \varphi(x,t)\big]\mathrm{d}v = -\kappa\mathcal{L}^*(\varphi).$$

where $\mathcal{L}$ is the operator defined in Theorem 1.1.4, formula (1.1.14). Moreover, the last limit is uniform with respect to x and t.

Proof of Proposition 1.3.3. The proof is done in two main steps. The first consists in showing that the small velocities do not participate in the limit, and therefore it will be the large velocities which gives the elliptic operator $\mathcal{L}$, which is the subject of the second step.

Recall that the limit to be studied is given by

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_{\mathbb{R}^d} \nu(x,v)F(v)\big[\chi^\varepsilon(t,x,v) - \varphi(t,x)\big]\mathrm{d}v,$$

for $\varphi \in \mathcal{D}([0,+\infty)\times\mathbb{R}^d)$ and χ^ε defined by (1.2.5).

First, let's decompose the integral of the previous limit into two parts, one for $\{|v| \leqslant C\}$ and the other for $\{|v| \geqslant C\}$ for some $C > 0$ large enough:

$$\int_{\mathbb{R}^d} \nu(x,v)F(v)\big(\chi^\varepsilon - \varphi\big)\mathrm{d}v = \int_{|v|\leqslant C} \nu(x,v)F(v)\big(\chi^\varepsilon - \varphi\big)dv + \int_{|v|\geqslant C} \nu(x,v)F(v)\big[\chi^\varepsilon - \varphi\big]\mathrm{d}v.$$

Also, each of the previous integrals can be written as follows

$$\int \nu(x,v)F(v)\big(\chi^\varepsilon-\varphi\big)\mathrm{d}v = \int\int_0^\infty \nu(x,v)F(v)\lceil v\rfloor^{-\beta}\tilde\nu(x,v,z)\ e^{-\int_0^z \lceil v\rfloor^{-\beta}\tilde\nu(x,v,s)\mathrm{d}s}\big[\tilde\varphi(t,x,z)-\varphi(t,x)\big]\mathrm{d}z\mathrm{d}v.$$

In order to simplify the computations and without loss of generality, we can replace the Assumptions (B1) and (B2) concerning the behavior of F and ν at infinity by:

$$F(v)=\frac{\kappa}{|v|^{\alpha+d}} \quad \text{and } \nu(x,v)=\nu_0(x)|v|^{-\beta}, \quad \text{for all } \ |v|\geqslant C \ \text{ and } \ x\in\mathbb{R}^d.$$

Finally, we intoduce two functions P_ε and P_0 defined by

$$P_\varepsilon(x,v,z):=\lceil v\rfloor^{-\beta}\tilde\nu(x,v,z)\,e^{-\int_0^z\lceil v\rfloor^{-\beta}\tilde\nu(x,v,s)\mathrm{d}s} \ \text{ and } \ P_0(x,v,z):=\lceil v\rfloor^{-\beta}\nu(x,v)\,e^{-z\lceil v\rfloor^{-\beta}\nu(x,v)}.$$

Step 1: Small velocities don't contribute to the limit.

Lemma 1.3.4. *Let $\varphi\in\mathcal{D}([0,+\infty)\times\mathbb{R}^d)$ and χ^ε defined by (1.2.5). Assume (A1-A2) and (B2). Then, the following estimate holds*

$$\bigg|\int_{|v|\leqslant C}\nu F\big[\chi^\varepsilon-\varphi\big]\mathrm{d}v\bigg|\leqslant C_0\varepsilon^2\|\mathrm{D}_x\varphi\|_{W^{2,\infty}},$$

where $C_0>0$ depends on ν_1, ν_2 and C, the constant of assumption (B2). Therefore,

$$\lim_{\varepsilon\to0}\ \varepsilon^{-\gamma}\int_{|v|\leqslant C}\nu(x,v)F(v)\big[\chi^\varepsilon(t,x,v)-\varphi(t,x)\big]\mathrm{d}v=0.$$

Proof of Lemma 1.3.4. By (A1) and (A2),

$$\nu(x,-v)F(-v)P_0(x,-v,z)=\nu(x,v)F(v)P_0(x,v,z).$$

Then, we can write

$$\begin{aligned}\int_{|v|\leqslant C}\nu F\big[\chi^\varepsilon-\varphi\big]\mathrm{d}v&=\int_{|v|\leqslant C}\int_0^\infty \nu F\big(P_\varepsilon-P_0\big)\big[\tilde\varphi(t,x,z)-\varphi(t,x)\big]\mathrm{d}z\mathrm{d}v\\&+\int_{|v|\leqslant C}\int_0^\infty \nu FP_0\big[\tilde\varphi(t,x,z)-\varphi(x,t)-\nabla_x\varphi\cdot\varepsilon v\lceil v\rfloor^{-\beta}z\big]\mathrm{d}z\mathrm{d}v.\end{aligned}$$

Let's start with the integral of the first line. We have

$$\begin{aligned}\big|P_\varepsilon-P_0\big|&=\bigg|\lceil v\rfloor^{-\beta}\tilde\nu(x,v,z)\ e^{-\int_0^z\lceil v\rfloor^{-\beta}\tilde\nu(x,v,s)\mathrm{d}s}-\lceil v\rfloor^{-\beta}\nu(x,v)\ e^{-z\lceil v\rfloor^{-\beta}}\nu(x,v)\bigg|\\&\leqslant\lceil v\rfloor^{-\beta}\big|\tilde\nu(x,v,z)-\nu(x,v)\big|e^{-\int_0^z\lceil v\rfloor^{-\beta}\tilde\nu(x,v,s)\mathrm{d}s}\\&+\lceil v\rfloor^{-\beta}\nu(x,v)\bigg|e^{-\int_0^z\lceil v\rfloor^{-\beta}[\tilde\nu(x,v,s)-\nu(x,v)]\mathrm{d}s}-1\bigg|e^{-z\lceil v\rfloor^{-\beta}\nu(x,v)}.\end{aligned}$$

Now, since $|e^{-y}-1| \leqslant y$ for all $y \geqslant 0$, then

$$\left| e^{-\int_0^z \lceil v \rfloor^{-\beta}[\tilde{\nu}(x,v,s)-\nu(x,v)]\mathrm{d}s} - 1 \right| \leqslant z\lceil v \rfloor^{-\beta}\big|\tilde{\nu}(x,v,s)-\nu(x,v)\big|.$$

And since ν is C^1 with respect to x and $\|\langle v\rangle^{-\beta}\partial_x\nu\|_{L^\infty} \leqslant C$, then

$$\big|\tilde{\nu}(x,v,z)-\nu(x,v)\big| := \big|\nu(x,x+\varepsilon\lceil v \rfloor^{-\beta}vz,v)-\nu(x,v)\big| \leqslant \varepsilon z|v|\|\lceil v \rfloor^{-\beta}\partial_x\nu\|_{L^\infty} \leqslant C\varepsilon z|v|.$$

Hence,

$$\big|P_\varepsilon - P_0\big| \leqslant C\varepsilon|v|\lceil v \rfloor^{-\beta}(z+z^2)e^{-\nu_1 z}.$$

We have also,

$$\big|\tilde{\varphi}(t,x,z)-\varphi(t,x)\big| := \big|\varphi(t,x+\varepsilon|v|^{-\beta}vz)-\varphi(t,x)\big| \leqslant C\varepsilon z|v|\lceil v \rfloor^{-\beta}\|\mathrm{D}_x\varphi\|_{L^\infty}.$$

Thus, on the one hand,

$$\begin{aligned}
&\left|\int_{|v|\leqslant C}\nu F\int_0^\infty \big(P_\varepsilon-P_0\big)\big[\tilde{\varphi}(t,x,z)-\varphi(t,x)\big]\mathrm{d}z\mathrm{d}v\right| \\
&\leqslant C\int_{|v|\leqslant C}\lceil v \rfloor^{\beta}F\int_0^\infty \varepsilon^2 e^{-\nu_1 z}|v|^2\lceil v \rfloor^{-2\beta}(z+z^2)\|\mathrm{D}_x\varphi\|_{L^\infty}\mathrm{d}z\mathrm{d}v \\
&\leqslant C\varepsilon^2\|\mathrm{D}_x\varphi\|_{L^\infty}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|\int_{|v|\leqslant C}\nu F\int_0^\infty P_0\big[\tilde{\varphi}(t,x,z)-\varphi(t,x)-\nabla_x\varphi\cdot\varepsilon v\lceil v \rfloor^{-\beta}z\big]\mathrm{d}z\mathrm{d}v\right| \\
&\leqslant \nu_2\int_{|v|\leqslant C}\lceil v \rfloor^{\beta}F\int_0^\infty \varepsilon^2 e^{-\nu_1 z}|v|^2\lceil v \rfloor^{-2\beta}z^2\|\mathrm{D}^2\varphi\|_{L^\infty}\mathrm{d}z\mathrm{d}v \\
&\leqslant C\varepsilon^2\|\mathrm{D}_x^2\varphi\|_{L^\infty}.
\end{aligned}$$

Hence, the estimate and limit of Lemma 1.3.4 hold. □

Step 2: Convergence to the elliptic operator.
We still have to deal with the following limit

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int_{|v|\geqslant C}\nu(x,v)F(v)\big[\chi^\varepsilon-\varphi\big]\mathrm{d}v.$$

For $|v| \geqslant C$, $\lceil v \rfloor^{-\beta} = |v|^{-\beta}$. By performing the change of variable $w = \varepsilon z|v|^{-\beta}v$, we obtain: $|w| = \varepsilon z|v|^{1-\beta}$, $|v| = \big(\frac{|w|}{\varepsilon z}\big)^{\frac{1}{1-\beta}}$, $v = \frac{w}{(\varepsilon z)^{\frac{1}{1-\beta}}|w|^{-\frac{\beta}{1-\beta}}}$ and $\mathrm{d}v = \frac{1}{1-\beta}\frac{\mathrm{d}w}{(\varepsilon z)^{\frac{d}{1-\beta}}|w|^{-\frac{\beta d}{1-\beta}}}$.

So the integral becomes

$$\begin{aligned}
&\int_{|v|\geqslant C} \nu(x,v)F(v)\big[\chi^\varepsilon - \varphi\big]\mathrm{d}v \\
&= \frac{\kappa}{1-\beta}\int_0^\infty \int_{|w|\geqslant C\varepsilon z} \nu_0(x)\nu_0(x+w)\ e^{-\int_0^z \nu_0(x+\frac{s}{z}w)\mathrm{d}s}\ \frac{\varphi(t,x+w)-\varphi(t,x)}{(\varepsilon z)^{\frac{\beta-\alpha}{1-\beta}}|w|^{d+\frac{\alpha-\beta}{1-\beta}}}\ \mathrm{d}w\mathrm{d}z \\
&= \frac{\kappa}{1-\beta}\ \varepsilon^\gamma \int_0^\infty \int_{|w|\geqslant C\varepsilon z} z^\gamma e^{-z\int_0^1 \nu_0(x+sw)\mathrm{d}s}\ \nu_0(x)\nu_0(x+w)\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\ \mathrm{d}w\mathrm{d}z \\
&=: \kappa\ L^\varepsilon_\varphi,
\end{aligned}$$

where

$$L^\varepsilon_\varphi := \frac{\kappa}{1-\beta}\ \int_0^\infty \int_{w\geqslant C\varepsilon z} z^\gamma e^{-\int_0^z \nu_0(x+\frac{s}{z}w)\mathrm{d}s}\ \nu_0(x)\nu_0(x+w)\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\ \mathrm{d}w\mathrm{d}z.$$

Hence,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\ \varepsilon^{-\gamma}\int_{|v|\geqslant C}\nu F\big[\chi^\varepsilon-\varphi\big]\mathrm{d}v = \kappa\ \lim_{\varepsilon\to 0}\ L^\varepsilon_\varphi &= \frac{\kappa}{1-\beta}\ \mathrm{PV}\int_{\mathbb{R}^d}\eta(x,y)\frac{\varphi(t,y)-\varphi(t,x)}{|x-y|^{\gamma+d}}\ \mathrm{d}y \\
&=: -\kappa\ \mathcal{L}^*(\varphi),
\end{aligned}$$

with

$$\eta(x,y) = \nu_0(x)\nu_0(y)\int_0^\infty z^\gamma e^{-z\int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s}\ \mathrm{d}z.$$

In order to justify this last limit rigorously and show that it holds uniformly with respect to x and t, we introduce the function

$$\bar\eta(x,y,z) = \nu_0(x)\nu_0(y)z^\gamma e^{-z\int_0^1 \nu_0(sx+(1-s)y)\mathrm{d}s}$$

and we split the integral L^ε_φ as follows:

$$\begin{aligned}
L^\varepsilon_\varphi &= \int_0^\infty \int_{|w|\geqslant C\varepsilon z}\bar\eta(x,x+w,z)\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\mathrm{d}w\mathrm{d}z \\
&= \int_0^{\frac{1}{C\varepsilon}}\int_{|w|\geqslant 1}\bar\eta(x,x+w,z)\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\mathrm{d}w\mathrm{d}z \\
&+ \int_0^{\frac{1}{C\varepsilon}}\int_{C\varepsilon z\leqslant|w|\leqslant 1}\big[\bar\eta(x,x+w,z)-\bar\eta(x,x,z)\big]\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\ \mathrm{d}w\mathrm{d}z \\
&+ \int_0^{\frac{1}{C\varepsilon}}\int_{C\varepsilon z\leqslant|w|\leqslant 1}\bar\eta(x,x,z)\ \frac{\varphi(t,x+w)-\varphi(t,x)-w\cdot\nabla_x\varphi}{|w|^{\gamma+d}}\ \mathrm{d}w\mathrm{d}z \\
&+ \int_{\frac{1}{C\varepsilon}}^\infty\int_{|w|\geqslant C\varepsilon z}\bar\eta(x,x+w,z)\ \frac{\varphi(t,x+w)-\varphi(t,x)}{|w|^{\gamma+d}}\ \mathrm{d}w\mathrm{d}z.
\end{aligned}$$

The function $\bar\eta$ plays the role of P_ε for small velocities and was introduced in order to

make the previous quantities integrable. Note that all the integrals above are defined in the classical sense without need for principal value and the fact that we have assumed that $\nu(x,v) = \nu_0(x)|v|^{-\beta}$ and $F(v) = C|v|^{-\alpha-d}$ for $|v| \geqslant C$, made passing to the limit much easier ($\lim_{\varepsilon\to 0} \; L^{\varepsilon}_{\varphi}$ is exactly the operator defined by the principal value).
This completes the proof of Theorem 1.1.4.

Fractional diffusion for the kinetic Fokker-Planck equation

CHAPTER 2

Construction of an eigen-solution for the Fokker-Planck operator with heavy tail equilibrium: an à la Koch method in dimension 1

Le travail de ce chapitre et le suivant a été fait en collaboration avec Marjolaine PUEL, ils font l'objet des papiers [DP23a] et [DP23b] respectivement.

Résumé

Ce chapitre est consacré à la construction d'une *solution propre* pour l'opérateur de Fokker-Planck avec équilibre à queue lourde. Nous proposons une méthode *alternative* en dimension 1, qui sera généralisable en dimension supérieure. Cette dernière méthode s'inspire des travaux de H. Koch sur l'équation KdV non linéaire [Koc15]. Comme conséquence de cette construction, on retrouve le résultat de G. Lebeau et M. Puel [LP19] sur la limite de diffusion fractionnaire pour l'équation de Fokker-Planck.

The work of this chapter and the following one was done in collaboration with Marjolaine PUEL, they are the subject of the papers [DP23a] and [DP23b] respectively.

Abstract

This chapter is devoted to the construction of an *eigen-solution* for the Fokker-Planck operator with heavy tail equilibrium. We propose an *alternative* method in dimension 1, which will be generalizable in higher dimension. The later method is inspired by the work of H. Koch on non-linear KdV equation [Koc15]. As a consequence of this construction, we recover the result of G. Lebeau and M. Puel [LP19] on the fractional diffusion limit for the Fokker-Planck equation.

Contents

2.1 Introduction

2.1.1 Setting of the problem

Our starting point is the kinetic Fokker-Planck (FP) equation, which describes in a deterministic way the *Brownian motion* of a set of particles. It is given by the following form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f), \quad t \geqslant 0, \; x \in \mathbb{R}^d, \; v \in \mathbb{R}^d, \\ \\ f(0,x,v) = f_0(x,v), \qquad x \in \mathbb{R}^d, \; v \in \mathbb{R}^d, \end{cases} \tag{2.1.1}$$

where the collisional Fokker-Planck operator Q is given by

$$Q(f) = \nabla_v \cdot \left(F \nabla_v \left(\frac{f}{F} \right) \right), \tag{2.1.2}$$

and F is the equilibrium of Q, a fixed function which depends only on v and satisfying

$$Q(F) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} F(v) \, \mathrm{d}v = 1.$$

For non-negative initial data f_0, the unknown $f(t,x,v) \geqslant 0$ can be interpreted as the density of particles occupying at time $t \geqslant 0$, the position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$.

The *classical* or *fractional diffusion limits* for kinetic equations have been studied in a series of papers in recent years. The principal motivation behind this study is to derive simpler models from kinetic equations with collision operators. Indeed, in the diffusion approximation, the velocity variable is involved only in the equilibrium. This approximation is a kind of hydrodynamic limit. When the interactions between particles are the dominant phenomena and when the time of observation is very large, it reflects the fact that we are between the mesoscopic and the macroscopic scale. More precisely, we introduce a small parameter $\varepsilon \ll 1$, the mean free path and we proceed to rescaling the distribution function $f(t, x, v)$ in time and space

$$t = \frac{t'}{\theta(\varepsilon)} \quad \text{and} \quad x = \frac{x'}{\varepsilon} \quad \text{with} \quad \theta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

which leads to the following rescaled equation (without primes)

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon), \quad t \geqslant 0, x \in \mathbb{R}^d, v \in \mathbb{R}^d, \\ \\ f^\varepsilon(0, x, v) = f_0(x, v), \quad x \in \mathbb{R}^d, v \in \mathbb{R}^d. \end{cases} \tag{2.1.3}$$

Note that taking initial conditions independent of ε means that we take well prepared initial conditions in the non rescaled variable. Now, in order to study the behaviour of the solution f^ε as $\varepsilon \to 0$, we are interested in the following question: what is the good time scale $\theta(\varepsilon)$ such that the limit of the solution f^ε is not trivial? Formally, passing to the limit when $\varepsilon \to 0$ in the equation (2.1.3), we obtain that the limit f^0 is in the kernel of Q which is spanned by the equilibrium F, which means that $f^0 = \rho(t, x)F(v)$. Thus, what is the equation satisfied by ρ and how do we justify this passage *rigorously*?

The limit of f^ε when ε goes to 0 may depend on the nature of the equilibrium considered. For Gaussian equilibria, the answers have been known for a long time and it is the *classical diffusion* that we obtain for a usual scaling $\theta(\varepsilon) = \varepsilon^2$. For slowly decreasing equilibria, or so-called *heavy-tailed equilibria* of the form $F(v) \sim \langle v \rangle^{-\beta}$, it is more complicated, and the answers to this question has been the interest of many papers in the last few years, with different methods and for different collision operators. This was initiated by Mischler, Mouhot and Mellet [MMM11] on the linear Boltzmann equation, for an ad hoc cross section, which does not depend on the spatial variable, where they obtained classical and fractional diffusion, according to the powers of $\langle v \rangle$ which appear in F. Their method is based on *Fourier-Laplace transformation*, with close links to earlier work by Milton, Komorowski and Olla [JKO09] on *Markov chains*. This was continued by A. Mellet [Mel10], still for the same equation, more general since it also applies to cross sections that depend on the position variable. See also the work of Ben Abdallah, Mellet and the second author [BAMP11a, BAMP11b], where they used a *Hilbert expansions approach* [BAMP11b] and obtained a strong convergence of the solution f^ε to ρF for initial data f_0 in $H^4(\mathbb{R}^{2d})$.

The diffusion limit for the FP equation has been studed after that of the linear Boltzmann one, and seems more complicated. Indeed, there is no *spectral gap*, thus makes the limiting process more difficult. In addition, for this equation, all the terms of the operator participate in the limit, i.e. the collision and advection parts. The limit was studed in [NP15] with $\theta(\varepsilon) = \varepsilon^2$ and $\beta > d+4$ using the *moment method*. Also in [CNP19] for the critical case $\beta = d+4$ with $\theta(\varepsilon) = \varepsilon^2 \ln(\varepsilon^{-1})$ using *a probabilistic approach*. In both cases, the authors obtained classical diffusion. Concerning the case $d < \beta < d+4$, it has been solved recently, by G. Lebeau and the second author in dimension 1 [LP19], with a *spectral approach*, then by N. Fournier and C. Tardif in dimension 1 [FT21] and then in dimension $d \geqslant 2$ [FT20] by *a probabilistic approach*. Depending on the range of the exponents, different regimes corresponding to Brownian processes, stable processes or integrated symmetric Bessel processes are obtained and described in this last paper (see also the paper of E. Barkai, E. Aghion, and D. A. Kessler for a physical interpretation of this equation[BAK14]). Recently, E. Bouin and C. Mouhot [BM22] have constructed what they call a '*fluid mode*' using a method that combines energy estimates and a quantitative spectral method. Their method was valid to scattering models, Fokker-Planck (by introducing a *weighted* density, $\tilde{\rho} := \int_{\mathbb{R}^d} \frac{f}{\langle v\rangle^2} \mathrm{d}v$) and Lévy-Foker-Planck operators. We refer also to the paper by E. Bouin, J. Dolbeau and L. Lafleche [BDL22], where the authors have developed an L^2-hypocoercivity approach and established an optimal decay rate, determined by a fractional Nash type inequality, compatible with the fractional diffusion limit.

In [LP19], the authors proposed to take the eigenfunction of the Fokker-Planck operator (with the advection part) as an adequate test function, i.e. take the solution of the spectral problem

$$(Q + \mathrm{i}\varepsilon\xi v)M_{\varepsilon,\mu}(v) = \mu M_{\varepsilon,\mu}(v), \quad v \in \mathbb{R},$$

written in Fourier variable, ξ, for the position x. Their construction consists in reconnecting two branches constructed as follows: They first constructed for each μ, ε (fixed) a branch in the half space $\mathbb{R}_+$, by introducing an approximate equation for large velocities. Then, by symmetry of the equation, they obtained a second branch in the other half space $\mathbb{R}_-$, and in order to reconnect the two functions to obtain a $C^1(\mathbb{R})$ solution, the reconnection of the derivatives implies a relation $\mu(\varepsilon)$. This method of reconnection seems very complicated to adapt in higher dimension, because it means that the implicit function theorem; used to study constraint; must be applied to the whole derivative operator.

The purpose of this chapter is to propose an alternative method, inspired by the work of Herbert Koch on the non-linear KdV equation [Koc15], where we solve the spectral problem associated to the Fokker-Planck operator, taking into account the advection part, working on the whole space $\mathbb{R}$ and avoiding the reconnection problems. This method is generalizable in higher dimension by PDE methods (the subject of the next chapter) and is probably interesting for generalized potential or non linear problems.

A key tool of the proof of the present chapter is the splitting of the Fokker-Planck operator is involved, which recalls the *enlargement theory* for nonlinear Boltzmann operator when there are spectral gap issues. This theory was developed by Gualdani, Miscler and Mouhot in [GMM10] whose key idea was based on the decomposition of the operator into two parts, a *dissipative* part plus a *regularizing* part. See also P. Gervais ([Ger20] and references therein) for a spectral study of the linearized Boltzmann operator in L^2 spaces with polynomial and Gaussian weights.

2.1.2 Setting of the result

In the present work, we consider for any $\beta > 1$, heavy tail equilibria

$$F(v) \sim \frac{1}{(1+|v|^2)^{\frac{\beta}{2}}}.$$

Before stating our main result, let us give some notations that we will use along this chapter.

Notations. As in [LP19], in order to simplify the computation and work with a self-adjoint operator in L^2, we proceed to a change of unknown by writing

$$f = F^{\frac{1}{2}} g = C_\beta M g$$

with

$$M = C_\beta^{-1} F^{\frac{1}{2}} = \frac{1}{(1+|v|^2)^{\frac{\gamma}{2}}},$$

since we impose $\gamma := \frac{\beta}{2} > \frac{1}{2}$, $F \in L^1(\mathbb{R})$ then, $M \in L^2(\mathbb{R})$ and C_β is chosen such that

$$\int_{\mathbb{R}} F \mathrm{d}v = 1.$$

The equation (2.1.3) becomes

$$\theta(\varepsilon)\partial_t g^\varepsilon + \varepsilon v \cdot \nabla_x g^\varepsilon = \frac{1}{M}\partial_v\left(M^2 \partial_v\left(\frac{g^\varepsilon}{M}\right)\right) = \partial_v^2 g^\varepsilon - W(v) g^\varepsilon,$$

with

$$W(v) = \frac{\partial_v^2 M}{M} = \frac{\gamma(\gamma+1)|v|^2 - \gamma}{(1+|v|^2)^2}.$$

We see the equation as

$$\theta(\varepsilon)\partial_t g^\varepsilon = -\mathcal{L}_\varepsilon g^\varepsilon,$$

where

$$\mathcal{L}_\varepsilon := -\partial_v^2 + W(v) + \varepsilon v \cdot \nabla_x = -(Q - \varepsilon v \cdot \nabla_x)$$

and

$$Q = -\Delta_v + W(v).$$

We operate a Fourier transform in x and since the operator Q has coefficient that do not depend on x, we get:

$$\theta(\varepsilon)\partial_t \hat{g}^\varepsilon = -\mathcal{L}_\eta \hat{g}^\varepsilon, \tag{2.1.4}$$

where

$$\mathcal{L}_\eta := -\partial_v^2 + W(v) + \mathrm{i}\eta v$$

and

$$\eta = \varepsilon\xi,$$

with ξ being the space Fourier variable.

The operator $\mathcal{L}_\eta$ is an unbounded self-adjoint operator acting on L^2. Its domain is given by

$$D(\mathcal{L}_\eta) = \big\{ g \in L^2(\mathbb{R}) \ ; \ \partial_v^2 g \in L^2(\mathbb{R}), vg \in L^2(\mathbb{R}) \big\}.$$

The aim of this chapter is to prove, with a geometry independent method, the following main theorem.

Main Theorem

Theorem 2.1.1 (Eigen-solution for the Fokker-Planck operator)**.** *Assume that $\beta \in]1,5[\setminus\{2\}$. Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Then, for all $\eta \in [0, \eta_0]$, there exists a unique eigen-couple $\big(\mu(\eta), M_\eta\big)$ in $\{\mu \in \mathbb{C}, |\mu| \leqslant \eta^{\frac{2}{3}}\lambda_0\} \times L^2(\mathbb{R},\mathbb{C})$, solution to the spectral problem*

$$\mathcal{L}_\eta(M_{\mu,\eta})(v) = \big[-\partial_v^2 + W(v) + \mathrm{i}\eta v\big] M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v), \ v \in \mathbb{R}. \tag{2.1.5}$$

Moreover, one has

1. *The following convergence in $L^2(\mathbb{R},\mathbb{C})$,*

$$\|M_\eta - M\|_{L^2} \underset{\eta\to 0}{\longrightarrow} 0. \tag{2.1.6}$$

2. *The relationship between the eigenvalue $\mu(\eta)$, the scale of the time variable $\theta(\varepsilon)$ and the coefficient κ is given by the following expansion:*

$$\mu(\eta) = \kappa \eta^{\frac{\beta+1}{3}} \big(1 + O(\eta^{\frac{\beta+1}{3}})\big), \tag{2.1.7}$$

where κ is a positive constant given by

$$\kappa = -2C_\beta^2 \int_0^\infty s^{1-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s, \tag{2.1.8}$$

and where H_0 *is the unique solution to the equation*

$$\big[-\partial_s^2+\frac{\gamma(\gamma+1)}{s^2}+\mathrm{i}s\big]H_0(s)=0,\quad \forall s\in\mathbb{R}^*, \tag{2.1.9}$$

satisfying

$$\int_{\{|s|\geqslant 1\}}|H_0(s)|^2\mathrm{d}s<\infty \quad \textit{and} \quad H_0(s)\underset{0}{\sim}|s|^{-\gamma}. \tag{2.1.10}$$

For $\eta\in[-\eta_0,0]$*, by complex conjugation on the equation, we get*

$$\mu(\eta)=\bar{\mu}(-\eta)=\kappa|\eta|^{\frac{\beta+1}{3}}(1+O(|\eta|^{\frac{\beta+1}{3}})).$$

Remark 2.1.2. The hypothesis $\beta\neq 2$ is technical. It avoids to introduce logarithmic terms in the expression of $\mu(\eta)$.

Idea of the proof

The proof of our main result is done in two main steps, both based on the Implicit Function Theorem. First, we consider what we call a *penalized equation.* We introduce an additional term that kills the M direction in the kernel of the linear operator computed at $\eta=0$. That gives the following equation

$$\begin{cases}\big[-\partial_v^2+W(v)+\mathrm{i}\eta v\big]M_{\mu,\eta}(v)=\mu M_{\mu,\eta}(v)-\langle M_{\mu,\eta}-M,\Phi\rangle\Phi,\ v\in\mathbb{R},\\ \\ M_{\mu,\eta}\in L^2(\mathbb{R}).\end{cases} \tag{2.1.11}$$

where Φ is a function that we will determine later. This penalized equation has a solution for any μ and η on the whole space and it allows us to avoid the problem of reconnection and to work directly on the whole space $\mathbb{R}$. This is one of the key points of this method and it iallows to generalize this construction in any dimension.

The objective of the first step is to show the existence of a unique solution for equation (2.1.11) for any η and μ. Indeed, as we said above, we will decompose the operator "$-\partial_v^2+W(v)+\mathrm{i}\eta v-\mu$" into two parts. The first part is chosen such that it admits "a right inverse" that is continuous as a linear operator between two suitable functional spaces, continuous with respect to the parameters η and μ and compact at $\eta=\mu=0$. The second part of the operator is left in the right-hand side of the equation, i.e. is considered as a source term. Find a solution to (2.1.11) remains to a fixed point process.

In the second step, to ensure that the additional term vanishes, we have to chose $\mu(\eta)$ obtained via the Implicit Function Theorem around the point $(\mu,\eta)=(0,0)$.

2.1.3 Relation to the fractional diffusion problem

In this subsection, we will first give a formal argument to obtain the eigenvalue and then, we will explain how one can recover the fractional diffusion limit for the Fokker-Planck equation (2.1.3), and how the eigenvalue is related to the diffusion coefficient in our main result.

Heuristic on the computation of the eigenvalue. With a formal calculation, we will present how the time scaling $\theta(\varepsilon)$ is chosen and how it appears in the spectral problem. Assume that the couple $(\mu(\eta), M_{\mu,\eta})$ is solution to the problem

$$\mathcal{L}_\eta(M_{\mu,\eta}) = [-\partial_v^2 + W(v) + \mathrm{i}\eta v]M_{\mu,\eta} = \mu(\eta)M_{\mu,\eta}, \quad v \in \mathbb{R}.$$

Then, integrating this equation against M and using the fact that $[-\partial_v^2 + W(v)]M = 0$, we get

$$\mathrm{i}\eta \int_{\mathbb{R}} vM_{\mu,\eta}M\mathrm{d}v = \mu(\eta)\int_{\mathbb{R}} M_{\mu,\eta}M\mathrm{d}v.$$

Therefore,

$$\mu(\eta) = -\mathrm{i}\eta\int_{\mathbb{R}} vM_{\mu,\eta}M\mathrm{d}v\Big(\int_{\mathbb{R}} M_{\mu,\eta}M\mathrm{d}v\Big)^{-1}.$$

If $M_{\mu,\eta} \to M$ when $\eta \to 0$ then, we get $\big(\int_{\mathbb{R}} M_{\mu,\eta}M\mathrm{d}v\big)^{-1} \to \|M\|_2^{-2} = C_\beta^2$. Formally, by a Hilbert expansion

$$M_{\mu,\eta} = M + \eta N + o(\eta), \text{ where } [-\partial_v^2 + W(v)]N = +\mathrm{i}vM,$$

we get

$$|M_{\mu,\eta}(v) - M(v)| \lesssim \eta\langle v\rangle^{3-\gamma} \quad \text{for } |v| \leqslant \eta^{-\frac{1}{3}}s_0,$$

and by rescaling the integral on large velocities by $v = \eta^{-\frac{1}{3}}s$ we finally obtain:

$$\mu(\eta) \sim \eta^2\int_{\{|v|\leqslant\eta^{-\frac{1}{3}}s_0\}} |v|\langle v\rangle^{3-2\gamma}\mathrm{d}v + 2\eta^{\frac{2\gamma+1}{3}}\int_{\{s\geqslant s_0\}} s^{1-\gamma}\mathrm{Im}H_0(s)\mathrm{d}s,$$

where H_0 is the limit of $M_{\mu,\eta}$ rescaled, solution to the limit "rescaled equation"

$$\big[-\partial_s^2 + \frac{\gamma(\gamma+1)}{s^2} + \mathrm{i}s\big]H_0(s) = 0, \quad \forall s \in \mathbb{R}^*.$$

Recall that $\eta = \varepsilon\xi$. Then, $\mu(\eta) = \mu(\varepsilon|\xi|)$. Thus, if $2\gamma > 5$ then we find classical diffusion with the usual scaling $\theta(\varepsilon) = \varepsilon^2$, and it would be the small velocities which give the diffusion coefficient. While if $2\gamma < 5$ then we get fractional diffusion with a power of $\frac{2\gamma+1}{6}$ for the Laplacian, the scaling is given by $\theta(\varepsilon) = \varepsilon^{\frac{2\gamma+1}{3}}$ and the diffusion coefficient is determined by the integral for large velocities in this case.

Heuristic of the diffusion approximation. Thanks to the main result of this chapter, we recover the following theorem (See Section 3 of [LP19] for more details).

Theorem 2.1.3 ([LP19]). *Assume that $1 < \beta < 5$ with $\beta \neq 2$. Assume that $f_0 \in L^1(\mathbb{R} \times \mathbb{R})$ is a non-negative function in $L^2_{F^{-1}}(\mathbb{R} \times \mathbb{R}) \cap L^\infty_{F^{-1}}(\mathbb{R} \times \mathbb{R})$. Let f^ε be the solution of (2.1.3) in Y with initial data f_0 and $\theta(\varepsilon) = \varepsilon^{\frac{\beta+1}{3}}$. Let κ be the constant given by (2.1.8). Then, f^ε converges weakly star in $L^\infty\left([0,T],\ L^2_{F^{-1}}(\mathbb{R}^2)\right)$ towards $\rho(t,x)F(v)$, where $\rho(t,x)$ is the solution to*

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\beta+1}{6}} \rho = 0, \quad \rho(0,x) = \int f_0 \mathrm{d}v. \tag{2.1.12}$$

Introduce g^ε solution to

$$\hat{g}^\varepsilon(t,\xi,v) = e^{-t\theta(\varepsilon)\mathcal{L}_\eta} \hat{g}(0,\xi,v),$$

which gives, going back to the rescaled space variable y, as follows

$$g^\varepsilon(t,x,v) = \frac{1}{2\pi} \int e^{\mathrm{i}x\xi} \hat{g}^\varepsilon(t,\xi,v) \mathrm{d}\xi$$

a solution to equation (2.1.4).

We consider the Fourier transform of ρ and we will establish that $\hat{\rho}(t,\xi) = \int e^{-\mathrm{i}x\xi}\rho(t,x)\mathrm{d}x$ satisfies

$$\partial_t \hat{\rho} + \kappa|\xi|^\alpha \hat{\rho} = 0. \tag{2.1.13}$$

Let M_η be the unique solution in L^2 of the equation $\mathcal{L}_\eta(M_\eta) = \mu(\eta)M_\eta$ given in Theorem 2.1.1. One has

$$\begin{aligned} \frac{\partial}{\partial t} \int \hat{g}^\varepsilon(t,\xi,v) M_\eta \mathrm{d}v &= \int \partial_t \hat{g}^\varepsilon M_\eta \mathrm{d}v = -\varepsilon^{-\alpha} \int \mathcal{L}_\varepsilon(\hat{g}^\varepsilon) M_\eta \mathrm{d}v \\ &= -\varepsilon^{-\alpha} \int \hat{g}^\varepsilon \mathcal{L}_\varepsilon(M_\eta) \mathrm{d}v = -\varepsilon^{-\alpha}\mu(\eta) \int \hat{g}^\varepsilon(t,\xi,v) M_\eta \mathrm{d}v. \end{aligned}$$

Recall that $\eta = \varepsilon\xi$ and by Theorem 2.1.1, $\varepsilon^{-\alpha}\mu(\eta) \to \kappa|\xi|^\alpha$ and $M_\eta \to M$ when $\varepsilon \to 0$. Therefore, passing to the limit in the previous equality formally, we obtain the equation

$$\partial_t \hat{\rho} = |\xi|^\alpha \kappa \hat{\rho}.$$

Which means that $\hat{\rho}$ satisfies (2.1.13). Thus, the solution f^ε of (2.1.3) converges, in some sense, towards $\rho(t,x)F(v)$, where ρ is the solution of the fractional diffusion equation

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}} \rho = 0, \quad \rho(0,x) = \int f_0 \mathrm{d}v. \tag{2.1.14}$$

Returning to the space variable, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is a non-local operator which can be defined by

$$(-\Delta)^{\frac{\alpha}{2}} \rho(x) := c_\alpha \text{ P.V.} \int_{\mathbb{R}} \frac{\rho(x) - \rho(y)}{|x-y|^{\alpha+1}} \mathrm{d}y,$$

which also can be seen as an α-stable Lévy process, and thus can be interpreted as a random trajectory, generalising the concept of Brownian motion, which may contain jump discontinuities.

2.1.4 Notations and definition of the considered operators

Let λ and η be fixed, where $\lambda \in \mathbb{C}$ being defined by $\lambda := \mu\eta^{-\frac{2}{3}}$. Let us denote by $L_{\lambda,\eta}$ the operator

$$L_{\lambda,\eta} := -\partial_v^2 + \tilde{W}(v) + \mathrm{i}\eta v - \lambda\eta^{\frac{2}{3}},$$

where

$$\tilde{W}(v) = \frac{\gamma(\gamma+1)}{1+v^2},$$

and let denote by $V := \tilde{W} - W$. We have

$$V(v) = \frac{\gamma(\gamma+2)}{(1+v^2)^2}.$$

We will rewrite the equation (2.1.11) as follows:

$$\begin{cases} L_{\lambda,\eta} M_{\lambda,\eta}(v) = V(v) M_{\lambda,\eta}(v) - \langle M_{\lambda,\eta} - M, \Phi \rangle \Phi(v), \ v \in \mathbb{R}, \\ \\ M_{\lambda,\eta} \in L^2(\mathbb{R}). \end{cases} \tag{2.1.15}$$

The two equations (2.1.11) and (2.1.15) are equivalent.

Remark 2.1.4.

1. Since $L_{\lambda,0}$ does not depend on λ, let's denote it by L_0, $L_0 := L_{\lambda,0}$.

2. If $\bar{\Phi}(-v) = \Phi(v)$ and $M_{\lambda,\eta}(v)$ satisfies the equation (2.1.15), then $\overline{M}_{\bar{\lambda},\eta}(-v)$ is also solution to (2.1.15), since the potential W is symmetric for a symmetric equilibrium M. Note that this is where the symmetry of the equilibrium M is used and therefore this is a "non-drift condition".

3. In the first step of our proof, the continuity of $L_{\lambda,\eta}^{-1}$ and the compactness of L_0^{-1} are among the most important and difficult points to prove. The splitting of the potential W into V and $\tilde{W}$ is the key idea in the proof of these last two points.

We will construct a right inverse to the operator $L_{\lambda,\eta}$ defined above. We are going to study the operator L_0 first, then we construct solutions for the equation $L_{\lambda,\eta}(\psi) = f$, with any source term f, from the solutions of $L_0(\psi) = f$, after noticing that $L_{\lambda,\eta}$ is in fact a small perturbation of L_0 for λ and η small enough.

2.2 Green function for the leading part of the limiting operator

We consider the equation:

$$L_0(\psi) = \big[-\partial_v^2 + \tilde{W}(v) \big]\psi = f, \quad v \in \mathbb{R} \tag{2.2.1}$$

We are going to construct solutions for this equation by the method of the variation of the constant, which allows us to define a right inverse to L_0. Let's start by looking at the homogeneous equation:

$$L_0(\psi) = \big[-\partial_v^2 + \tilde{W}(v) \big]\psi = 0, \quad v \in \mathbb{R}. \tag{2.2.2}$$

2.2.1 Construction of an intermediate solution to the homogeneous equation

In order to construct the basis of solution to the homogeneous equation, we need the following two lemmas:

Lemma 2.2.1 (Behaviour of the solutions of (2.2.2))**.** *If $\tilde{\psi}$ is a solution for the equation (2.2.2) then, either $\tilde{\psi}(v) \underset{+\infty}{\sim} |v|^{-\gamma}$ or $\tilde{\psi}(v) \underset{+\infty}{\sim} |v|^{\gamma+1}$. Thus, for a positive constant v_0 large enough, there exists a unique function $R_1 \in L^\infty([v_0, +\infty[)$, with $R_1 = O(v^{-2})$ such that*

$$\tilde{\psi} = |v|^{-\gamma}(1 + R_1) \quad \text{or} \quad \tilde{\psi} = |v|^{\gamma+1}(1 + R_1).$$

Similarly, either $\psi(v) \underset{-\infty}{\sim} |v|^{-\gamma}$ or $\psi(v) \underset{-\infty}{\sim} |v|^{\gamma+1}$. Then, for a positive constant v_0 large enough, there exists a unique function $R_2 \in L^\infty(]-\infty, -v_0])$, with $R_2 = O(|v|^{-2})$) such that

$$\tilde{\psi} = |v|^{-\gamma}(1 + R_2) \quad \text{or} \quad \tilde{\psi} = |v|^{\gamma+1}(1 + R_2).$$

Lemma 2.2.2 (The existence of an intermediate solution for (2.2.2))**.**
There is a unique positive smooth function ψ, solution to the Cauchy problem:

$$\begin{cases} L_0(\psi) = 0, \\ \psi(0) = 1, \\ \psi'(0) = 0. \end{cases} \tag{2.2.3}$$

Moreover,

1. *For all v in $\mathbb{R}$: $\psi(-v) = \psi(v) > 0$.*

2. *There exists a positive constant v_0 large enough, there exists a unique function R belongs to $L^\infty([v_0, +\infty[)$, with $R(v) = O(|v|^{-2})$ such that*

$$\psi(v) = c|v|^{\gamma+1}\big(1 + R(v)\big), \quad \forall |v| \geqslant v_0, \tag{2.2.4}$$

where c is a given positive constant.

Proof of Lemma 2.2.1. Since $\tilde{W}(v) \underset{|v|\sim\infty}{\sim} \frac{\gamma(\gamma+1)}{v^2}$ for $v \gg 1$, we consider the approximate equation

$$\tilde{L}_0(\psi) := \big[-\partial_v^2 + \frac{\gamma(\gamma+1)}{v^2}\big]\psi = 0. \tag{2.2.5}$$

The two functions $\psi_1^* = |v|^{-\gamma}$ and $\psi_2^* = |v|^{\gamma+1}$ are solutions to (2.2.5) in $\mathbb{R}^*$, and form a basis of solutions on $\mathbb{R}^*_+$ and $\mathbb{R}^*_-$ respectively. Now, since we know the asymptotic profile of the solutions to (2.2.2), on $\{|v| \geqslant v_0\}$, we will look for solutions via the following change of unknown

$$\tilde{\psi}_1 = \psi_1^*(1+R_1) \text{ on } [v_0,+\infty) \quad \text{and} \quad \tilde{\psi}_2 = \psi_2^*(1+R_2) \text{ on } (-\infty,-v_0].$$

We show the existence and uniqueness of R_i for $i = 1,2$ by a fixed point argument performed on R_i which satisfy the equation

$$2(\psi_i^*)'R_i' + \psi_i^* R_i'' = [\tilde{W} - \frac{\gamma(\gamma+1)}{v^2}]\psi_i^*(1+R_i),$$

which can be written (after multiplication by ψ_i^*) as follows

$$\big([\psi_i^*]^2 R_i'\big)' = [\tilde{W} - \frac{\gamma(\gamma+1)}{v^2}][\psi_i^*]^2(1+R_i)$$

and this leads to the implicit equations

$$\begin{cases} R_1(v) = \displaystyle\int_v^\infty \int_v^w \frac{[\psi_j^*]^2(w)}{[\psi_j^*]^2(u)}\mathrm{d}u\big[\tilde{W}(w) - \frac{\gamma(\gamma+1)}{w^2}\big](1+R_1(w))\mathrm{d}w, \ v \geqslant v_0, \ j=1,2, \\ R_2(v) = \displaystyle\int_{-\infty}^v \int_w^v \frac{[\psi_j^*]^2(w)}{[\psi_j^*]^2(u)}\mathrm{d}u\big[\tilde{W}(w) - \frac{\gamma(\gamma+1)}{w^2}\big](1+R_2(w))\mathrm{d}w, \ v \leqslant -v_0, \ j=1,2. \end{cases}$$

Note that the two previous equations are of the form: $(Id - \mathbb{K}_i)(R_i) = \mathbb{K}_i(1)$ for $i = 1,2$ with

$$\begin{cases} \mathbb{K}_1(g) = \displaystyle\int_v^\infty \int_v^w \frac{[\psi_j^*]^2(w)}{[\psi_j^*]^2(u)}\mathrm{d}u\big[\tilde{W}(w) - \frac{\gamma(\gamma+1)}{w^2}\big]g(w)\mathrm{d}w, \ v \geqslant v_0, \ j=1,2, \\ \mathbb{K}_2(g) = \displaystyle\int_{-\infty}^v \int_w^v \frac{[\psi_j^*]^2(w)}{[\psi_j^*]^2(u)}\mathrm{d}u\big[\tilde{W}(w) - \frac{\gamma(\gamma+1)}{w^2}\big]g(w)\mathrm{d}w, \ v \leqslant -v_0, \ j=1,2. \end{cases}$$

We have for $v \geqslant v_0 > 0$,

- for $j = 1$:

$$\begin{aligned} \big|v^{n+2}\mathbb{K}_1(g)(v)\big| &\leqslant v^{n+2}\int_v^\infty \int_v^w \frac{u^{2\gamma}}{w^{2\gamma}}\mathrm{d}u \frac{\gamma(\gamma+1)}{w^{n+4}} w^n |g(w)|\mathrm{d}w \\ &\leqslant \frac{\gamma(\gamma+1)}{2\gamma+1} v^{n+2}\int_v^\infty w^{-3-n}\mathrm{d}w \|v^n g\|_{L^\infty(v_0,\infty)}. \end{aligned}$$

Hence,

$$\left|v^{n+2}\mathbb{K}_1(g)(v)\right| \leqslant \frac{\gamma(\gamma+1)}{2\gamma+1}\frac{1}{n+2}\|v^n g\|_{L^\infty(v_0,\infty)}.$$

• for $j=2$, we take $n>2\gamma$ and we write:

$$\begin{aligned}\left|v^{n+2}\mathbb{K}_1(g)(v)\right| &\leqslant v^{n+2}\int_v^\infty\int_v^w \frac{w^{2\gamma+2}}{u^{2\gamma+2}}\mathrm{d}u\frac{\gamma(\gamma+1)}{w^{n+4}}w^n|g(w)|\mathrm{d}w\\ &\leqslant \frac{\gamma(\gamma+1)}{2\gamma+1}v^{n-2\gamma-1}\int_v^\infty w^{2\gamma-n-2}\mathrm{d}w\|v^n g\|_{L^\infty(v_0,\infty)}\\ &\leqslant \frac{\gamma(\gamma+1)}{2\gamma+1}\frac{1}{n+2-2\gamma}\|v^n g\|_{L^\infty(v_0,\infty)}.\end{aligned}$$

Finally, in both cases, $\mathbb{K}_1$ is bounded in $L^\infty(v_0,\infty)$ with

$$\|\mathbb{K}_1\|_{\mathcal{L}(L^\infty(v_0,\infty))} \leqslant \frac{\gamma(\gamma+1)}{2(2\gamma+1)}v_0^{-2} \leqslant \frac{1}{2} \quad \text{for } v_0 \geqslant \sqrt{\frac{\gamma(\gamma+1)}{2(2\gamma+1)}}.$$

Hence the existence and uniqueness of R_1 in $L^\infty(v_0,\infty)$ with the asymptotic $R_1=O(v^{-2})$ since $\mathbb{K}_1(1)=O(v^{-2})$.

By doing exactly the same computations for R_2, as in R_1 (after change of variable $v=-v'$), we get

$$\left||v|^n\mathbb{K}_2(g)(v)\right| \leqslant \frac{\gamma(\gamma+1)}{2(2\gamma+1)}v_0^{-2}\left\||v|^n g\right\|_{L^\infty(-\infty,-v_0)}, \quad \text{for all } j=1,2 \ \text{ if } n>2\gamma.$$

The rest of the proof is done in the same way as for R_1. □

Remark 2.2.3. Note that 0 is the unique solution to (2.2.2) in $L^2(\mathbb{R})$. It can be seen by multiplying the equation by ψ and integrating by parts over $\mathbb{R}$. Therefore, we cannot have a solution like $|v|^{-\gamma}$ in $+\infty$ and $-\infty$ at the same time.

Proof of Lemma 2.2.2.

Existence: The existence of a unique global solution ψ for the Cauchy problem (2.2.3) follows from the Cauchy-Lipschitz theorem. Concerning regularity, we have $\partial_v^2\psi=\tilde{W}\psi\in C^0(\mathbb{R})$, then ψ is in $C^2(\mathbb{R})$, and by derivation of the equation, we show that the solution belongs to $C^\infty(\mathbb{R})$.

Symmetry and positivity: The function $\psi(-v)$ satisfies the Cauchy problem (2.2.3), with the same initial condition, hence we get the symmetry $\psi(-v)=\psi(v)$ on $\mathbb{R}$, by uniqueness of the solution.
Let us now show that $\psi(v)>0$ for all v in $\mathbb{R}$. We check it on $\mathbb{R}^+$ since ψ is even. Suppose there is $v_1>0$ such that $\psi(v_1)=0$. Since $\psi(-v)=\psi(v)$ then, $\psi'(0)=0$. So, by multiplying the equation (2.2.2) by ψ and integrating it over $(0,v_1)$, we obtain:

$\int_0^{v_1} \big[|\psi'|^2 + \tilde{W}|\psi|^2\big]\mathrm{d}v = 0$. Therfore, $\psi(v) = 0$ for all $v \in [0, v_0]$, which contradicts the fact that $\psi(0) = 1$. Thus, $\psi(v) > 0$ for all v in $\mathbb{R}$.

Behavior at infinity: Since ψ is even, the behavior $|v|^{-\gamma}$ is excluded, because otherwise ψ will be in $L^2(\mathbb{R})$, which implies thereafter that $\psi = 0$. Finally, the Lemma 2.2.1 gives us the existence of a unique function $R \in L^\infty([v_0, +\infty[)$ for v_0 large enough, with $R(v) = O(v^{-2})$ such that: $\psi(v) = |v|^{\gamma+1}\big(1 + R(v)\big)$, up to a multiplicative constant. This constant is determined by the uniqueness of ψ and it is positive since ψ is positive. □

2.2.2 Construction of a adequate basis of solutions

Proposition 2.2.4 (A basis of solutions to (2.2.2))**.** *There are two positive functions ψ_1 and ψ_2, of class C^∞, solutions to the equation (2.2.2) and satisfying*

1. *For all v in $\mathbb{R}$: $\psi_1(-v) = \psi_2(v) > 0$.*
2. *$\{\psi_1, \psi_2\}$ forms a basis of solutions for the differential equation (2.2.2). Moreover, the Wronskian is given by: $W\{\psi_1, \psi_2\} = \psi_1\psi_2' - \psi_1'\psi_2 = 1$, for all v in $\mathbb{R}$.*
3. *we have the following estimate*

$$\psi_1(v) \lesssim \begin{cases} |v|^{-\gamma} & \text{for } v \geqslant v_0, \\ |v|^{1+\gamma} & \text{for } v \leqslant -v_0. \end{cases} \qquad \text{moreover} \qquad \psi_1(v) \sim \begin{cases} c_1|v|^{-\gamma} & \text{at } +\infty, \\ c_2|v|^{1+\gamma} & \text{at } -\infty. \end{cases} \tag{2.2.6}$$

Hence, the estimate and the behavior of ψ_2 thanks to the symmetry:

$$\psi_2(v) \lesssim \begin{cases} |v|^{\gamma+1} & \text{for } v \geqslant v_0, \\ |v|^{-\gamma} & \text{for } v \leqslant -v_0, \end{cases} \qquad \text{and} \qquad \psi_2(v) \sim \begin{cases} c_2|v|^{1+\gamma} & \text{at } +\infty, \\ c_1|v|^{-\gamma} & \text{at } -\infty, \end{cases} \tag{2.2.7}$$

with $v_0 > 0$ large enough, and where c_1 and c_2 are two given positive constants.

Proof. Let ψ be the function constructed in the previous Lemma 2.2.2. We define the two functions ψ_1 and ψ_2 by:

$$\psi_1(v) := \psi(v)\int_v^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} \quad \text{and} \quad \psi_2(v) = \psi(v)\int_{-\infty}^{v} \frac{\mathrm{d}w}{\psi^2(w)} = \psi_1(-v). \tag{2.2.8}$$

We are going to establish the properties of ψ_1 and those of ψ_2 are obtained by symmetry. The function ψ_1 is well defined and positive since $\psi(v) > 0$ for all v in $\mathbb{R}$ and

$$\frac{1}{\psi^2(v)} \underset{|v|\sim\infty}{\sim} c^{-2}|v|^{-2(\gamma+1)} \in L^1(\mathbb{R}).$$

Moreover,

$$\psi_1(v) \sim \begin{cases} c_1|v|^{-\gamma} & \text{at } +\infty, \\ c_2|v|^{\gamma+1} & \text{at } -\infty. \end{cases}$$

Indeed, since $\psi(v) = c|v|^{\gamma+1}\big(1+R(v)\big)$ for $|v| \geqslant v_0$ with $R(v) = O(v^{-2})$. Then, for $|v| \geqslant v_*$ big enough

$$C_1|v|^{\gamma+1} \leqslant \psi(v) \leqslant C_2|v|^{\gamma+1}.$$

Therefore,

$$C_1'|v|^{\gamma+1} \int_v^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} \leqslant \psi_1(v) \leqslant C_2'|v|^{\gamma+1} \int_v^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} \quad \text{for all } |v| \geqslant v_*.$$

Now, for $v \geqslant v_*$

$$\int_v^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} \sim c' \int_v^{+\infty} \frac{\mathrm{d}w}{|w|^{2(\gamma+1)}} \sim c''|v|^{-2\gamma-1}$$

and for $v \leqslant -v_*$

$$\int_v^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} \leqslant \int_{-\infty}^{+\infty} \frac{\mathrm{d}w}{\psi^2(w)} = C > 0.$$

Hence the estimate (2.2.6) hold.
The functions ψ_1 and ψ_2 are two linearly independent solutions of (2.2.2) and their Wronskian is given by:

$$W\{\psi_1, \psi_2\} = \int_{\mathbb{R}} \frac{\mathrm{d}w}{\psi^2(w)} = \big\|1/\psi\big\|_2^2.$$

Observe that $\|1/\psi\|_2$ is a positive constant by which we can divide in the definition of ψ_1 and ψ_2 in order to obtain $W\{\psi_1, \psi_2\} = 1$. ☐

2.2.3 Solutions of $L_0(\psi) = f$ and definition of T_0.

Let's go back to the equation with a general source term (2.2.1). We have the following Definition/Proposition:

Proposition 2.2.5 (Definition of T_0). *There exists a "right inverse operator" of L_0, denoted by T_0, with integral kernel K_0 such that, for all f in $L^\infty(\mathbb{R}; \langle v\rangle^{-\sigma}\mathrm{d}v)$ with $\sigma > \gamma+2$,*

$$T_0(f)(v) := \int_{\mathbb{R}} K_0(v, w) f(w)\mathrm{d}w, \tag{2.2.9}$$

is solution to the equation (2.2.1), where $K_0(v, w)$ is given by:

$$K_0(v, w) := \psi_1(v)\psi_2(w)\chi_{\{w<v\}} + \psi_1(w)\psi_2(v)\chi_{\{w>v\}}. \tag{2.2.10}$$

Thus,

$$T_0(f)(v) = \int_{-\infty}^{v} \psi_1(v)\psi_2(w)f(w)\mathrm{d}w + \int_v^{\infty} \psi_1(w)\psi_2(v)f(w)\mathrm{d}w. \tag{2.2.11}$$

Proof. For f in $L^\infty(\mathbb{R}; \langle v\rangle^{-\sigma}\mathrm{d}v)$ with $\sigma > \gamma + 2$, the integral (2.2.5) makes sense and T_0 is well defined. The function $T_0(f)$ belongs to $C^\infty(\mathbb{R})$ since ψ_1 and ψ_2 are in $C^\infty(\mathbb{R})$, in particular $T_0(f)$ is in $C^2(\mathbb{R})$. Moreover, a simple computation gives:

$$L_0\big[T_0(f)\big] = [-\partial_v^2 + \tilde{W}(v)]T_0(f) = f.$$

□

2.3 Green function for the leading part of the perturbed operator

We will proceed as for L_0, we start by studying the homogeneous equation and constructing a basis of solutions, and establishing the behavior of the latter, then we return to the equation with right order terme by defining a suitable right inverse operator.

2.3.1 Approximation for large velocities

Since $\tilde{W}(v) \underset{|v|\to\infty}{\longrightarrow} 0$, we will approximate the homogeneous equation

$$L_{\lambda,\eta}(\psi) = \big[-\partial_v^2 + \tilde{W}(v) + \mathrm{i}\eta v - \lambda\eta^{\frac{2}{3}}\big]\psi = 0 \tag{2.3.1}$$

by the equation

$$\tilde{L}_{\lambda,\eta}(\psi) := \big[-\partial_v^2 + \mathrm{i}\eta v - \lambda\eta^{\frac{2}{3}}\big]\psi = 0, \tag{2.3.2}$$

which becomes after the rescaling $v = \eta^{-\frac{1}{3}}s$:

$$\tilde{L}_{\lambda}(\phi) := \big[-\partial_s^2 + \mathrm{i}s - \lambda\big]\phi = 0, \tag{2.3.3}$$

where $\phi(s) := \psi(\eta^{-\frac{1}{3}}s)$.

Note that the operator $\tilde{L}_\lambda$ is nothing but Airy's operator "$-\partial_z^2 + z$" modified. Let's give a little reminder about the Airy function.

Airy's equation. We collect the following from [Ler17] and [VS].

Lemma 2.3.1 (Fundamental solutions to the Airy equation)**.**

Let us denote by Ai the Airy function given by

$$Ai(z) = \frac{1}{2\pi}\int_{\mathbb{R}+\mathrm{i}\zeta} e^{\mathrm{i}(\frac{t^3}{3}+zt)}\mathrm{d}t,\ \zeta > 0. \tag{2.3.4}$$

Ai is independent of $\zeta > 0$ and has the following properties:

1. *It defines an entire function of $z \in \mathbb{C}$ and solves the Airy equation:*

$$\partial_z^2 Ai(z) = zAi(z). \tag{2.3.5}$$

2. *There holds the symmetry:*
$$\forall z \in \mathbb{C}, \ \ Ai(\overline{z}) = \overline{Ai(z)}.$$

3. $Ai(jz)$ *and* $Ai(j^2 z)$, *where* $j = e^{\frac{2\pi \mathrm{i}}{3}}$, *are two other fundamental solutions of* (2.3.5) *with*
$$\forall z \in \mathbb{C}, \ \ Ai(z) + jAi(jz) + j^2 Ai(j^2) = 0.$$

4. *There holds the asympotic expansion:*
$$Ai(z) = \begin{cases} \dfrac{1}{\pi 3^{\frac{2}{3}}} \displaystyle\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{3}\right)}{n!} \sin\left(\frac{2}{3}(n+1)\pi\right) \left(3^{\frac{1}{3}} z\right)^n, \ |z| \leqslant 1, \\ \\ \dfrac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3} z^{\frac{3}{2}}} \left(1 + \mathcal{O}\left(\dfrac{1}{|z|^{\frac{3}{2}}}\right)\right), \ \forall z \in \mathbb{C} \setminus \mathbb{R}_-; |z| \geqslant 1, \end{cases} \tag{2.3.6}$$
and simarily for derivatives.

We will look for solutions for the equation (2.3.3) under the form:
$$\phi(s) = Ai\big(as + b\big).$$
We have:
$$\partial_s^2 \phi(s) = a^2 \partial_s^2 Ai(as + b) = a^2 \big(as + b\big) \phi(s).$$
Then, by identification we get
$$a^3 = \mathrm{i} \quad \text{and} \quad ba^2 = -\lambda.$$
Which implies that
$$a_k = e^{\mathrm{i}\frac{\pi}{6}} e^{\mathrm{i}\frac{2\pi k}{3}}, \quad b_k = \frac{-\lambda}{a_k^2} \quad \text{with } k = 0, 1, 2.$$
Hence, each of the following three functions is a solution of (2.3.3):
$$\mathfrak{a}_\lambda(s) = Ai\big[e^{\mathrm{i}\frac{\pi}{6}}(s + \mathrm{i}\lambda)\big], \mathfrak{b}_\lambda(s) = Ai\big[e^{\mathrm{i}\frac{\pi}{6}} j(s + \mathrm{i}\lambda)\big] \text{ and } \mathfrak{c}_\lambda(s) = Ai\big[e^{\mathrm{i}\frac{\pi}{6}} j^2(s + \mathrm{i}\lambda)\big].$$
Note that $\{\mathfrak{a}_\lambda, \mathfrak{b}_\lambda\}$ form a basis of solutions to the differential equation of order 2 (2.3.3).

Back to equation (2.3.1)**.** The equation (2.3.1) admits two possible behaviors at infinity, $\mathfrak{a}_{\lambda,\eta}$ and $\mathfrak{b}_{\lambda,\eta}$, which form a basis of solutions for the approximate equation (2.3.2). We have the following Lemma:

Lemma 2.3.2 (Airy behavior for large velocities for (2.3.1))**.** *For* $s_0 > 0$ *large enough, there exists a unique function* $R_1^{\lambda,\eta} \in L^\infty(\{|s| \geqslant s_0\})$ *(respectively, there exists a unique* *function $R_2^{\lambda,\eta} \in L^\infty(\{|s| \geqslant s_0\})$) such that, the functions*

$$\tilde{\psi}_1^{\lambda,\eta}(v) = \mathfrak{a}_{\lambda,\eta}(v)\big(1 + R_1^{\lambda,\eta}(\eta^{\frac{1}{3}}v)\big), \quad |v| \geqslant s_0\eta^{-\frac{1}{3}}, \tag{2.3.7}$$

$$\tilde{\psi}_2^{\lambda,\eta}(v) = \mathfrak{b}_{\lambda,\eta}(v)\big(1 + R_2^{\lambda,\eta}(\eta^{\frac{1}{3}}v)\big), \quad |v| \geqslant s_0\eta^{-\frac{1}{3}} \tag{2.3.8}$$

are solutions to the homogeneous equation (2.3.1) in $\{|v| \geqslant s_0\eta^{-\frac{1}{3}}\}$, unique up to two complex multiplicative constants which can depend on λ and η, and where $\mathfrak{a}_{\lambda,\eta}(v) := \mathfrak{a}_\lambda(\eta^{\frac{1}{3}}v)$ and $\mathfrak{b}_{\lambda,\eta}(v) := \mathfrak{b}_\lambda(\eta^{\frac{1}{3}}v)$. Moreover, $R_i^{\lambda,\eta}$ are holomorphic in $\{|\lambda| \leqslant \lambda_0\}$ for $i = 1, 2$, with $R_1^{\lambda,\eta}(s) = O(|s|^{-\frac{3}{2}})$ and $R_2^{\lambda,\eta}(s) = O(|s|^{-\frac{3}{2}}|\mathfrak{a}_0(s)|^2)$ uniformly in $\{|\lambda| \leqslant \lambda_0\}$ and $\eta \in [0, \eta_0]$.

Remark 2.3.3.

1. Thanks to the second point of Lemma 2.3.1, the functions $\mathfrak{a}_{\lambda,\eta}$ and $\mathfrak{b}_{\lambda,\eta}$ are linked by the following identity:

$$\bar{\mathfrak{a}}_{\bar{\lambda},\eta}(-v) = \mathfrak{b}_{\lambda,\eta}(v), \quad \forall v \in \mathbb{R}. \tag{2.3.9}$$

2. The Airy function $\mathfrak{a}_\lambda$ decreases exponentially in $+\infty$ and increases exponentially in $-\infty$ and inversely for $\mathfrak{b}_\lambda$:

$$\mathfrak{a}_\lambda(s) \sim \begin{cases} c_1 s^{-\frac{1}{4}} e^{-\frac{\sqrt{2}}{3}s^{\frac{3}{2}}} & \text{at } +\infty, \\ c_2 |s|^{-\frac{1}{4}} e^{\frac{\sqrt{2}}{3}|s|^{\frac{3}{2}}} & \text{at } -\infty. \end{cases}$$

Proof. The proof is identical to that of Lemma 2.2.1, it relies on the fixed point theorem.
• Let's start by showing (2.3.7). If $\tilde{\psi}_1^{\lambda,\eta}$ is a solution to (2.3.1) of the form $\mathfrak{a}_{\lambda,\eta}(v)\big(1 + R_1^{\lambda,\eta}(\eta^{\frac{1}{3}}v)\big)$ then, after rescaling the equation (2.3.1) by $v = s\eta^{-\frac{1}{3}}$, $R_1^{\lambda,\eta}$ satisfies

$$2\mathfrak{a}'_\lambda(s)(R_1^{\lambda,\eta})'(s) + \mathfrak{a}_\lambda(s)(R_1^{\lambda,\eta})''(s) = \tilde{W}_\eta(s)\mathfrak{a}_\lambda(s)(1 + R_1^{\lambda,\eta}(s)),$$

where $\tilde{W}_\eta(s) = \eta^{-\frac{2}{3}}\tilde{W}(\eta^{-\frac{1}{3}}s) = \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}}+s^2}$. Define $\mathbb{K}_1^{\lambda,\eta}$ by

$$\mathbb{K}_1^{\lambda,\eta}(g) := \begin{cases} \displaystyle\int_s^\infty \int_s^t \frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(u)} \mathrm{d}u \tilde{W}_\eta(t) g(t) \mathrm{d}t, \ s \geqslant s_0, \\ \displaystyle\int_{-\infty}^s \int_t^s \frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(u)} \mathrm{d}u \tilde{W}_\eta(t) g(t) \mathrm{d}t, \ s \leqslant -s_0. \end{cases}$$

The previous equation on $R_1^{\lambda,\eta}$ is equivalent to $(Id - \mathbb{K}_1^{\lambda,\eta})(R_1^{\lambda,\eta}) = \mathbb{K}_1^{\lambda,\eta}(1)$.
The computations for the fixed point for large velocities come from [LP19], and which we will recall here for consistency. Let's start by introducing the set

$$U := \{e^{\mathrm{i}\frac{\pi}{6}}(x + \mathrm{i}\lambda); x \geqslant 0, |\lambda| \leqslant \lambda_0\}.$$

We have from [ZQ07] or also from the equation (2.3.6), for $z \in U$; $|z| \geqslant \frac{1}{2}$:

$$Ai(z) = \tau(z)e^{-\frac{2}{3}z^{\frac{3}{2}}} \quad \text{with} \quad \frac{c_0}{(1+|z|)^{\frac{1}{4}}} \leqslant |\tau(z)| \leqslant \frac{c_1}{(1+|z|)^{\frac{1}{4}}}.$$

Let's show that for s_0 large enough,

$$\left\| |s|^{n+\frac{3}{2}} \mathbb{K}_1^{\lambda,\eta}(g) \right\|_{L^\infty(\{|s| \geqslant s_0\})} \leqslant \frac{1}{2} \left\| |s|^n g \right\|_{L^\infty(\{|s| \geqslant s_0\})}.$$

We are going to establish the previous inequality for $s \geqslant s_0$ and for $s \leqslant -s_0$ it is obtained in the same way, by symmetry thanks to the parity of $\tilde{W}_\eta$ and the identity (2.3.9). We have

$$\left| s^{n+\frac{3}{2}} \mathbb{K}_1^{\lambda,\eta}(g)(s) \right| \leqslant \gamma(\gamma+1) s^{n+\frac{3}{2}} \int_s^\infty \int_s^t \left| \frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(u)} \right| \mathrm{d}u \frac{|t^n g(t)|}{t^{n+2}} \mathrm{d}t.$$

First, there exists a constant $C > 0$ such that for all $s \geqslant 1$, $|\lambda| \leqslant \lambda_0$ and $t \geqslant s$, we have

$$\int_s^t \left| \frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(u)} \right| \mathrm{d}u \leqslant C t^{-\frac{1}{2}}.$$

Indeed, by (2.3.6): fourth point of Lemma 2.3.1, and for $|\lambda| \leqslant \lambda_0$ with λ_0 small enough

$$\begin{aligned} \int_s^t \left| \frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(u)} \right| \mathrm{d}u &\leqslant C \int_s^t e^{-\frac{4}{3}\mathrm{Re}([(t+\mathrm{i}\lambda)^{\frac{3}{2}} - (u+\mathrm{i}\lambda)^{\frac{3}{2}}] e^{\mathrm{i}\frac{\pi}{4}})} du \\ &= Ct \int_{\frac{s}{t}}^1 e^{-\frac{4}{3}t^{\frac{3}{2}}\mathrm{Re}([(1+\mathrm{i}\frac{\lambda}{t})^{\frac{3}{2}} - (x+\mathrm{i}\frac{\lambda}{t})^{\frac{3}{2}}] e^{\mathrm{i}\frac{\pi}{4}})} \mathrm{d}x \\ &\leqslant Ct \int_0^\infty e^{-xt^{\frac{3}{2}}} \mathrm{d}x \sim C t^{-\frac{1}{2}} \quad \text{if } t \geqslant s \geqslant 1, \end{aligned}$$

where we performed the change of variable $x = \frac{u}{t}$ in the second line and used the mean value theorem to estimate the real part of the exponent. Thus for $|\lambda| \leqslant \lambda_0$,

$$\left| s^{n+\frac{3}{2}} \mathbb{K}_1^{\lambda,\eta}(g)(s) \right| \leqslant C\gamma(\gamma+1) s^{n+\frac{3}{2}} \int_s^\infty t^{-(n+\frac{5}{2})} \mathrm{d}t \|s^n g\|_{L^\infty(1,\infty)}.$$

Then,

$$\|s^{n+\frac{3}{2}} \mathbb{K}_1^{\lambda,\eta}(g)\|_{L^\infty(1,\infty)} \leqslant C \frac{\gamma(\gamma+1)}{(n+\frac{3}{2})} \|s^n g\|_{L^\infty(1,\infty)}.$$

Finally, $\mathbb{K}_1^{\lambda,\eta}$ is bounded in $L^\infty(s_0,\infty)$ with

$$\|\mathbb{K}_1^{\lambda,\eta}\|_{\mathcal{L}(L^\infty(s_0,\infty))} \leqslant \frac{2C}{3}\gamma(\gamma+1) s_0^{-\frac{3}{2}} \leqslant \frac{1}{2} \quad \text{if } s_0 \text{ is big enough.}$$

The rest of the proof concerning (2.3.7) is identical to that of Lemma 2.2.1. Holomorphy of $R_1^{\lambda,\eta}$ comes from the fact that $\mathfrak{a}_\lambda$ is holomorphic on $\{|\lambda| \leqslant \lambda_0\}$, since Airy's function is an entire function in $\mathbb{C}$.

• Let us study (2.3.8). For a solution $\tilde{\psi}_2^{\lambda,\eta}$ of the form $\mathfrak{b}_{\lambda,\eta}(v)\big(1+R_2^{\lambda,\eta}(\eta^{\frac{1}{3}}v)\big)$, the same computations as for (2.3.7) lead to find $R_2^{\lambda,\eta}$ solution to

$$\begin{cases} (Id-\mathbb{K}_2^{\lambda,\eta})(R_2^{\lambda,\eta}) = \mathbb{K}_2^{\lambda,\eta}(1), \\ \mathbb{K}_2^{\lambda,\eta}(g) := \displaystyle\int_s^\infty \int_s^t \frac{\mathfrak{b}_\lambda^2(t)}{\mathfrak{b}_\lambda^2(u)} \mathrm{d}u \tilde{W}_\eta(t) g(t)\mathrm{d}t, \quad s \geqslant s_0, \\ \mathbb{K}_2^{\lambda,\eta}(g) := \displaystyle\int_{-\infty}^s \int_t^s \frac{\mathfrak{b}_\lambda^2(t)}{\mathfrak{b}_\lambda^2(u)} \mathrm{d}u \tilde{W}_\eta(t) g(t)\mathrm{d}t, \ s \leqslant -s_0. \end{cases}$$

As in the previous point, we establish the inequality on the norm of $\mathbb{K}_2^{\lambda,\eta}$ on $[s_0,+\infty[$, and on $]-\infty,-s_0]$ it is obtained by symmetry. We have

$$\frac{W\{\mathfrak{a}_\lambda,\mathfrak{b}_\lambda\}}{\mathfrak{b}_\lambda^2(u)} = \left(\frac{\mathfrak{a}_\lambda(u)}{\mathfrak{b}_\lambda(u)}\right)',$$

where $W\{\mathfrak{a}_\lambda,\mathfrak{b}_\lambda\}$ is the Wronskian of $\mathfrak{a}_\lambda$ and $\mathfrak{b}_\lambda$ given by $W\{\mathfrak{a}_\lambda,\mathfrak{b}_\lambda\} = \mathfrak{a}_\lambda\mathfrak{b}_\lambda' - \mathfrak{a}_\lambda'\mathfrak{b}_\lambda = \frac{1}{2\pi}$. Therefore,

$$\begin{aligned} \left|\frac{\mathbb{K}_2^{\lambda,\eta}(g)(s)}{s^{-(n+\frac{3}{2})}\mathfrak{a}_\lambda^2(s)}\right| &\leqslant Cs^{n+\frac{3}{2}} \int_s^\infty \left(\left|\frac{\mathfrak{a}_\lambda(s)}{\mathfrak{b}_\lambda(s)}\right| + \left|\frac{\mathfrak{a}_\lambda(t)}{\mathfrak{b}_\lambda(t)}\right|\right) \left|\frac{\mathfrak{b}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(s)}\right| |g(t)| \frac{\mathrm{d}t}{t^2} \\ &\leqslant Cs^{n+\frac{3}{2}} \int_s^\infty \left(\left|\frac{\mathfrak{a}_\lambda(t)\mathfrak{b}_\lambda(t)}{\mathfrak{a}_\lambda(s)\mathfrak{b}_\lambda(s)}\right| + \left|\frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(s)}\right|\right) |\mathfrak{a}_\lambda(t)\mathfrak{b}_\lambda(t)| \left|\frac{g(t)}{t^{-n}\mathfrak{a}_\lambda^2(t)}\right| \frac{\mathrm{d}t}{t^{n+2}}, \end{aligned}$$

where $C = 2\pi\gamma(\gamma+1)$. Now, since we have $|Ai(z)Ai(jz)| = |\tau(z)|^2$ (note that for the following choice of determination of the complex logarithm: $\log(re^{\mathrm{i}\theta}) = \log(r) + \mathrm{i}\theta$, $(re^{\mathrm{i}\theta})^\alpha = r^\alpha e^{\mathrm{i}\alpha\theta}$ for $\theta \in (-\pi,\pi)$, one has $(e^{\mathrm{i}\frac{\pi}{6}}j)^{\frac{3}{2}} = -(e^{\mathrm{i}\frac{\pi}{6}})^{\frac{3}{2}}$). Then, for $t \geqslant s \geqslant 1$ and $|\lambda| \leqslant \lambda_0$ with λ_0 small enough

$$|\mathfrak{a}_\lambda(t)\mathfrak{b}_\lambda(t)| = \frac{1}{4\pi}|t+\mathrm{i}\lambda|^{-\frac{1}{2}} \quad \text{and} \quad \left|\frac{\mathfrak{a}_\lambda^2(t)}{\mathfrak{a}_\lambda^2(s)}\right| \leqslant \frac{|t+\mathrm{i}\lambda|^{-\frac{1}{2}}}{|s+\mathrm{i}\lambda|^{-\frac{1}{2}}} \leqslant 2. \tag{2.3.10}$$

Then,

$$\left|\frac{\mathbb{K}_2^{\lambda,\eta}(g)(s)}{s^{-(n+\frac{3}{2})}\mathfrak{a}_\lambda^2(s)}\right| \leqslant C\gamma(\gamma+1)s^{n+\frac{3}{2}} \int_s^\infty \frac{\mathrm{d}t}{t^{n+\frac{5}{2}}} \left\|\frac{g}{s^{-n}\mathfrak{a}_\lambda^2}\right\|_{L^\infty(s_0,\infty)}.$$

Which implies that

$$\left\|\frac{\mathbb{K}_2^{\lambda,\eta}(g)}{s^{-n}\mathfrak{a}_\lambda^2}\right\|_{L^\infty(s_0,\infty)} \leqslant C\frac{\gamma(\gamma+1)}{(n+\frac{3}{2})} s_0^{-\frac{3}{2}} \left\|\frac{g}{s^{-n}\mathfrak{a}_\lambda^2}\right\|_{L^\infty(s_0,\infty)} \leqslant \frac{1}{2}\left\|\frac{g}{s^{-n}\mathfrak{a}_\lambda^2}\right\|_{L^\infty(s_0,\infty)}$$

for s_0 large enough. The rest of the proof is similar to the previous point, which completes the proof of the Lemma. □

2.3.2 Approximation for all ranges of velocities

As for $\eta = 0$, we are going to construct an intermediate function, solution to (2.3.1), with good properties, and which will allow us thereafter to define two other solutions forming a basis of solutions for the ODE (2.3.1) as well as a right inverse operator for $L_{\lambda,\eta}$, which gives us solutions for the equation with right-hand side. For this, consider the following Cauchy problem:

$$\begin{cases} L_{\lambda,\eta}(\psi) = 0, \\ \psi(0) = 1, \quad \psi'(0) = 0. \end{cases} \tag{2.3.11}$$

Lemma 2.3.4 (Existence of a global solution and properties)**.** *There exists a unique global solution $\psi^{\lambda,\eta}$ for the Cauchy problem (2.3.11), of class C^∞ with respect to v, holomorphic in $\{|\lambda| \leqslant \lambda_0\}$ and continuous with respect to $\eta \in [0, \eta_0]$. Furthermore,*

1. *For all $v \in \mathbb{R}$, $\bar{\psi}^{\bar{\lambda},\eta}(-v) = \psi^{\lambda,\eta}(v)$ and $\psi^{\lambda,\eta}(v) \neq 0$.*
2. *For $s_0 > 0$ large enough, there is a unique function $R^{\lambda,\eta} \in L^\infty[s_0, +\infty[$, holomorphic in $\{|\lambda| \leqslant \lambda_0\}$ with $R^{\lambda,\eta}(s) = O(|s|^{-\frac{3}{2}}|\mathfrak{a}_0(s)|^2)$ uniformly in $\{|\lambda| \leqslant \lambda_0\}$ and η, such that*
$$\psi^{\lambda,\eta}(v) = c_\lambda \eta^{-\frac{\gamma+1}{3}} \mathfrak{b}_{\lambda,\eta}(v)\big(1 + R^{\lambda,\eta}(\eta^{\frac{1}{3}} v)\big), \quad \forall v \geqslant s_0 \eta^{-\frac{1}{3}}, \tag{2.3.12}$$
where c_λ is a holomorphic function in $\{|\lambda| \leqslant \lambda_0\}$, bounded by two positive constants.

Proof. 1. **Existence:** The existence of a unique global solution $\psi^{\lambda,\eta}$ for the Cauchy problem (2.3.11) follows from the Cauchy-Lipschitz theorem.

Symmetry: For all v in $\mathbb{R}$, $\bar{\psi}^{\bar{\lambda},\eta}(-v)$ satisfies the equation (2.3.11). Hence, $\bar{\psi}^{\bar{\lambda},\eta}(-v) = \psi^{\lambda,\eta}(v)$ by uniqueness of the solution. Let us now show that the solution $\psi^{\lambda,\eta}$ does not vanish on $\mathbb{R}$, for all $\lambda \in \mathbb{C}; |\lambda| \leqslant \lambda_0$ and $\eta > 0$. We verify it only on $\mathbb{R}^+$ thanks to symmetry. The idea of the proof was inspired from [LP19] to prove that the rescaled function does not vanish on $\mathbb{R}^+$. Suppose there is $v_1 > 0$ such that $\psi^{\lambda,\eta}(v_1) = 0$. Set $v_1 = s_1 \eta^{-\frac{1}{3}}$. By integrating the equation of $\psi^{\lambda,\eta}$ rescaled by $v = s\eta^{-\frac{1}{3}}$, and multiplied by $\Psi_{\lambda,\eta}(s) := \psi^{\lambda,\eta}(s\eta^{-\frac{1}{3}})$, between 0 and s_1, we obtain:

$$\int_0^{s_1} \left[|(\Psi_{\lambda,\eta})'|^2 + \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} |\Psi_{\lambda,\eta}|^2 \mathrm{d}s + \mathrm{i}s|\Psi_{\lambda,\eta}|^2 \right] \mathrm{d}s = \lambda \int_0^{s_1} |\Psi_{\lambda,\eta}|^2 \mathrm{d}s.$$

Which implies that

$$\mathrm{Re}\lambda \geqslant \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} \quad \text{and} \quad \mathrm{Im}\lambda \geqslant s \geqslant 0.$$

Then,

$$\mathrm{Re}\lambda + \mathrm{Im}\lambda \geqslant \min_{s \in [0,s_1]} \left(s + \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} \right) \geqslant c_0 > 0 \quad \text{for } \eta \in [0, \eta_0].$$

Which contradicts the fact that $|\lambda| \leqslant \lambda_0$, since c_0 does not depend on λ_0 and we can choose λ_0 as small as we want.

2. **Airy behavior:** Since there are only two possible behaviors at infinity [LP19]: $\mathfrak{a}_{\lambda,\eta}$ and $\mathfrak{b}_{\lambda,\eta}$, using exactly the same argument that we used to show that $\psi^{\lambda,\eta} \neq 0$, we can show that the profile $\mathfrak{a}_{\lambda,\eta}$, Airy which decreases exponentially at $+\infty$, so in $L^2(1,\infty)$, is excluded. Indeed, by rescaling the equation by $v = \eta^{-\frac{1}{3}}s$ and by multiplying it by $\Psi_{\lambda,\eta}(s) := \psi^{\lambda,\eta}(\eta^{-\frac{1}{3}}s)$ and integrating it over $(0,\infty)$, thanks to the condition $(\psi^{\lambda,\eta})'(0) = 0$ (recalling that $\psi^{\lambda,\eta}$ satisfies the Cauchy problem (2.3.11)), we obtain:

$$\int_0^\infty \left[|(\Psi_{\lambda,\eta})'|^2 + \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} |\Psi_{\lambda,\eta}|^2 \mathrm{d}s + \mathrm{i}s|\Psi_{\lambda,\eta}|^2 \right] \mathrm{d}s = \lambda \int_0^\infty |\Psi_{\lambda,\eta}|^2 \mathrm{d}s.$$

Then,

$$\mathrm{Re}\lambda + \mathrm{Im}\lambda \geqslant \min_{s\geqslant 0} \left(s + \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} \right) \geqslant c_0 > 0 \quad \text{for } \eta \in [0,\eta_0].$$

Hence the contradiction with $|\lambda| \leqslant \lambda_0$, with λ_0 small enough. Thus, the function $\psi^{\lambda,\eta}$ takes the profile of Airy which explodes at $+\infty$, and therefore $\psi^{\lambda,\eta}(v) \underset{+\infty}{\sim} \mathfrak{b}_{\lambda,\eta}(v)$. To obtain the equation (2.3.12), by the previous Lemma, there exists a unique function $R^{\lambda,\eta} \in L^\infty(\{|s| \geqslant s_0\})$ such that $\psi^{\lambda,\eta}$ takes the form (2.3.8), up to a multiplicative constant. This constant is determined by the uniqueness of $\psi^{\lambda,\eta}$. To calculate it, we use the continuity of $\psi^{\lambda,\eta}$ with respect to v at the point $s_0\eta^{-\frac{1}{3}}$ and the fact that $\psi^{\lambda,\eta}$ is close to ψ on $[-s_0\eta^{-\frac{1}{3}}, s_0\eta^{-\frac{1}{3}}]$. For this last point, we will prove it in the following lemma. □

Lemma 2.3.5 (Some estimates on $\psi^{\lambda,\eta}$)**.**

1. *For all $\sigma \in (0,\frac{1}{3})$, for all $a > 0$ we have the following uniform convergence:*

$$\left\| \frac{(\psi^{\lambda,\eta} - \psi)}{\psi} \right\|_{L^\infty\left(-a\eta^{\sigma-\frac{1}{3}}, a\eta^{\sigma-\frac{1}{3}}\right)} \underset{\eta\to 0}{\longrightarrow} 0. \tag{2.3.13}$$

2. *There exist two positive constants C_1 and C_2 such that, the following estimates holds:*
   - *For all $|v| \leqslant s_0\eta^{-\frac{1}{3}}$, we have*

$$C_1\psi(v) \leqslant |\psi^{\lambda,\eta}(v)| \leqslant C_2\psi(v). \tag{2.3.14}$$

   - *For all v in $\mathbb{R}$, we have*

$$\psi(v) \leqslant C_2|\psi^{\lambda,\eta}(v)|. \tag{2.3.15}$$

3. *We have the limit*

$$\int_{\mathbb{R}} \frac{\mathrm{d}w}{[\psi^{\lambda,\eta}(w)]^2} \underset{\eta\to 0}{\longrightarrow} \int_{\mathbb{R}} \frac{\mathrm{d}w}{[\psi(w)]^2}. \tag{2.3.16}$$

Proof. 1. Let $\sigma \in (0,\frac{1}{3})$ and $a > 0$. Thanks to the symmetry, we show the limit (2.3.13) on $[0, a\eta^{\sigma-\frac{1}{3}}]$. Let's set $\phi^{\lambda,\eta} := \psi^{\lambda,\eta} - \psi$. Then, $\phi^{\lambda,\eta}$ satisfies the equation:

$$L_0[\phi^{\lambda,\eta}](v) = [-\partial_v^2 + \tilde{W}(v)]\phi^{\lambda,\eta}(v) = (\lambda\eta^{\frac{2}{3}} - \mathrm{i}\eta v)[\phi^{\lambda,\eta}(v) + \psi(v)]$$

First, recall that $\psi^{\lambda,\eta}(0) = \psi(0) = 1$ and $(\psi^{\lambda,\eta})'(0) = \psi'(0) = 0$. Thus, $\phi^{\lambda,\eta}$ satisfies the initial conditions $\phi^{\lambda,\eta}(0) = (\psi^{\lambda,\eta})'(0) = 0$. The solution of the equation $L_0(\phi) = S$ with $\phi(0) = \phi'(0) = 0$ is given by:

$$\phi(v) = \psi_1(v)\int_0^v \psi_2(w)S(w)\mathrm{d}w - \psi_2(v)\int_0^v \psi_1(w)S(w)\mathrm{d}w$$

taking into account the fact that $\psi_1(-v) = \psi_2(v)$. Therefore, $\phi^{\lambda,\eta}/\psi$ satisfies:

$$\begin{aligned}\left|\frac{\phi^{\lambda,\eta}(v)}{\psi(v)}\right| &\leqslant \frac{1}{\psi(v)}\int_0^v \big[\psi_1(v)\psi_2(w) + \psi_1(w)\psi_2(v)\big]\big|(\lambda\eta^{\frac{2}{3}} - \mathrm{i}\eta w)[\phi^{\lambda,\eta}(w) + \psi(w)]\big|\mathrm{d}w \\ &\leqslant \frac{2}{\psi(v)}\bigg(\int_0^v (|\lambda|\eta^{\frac{2}{3}} + \eta|w|)\big[\psi_1(v)\psi_2(w) + \psi_1(w)\psi_2(v)\big]\psi(w)\mathrm{d}w\bigg)\bigg(1 + \bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty\bigg).\end{aligned}$$

Now we split the two cases, $v \in [0, v_0]$ and $v \in [v_0, a\eta^{\sigma-\frac{1}{3}}]$ where $v_0 > 0$ is large enough such that ψ_1 and ψ_2 satisfy (2.2.6) and (2.2.7).

Step 1: $v \in [0, v_0]$. We have in this case, ψ, ψ_1 and ψ_2 are bounded above and below by positive constants. Therefore,

$$\begin{aligned}\left|\frac{\phi^{\lambda,\eta}(v)}{\psi(v)}\right| &\lesssim \bigg(\int_0^{v_0} (|\lambda|\eta^{\frac{2}{3}} + \eta|w|)\big[\psi_2(w) + \psi_1(w)\big]\psi(w)\mathrm{d}w\bigg)\bigg(1 + \bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty\bigg). \\ &\lesssim (|\lambda|\eta^{\frac{2}{3}} + \eta)\bigg(1 + \bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty\bigg).\end{aligned}$$

Therefore, for η small enough

$$\bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty \lesssim |\lambda|\eta^{\frac{2}{3}} + \eta. \tag{2.3.17}$$

Step 2: $v \in [v_0, a\eta^{\sigma-\frac{1}{3}}]$. We have in this case, $\psi(v) \sim v^{\gamma+1} \sim \psi_2(v)$ and $\psi_1(v) \sim v^{-\gamma}$, up to a multiplicative constants, and since these three functions are bounded on the compact $[0, v_0]$ then,

$$\begin{aligned}\left|\frac{\phi^{\lambda,\eta}(v)}{\psi(v)}\right| \lesssim \bigg(&\left|\frac{\psi_1(v)}{\psi(v)}\right|\int_0^v (|\lambda|\eta^{\frac{2}{3}} + \eta|w|)\psi_2(w)\psi(w)\mathrm{d}w \\ &+ \left|\frac{\psi_2(v)}{\psi(v)}\right|\int_0^v (|\lambda|\eta^{\frac{2}{3}} + \eta|w|)\psi_1(w)\psi(w)\mathrm{d}w\bigg)\bigg(1 + \bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty\bigg).\end{aligned}$$

Therefore,

$$\left|\frac{\phi^{\lambda,\eta}(v)}{\psi(v)}\right| \lesssim \big[(|\lambda|\eta^{\frac{2}{3}} + \eta) + (|\lambda|\eta^{\frac{2}{3}}v^2 + \eta v^3)\big]\bigg(1 + \bigg\|\frac{\phi^{\lambda,\eta}}{\psi}\bigg\|_\infty\bigg).$$

Thus,

$$\left\| \frac{\phi^{\lambda,\eta}}{\psi} \right\|_\infty \lesssim a^2|\lambda|\eta^{2\sigma} + a^3\eta^{3\sigma}, \quad \forall v \in [v_0, a\eta^{\sigma-\frac{1}{3}}]. \tag{2.3.18}$$

From where, with (2.3.17), we get the limit (2.3.13).

2. From (2.3.17) and (2.3.18), and for $\sigma = 0$ and a small enough such that, for all v in $[0, a\eta^{-\frac{1}{3}}]$

$$\left| \frac{\phi^{\lambda,\eta}(v)}{\psi(v)} \right| \leqslant \frac{1}{2}\left(1 + \left\| \frac{\phi^{\lambda,\eta}}{\psi} \right\|_\infty\right)$$

we get

$$\left\| \frac{\phi^{\lambda,\eta}}{\psi} \right\|_\infty \leqslant 1, \quad \forall v \in [0, a\eta^{-\frac{1}{3}}],$$

Then,

$$\psi(v) \leqslant |\psi^{\lambda,\eta}(v)| \leqslant 2\psi(v), \quad \forall v \in [0, a\eta^{-\frac{1}{3}}]. \tag{2.3.19}$$

It remains to establish the inequality (2.3.14) on $[a\eta^{-\frac{1}{3}}, s_0\eta^{-\frac{1}{3}}]$, which is obtained by rescaling the equation of $\psi^{\lambda,\eta}$ by $v = \eta^{-\frac{1}{3}}s$. Indeed, if we denote $\Psi_{\lambda,\eta}(s) := \psi^{\lambda,\eta}(\eta^{-\frac{1}{3}}s)$, then $\Psi_{\lambda,\eta}$ satisfies

$$\Big[-\partial_s^2 + \frac{\gamma(\gamma+1)}{\eta^{\frac{2}{3}} + s^2} + \mathrm{i}s - \lambda \Big] \Psi_{\lambda,\eta}(s) = 0.$$

We have by a fixed point argument as in Lemma 2.3.4, $\Psi_{\lambda,\eta}(s) = \Psi_{\lambda,0}(s)(1 + r_{\lambda,\eta}(s))$ up to a multiplicative constant, which may depend on λ and η, and where $\Psi_{\lambda,0}$ is the solution of the last equation with $\eta = 0$:

$$\Big[-\partial_s^2 + \frac{\gamma(\gamma+1)}{s^2} + \mathrm{i}s - \lambda \Big] \Psi_{\lambda,0}(s) = 0,$$

with $r_{\lambda,\eta}$ tends to 0 when η tends to 0. Now, since the function $\Psi_{\lambda,0}$ is continuous on the compact $[a, s_0]$ and holomorphic in $\{|\lambda| \leqslant \lambda_0\}$ then, since $\psi_{\lambda,\eta}$ does not vanish, $|\Psi_{\lambda,0}|$ is bounded from below and above by two positive constants, uniformly with respect to λ and η. Then, since $r_{\lambda,\eta}$ is also bounded on $[a, s_0]$, uniformly with respect to λ and η then, by continuity of $\psi^{\lambda,\eta}$ at $a\eta^{-\frac{1}{3}}$ and the inequality (2.3.19), we get $\psi^{\lambda,\eta}(v) = c_\lambda\eta^{-\frac{\gamma+1}{3}}\Psi_{\lambda,0}(s)(1 + r_{\lambda,\eta}(s))$ with c_λ holomorphic in $\{|\lambda| \leqslant \lambda_0\}$. Thus, $\psi^{-1}\psi^{\lambda,\eta} \sim \eta^{\frac{\gamma+1}{3}}\psi^{\lambda,\eta}$ is bounded on $[a\eta^{-\frac{1}{3}}, s_0\eta^{-\frac{1}{3}}]$ from below and above. Hence the inequality (2.3.14) holds on $[-s_0\eta^{-\frac{1}{3}}, s_0\eta^{-\frac{1}{3}}]$.

For the inequality (2.3.15), it comes from (2.3.14) for $|v| \leqslant s_0\eta^{-\frac{1}{3}}$ and the fact that for $v \geqslant s_0\eta^{-\frac{1}{3}}$, $1 \lesssim |\mathfrak{b}_{\lambda,\eta}(v)|$ and $\psi(v) \lesssim \eta^{-\frac{\gamma+1}{3}}$. The case $v \leqslant -s_0\eta^{-\frac{1}{3}}$ is obtained by symmetry.

3. This limit is a direct consequence of the second point, inequality (2.3.15), and Lebesgue's theorem, since $\psi^{\lambda,\eta}$ is continuous with respect to η, which gives us simple convergence. □

Proposition 2.3.6 (Basis of solutions and definition of $T_{\lambda,\eta}$).

• *There are two functions* $\psi_1^{\lambda,\eta}$ *and* $\psi_2^{\lambda,\eta}$*, solutions to the equation* (2.3.1)*, belonging to* $C^\infty(\mathbb{R},\mathbb{C})$*, continuous with respect to* $\eta \in [0,\eta_0]$ *and holomorphic in* $\{|\lambda| \leqslant \lambda_0\}$*. Furthermore,*

1. $\{\psi_1^{\lambda,\eta}, \psi_2^{\lambda,\eta}\}$ *forms a basis of solutions for the equation* (2.3.1) *with* $W\{\psi_1^{\lambda,\eta}, \psi_2^{\lambda,\eta}\} = 1$.
2. *For all* v *in* $\mathbb{R}$, $\bar{\psi}_1^{\lambda,\eta}(-v) = \psi_2^{\lambda,\eta}(v)$.

• *The operator* $T_{\lambda,\eta}$ *defined by the integral kernel* $K_{\lambda,\eta}$*, is a right inverse of* $L_{\lambda,\eta}$*:*

$$T_{\lambda,\eta}(f)(v) = \int_{\mathbb{R}} K_{\lambda,\eta}(v,w) f(w) \mathrm{d}w, \tag{2.3.20}$$

with

$$K_{\lambda,\eta}(v,w) = \psi_1^{\lambda,\eta}(v)\psi_2^{\lambda,\eta}(w)\chi_{\{w<v\}} + \psi_1^{\lambda,\eta}(w)\psi_2^{\lambda,\eta}(v)\chi_{\{w>v\}}, \tag{2.3.21}$$

continuous with respect to $(v,w) \in \mathbb{R}\times\mathbb{R}$ *and* $\eta \in [0,\eta_0]$ *and holomorphic in* $\{|\lambda| \leqslant \lambda_0\}$*. Thus,* $T_{\lambda,\eta}(f)$ *is solution to the equation* $L_{\lambda,\eta}(\psi) = f$*.*

Proof of Proposition 2.3.6.

• Let $\psi^{\lambda,\eta}$ be the function of lemma 2.3.4. We define the two functions $\psi_1^{\lambda,\eta}$ and $\psi_2^{\lambda,\eta}$, for $v \in \mathbb{R}$, by:

$$\psi_1^{\lambda,\eta}(v) := \frac{\psi^{\lambda,\eta}(v)}{\|1/\psi^{\lambda,\eta}\|_2} \int_v^{+\infty} \frac{\mathrm{d}w}{[\psi^{\lambda,\eta}(w)]^2} \quad \text{and} \quad \psi_2^{\lambda,\eta}(v) = \frac{\psi^{\lambda,\eta}(v)}{\|1/\psi^{\lambda,\eta}\|_2} \int_{-\infty}^{v} \frac{\mathrm{d}w}{[\psi^{\lambda,\eta}(w)]^2}. \tag{2.3.22}$$

The functions $\psi_1^{\lambda,\eta}$ and $\psi_2^{\lambda,\eta}$ are both well defined thanks to the inequality (2.3.15) which guarantees that the integral is indeed defined in both cases, and they are solutions to (2.3.1) having the same regularity as $\psi^{\lambda,\eta}$, i.e. $\psi_i^{\lambda,\eta} \in C^\infty(\mathbb{R})$ for $i = 1,2$. The continuity/holomorphy of $\psi_i^{\lambda,\eta}$ for $i = 1,2$ is obtained by Lebesgue's theorem thanks to the continuity/holomorphy of $\psi^{\lambda,\eta}$, the limit (2.3.16) and the domination $\frac{1}{|\psi^{\lambda,\eta}(v)|^2} \lesssim \frac{1}{\psi^2(v)}$ by (2.3.15). Moreover,
1. We have: $W\{\psi_1^{\lambda,\eta}, \psi_2^{\lambda,\eta}\} = \psi_1^{\lambda,\eta}(\psi_2^{\lambda,\eta})' - (\psi_1^{\lambda,\eta})'\psi_2^{\lambda,\eta} = \|1/\psi^{\lambda,\eta}\|_2^{-2} \int_{\mathbb{R}} \frac{\mathrm{d}w}{[\psi^{\lambda,\eta}(w)]^2} = 1$. Thus, $\psi_1^{\lambda,\eta}$ and $\psi_2^{\lambda,\eta}$ are linearly independent.
2. The second point comes from the symmetry of $\psi^{\lambda,\eta}$ (first point of the lemma 2.3.4).
• For f in $L^\infty\big(\mathbb{R}; \frac{\langle v\rangle^{-\sigma}\mathrm{d}v}{|\psi_1^{\lambda,\eta}|+|\psi_2^{\lambda,\eta}|}\big)$ with $\sigma > 2$, the integral (2.3.20) is well defined, $T_{\lambda,\eta}(f)$ belongs to $C^2(\mathbb{R},\mathbb{C})$ and we have: $L_{\lambda,\eta}[T_{\lambda,\eta}(f)] = f$. □

Remark 2.3.7.

1. Note that $T_{\lambda,0} =: T_0$ does not depend on λ, since the functions $\psi_i^{\lambda,\eta}$ for $i = 1,2$ are continuous with respect to η and their limits, when $\eta \to 0$, ψ_i do not depend on λ.

2. Thanks to the identity $\bar{\psi}_1^{\bar{\lambda},\eta}(-v) = \psi_2^{\lambda,\eta}(v)$, $K_{\lambda,\eta}$ satisfies

$$\bar{K}_{\bar{\lambda},\eta}(-v,-w) = K_{\lambda,\eta}(v,w), \quad \forall v,w \in \mathbb{R}.$$

Thus, for $\bar{f}_{\bar{\lambda},\eta}(-v) = f_{\lambda,\eta}(v)$, we get the following identity on $T_{\lambda,\eta}$

$$\bar{T}_{\bar{\lambda},\eta}[f](-v) = T_{\lambda,\eta}[f](v), \quad \forall v \in \mathbb{R}.$$

2.4 Properties of the Green functions corresponding to the leading part of the operator

2.4.1 Some complementary estimates of the the elements of the basis

Lemma 2.4.1 (Some estimates on $\psi_1^{\lambda,\eta}$ and $\psi_2^{\lambda,\eta}$).

1. *We have the following estimates:*

$$|\psi_1^{\lambda\eta}(v)| \lesssim \begin{cases} \eta^{\frac{\gamma}{3}}|\mathfrak{a}_{\lambda,\eta}(v)|, & v \geqslant s_0\eta^{-\frac{1}{3}}, \\ \psi_1(v), & |v| \leqslant s_0\eta^{-\frac{1}{3}}, \\ \eta^{-\frac{\gamma+1}{3}}|\mathfrak{a}_{\lambda,\eta}(v)|, & v \leqslant -s_0\eta^{-\frac{1}{3}}. \end{cases} \tag{2.4.1}$$

Similarly for $\psi_2^{\lambda,\eta}$,

$$|\psi_2^{\lambda\eta}(v)| \lesssim \begin{cases} \eta^{-\frac{\gamma+1}{3}}|\mathfrak{b}_{\lambda,\eta}(v)|, & v \geqslant s_0\eta^{-\frac{1}{3}}, \\ \psi_2(v), & |v| \leqslant s_0\eta^{-\frac{1}{3}}, \\ \eta^{\frac{\gamma}{3}}|\mathfrak{b}_{\lambda,\eta}(v)|, & v \leqslant -s_0\eta^{-\frac{1}{3}}. \end{cases} \tag{2.4.2}$$

2. *There is a constant $C > 0$ such that, for all $\sigma \in (0, \frac{1}{3})$, $a > 0$ and for all $v \in [-a\eta^{-\frac{1}{3}}, a\eta^{-\frac{1}{3}}]$ one has*

$$\left|\frac{\psi_i^{\lambda,\eta}(v) - \psi_i(v)}{\psi_i(v)}\right| \leqslant C(a^2|\lambda|\eta^{2\sigma} + a^3\eta^{3\sigma}), \qquad \forall i = 1,2. \tag{2.4.3}$$

Proof. 1. We are going to show the inequality (2.4.1), and that on $\psi_2^{\lambda,\eta}$ is obtained by symmetry.

• **Case 1:** $v \geqslant s_0\eta^{-\frac{1}{3}}$. We have $C_1 \leqslant \left|1+R^{\lambda,\eta}(\eta^{\frac{1}{3}}v)\right| \leqslant C_2$ since $R^{\lambda,\eta}(s) = O(|s|^{-\frac{3}{2}}|\mathfrak{a}_0(|s|)|^2)$ for $|s| \geqslant s_0$ with s_0 big enough. Then,

$$\left|\frac{\psi_1^{\lambda,\eta}(v)}{\eta^{\frac{\gamma}{3}}\mathfrak{a}_{\lambda,\eta}(v)}\right| \lesssim \eta^{-\frac{2\gamma+1}{3}}\left|\frac{\mathfrak{b}_{\lambda,\eta}(v)}{\mathfrak{a}_{\lambda,\eta}(v)}\right|\int_v^\infty \frac{\mathrm{d}w}{|\mathfrak{b}_{\lambda,\eta}(w)|^2}\eta^{\frac{2(\gamma+1)}{3}}.$$

By performing the changes of variables $v = s\eta^{-\frac{1}{3}}$ and $w = t\eta^{-\frac{1}{3}}$, and for $|\lambda| \leqslant \lambda_0$ with λ_0

small enough, we obtain

$$\begin{aligned}\left|\frac{\psi_1^{\lambda,\eta}(v)}{\eta^{\frac{\gamma}{3}}\mathfrak{a}_{\lambda,\eta}(v)}\right| &\lesssim \left|\frac{\mathfrak{b}_\lambda(s)}{\mathfrak{a}_\lambda(s)}\right|\int_s^\infty \frac{\mathrm{d}t}{|\mathfrak{b}_\lambda(t)|^2}\\ &= 4\pi e^{\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(s+\mathrm{i}\lambda)^{\frac{3}{2}}]}\int_s^\infty |t+\mathrm{i}\lambda|^{\frac{1}{2}}e^{-\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{3}{2}}]}\mathrm{d}t\\ &= e^{\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(s+\mathrm{i}\lambda)^{\frac{3}{2}}]}\int_s^\infty \mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{1}{2}}]\frac{|t+\mathrm{i}\lambda|^{\frac{1}{2}}}{\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{1}{2}}]}e^{-\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{3}{2}}]}\mathrm{d}t\\ &\lesssim e^{\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(s+\mathrm{i}\lambda)^{\frac{3}{2}}]}\int_s^\infty \mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{1}{2}}]e^{-\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{3}{2}}]}\mathrm{d}t\\ &\lesssim 1\end{aligned}$$

since for $t \geqslant s \geqslant s_0$ and $|\lambda| \leqslant \lambda_0$:

$$\frac{|t+\mathrm{i}\lambda|^{\frac{1}{2}}}{\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{1}{2}}]} \lesssim 1 \quad \text{and} \quad \int_s^\infty \mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{1}{2}}]e^{-\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(t+\mathrm{i}\lambda)^{\frac{3}{2}}]}\mathrm{d}t = \frac{3}{4}e^{-\frac{4}{3}\mathrm{Re}[e^{\mathrm{i}\frac{\pi}{4}}(s+\mathrm{i}\lambda)^{\frac{3}{2}}]}.$$

• **Case 2:** $v \leqslant -s_0\eta^{-\frac{1}{3}}$. As in the previous case, $|1 + R^{\lambda,\eta}(\eta^{\frac{1}{3}}|v|)|$ is bounded below and above and since $\psi^{\lambda,\eta}(v) = \bar{c}_{\bar{\lambda}}\eta^{-\frac{\gamma+1}{3}}\mathfrak{a}_{\lambda,\eta}(v)\big(1+\bar{R}^{\bar{\lambda},\eta}(-\eta^{\frac{1}{3}}v)\big)$ then, by (2.3.13)

$$\left|\frac{\psi_1^{\lambda,\eta}(v)}{\eta^{-\frac{\gamma+1}{3}}\mathfrak{a}_{\lambda,\eta}(v)}\right| \lesssim \int_v^\infty \frac{\mathrm{d}w}{|\psi^{\lambda,\eta}(w)|^2} \lesssim \int_v^\infty \frac{\mathrm{d}w}{|\psi(w)|^2} \lesssim \int_{\mathbb{R}} \frac{\mathrm{d}w}{|\psi(w)|^2} \lesssim 1.$$

• **Case 3:** $|v| \leqslant s_0\eta^{-\frac{1}{3}}$. We have in this case by (2.3.14), $C_1\psi(v) \leqslant |\psi^{\lambda,\eta}(v)| \leqslant C_2\psi(v)$ and thanks to (2.3.15) we get

$$\left|\frac{\psi_1^{\lambda,\eta}(v)}{\psi_1(v)}\right| = \left|\frac{\psi^{\lambda,\eta}(v)}{\psi(v)}\left(\int_v^\infty \frac{\mathrm{d}w}{|\psi^{\lambda,\eta}(w)|^2}\right)\left(\int_v^\infty \frac{\mathrm{d}w}{|\psi(w)|^2}\right)^{-1}\right| \lesssim 1.$$

2. The proof of this point is the same as for (2.3.13). □

We will end this subsection with a lemma on the estimation of the kernel $K_{\lambda,\eta}$.

Lemma 2.4.2. *Let $\sigma \in (0, \frac{1}{3})$ and let $a > 0$. The following estimate*

$$|K_{\lambda,\eta}(v,w) - K_0(v,w)| \lesssim \big(a^2|\lambda|\eta^{2\sigma} + a^3\eta^{3\sigma}\big)K_0(v,w) \tag{2.4.4}$$

holds for $|v|, |w| \leqslant a\eta^{\sigma-\frac{1}{3}}$. Therefore, for η small enough or for $\sigma = 0$ and a small enough, we have the estimate

$$|K_{\lambda,\eta}(v,w)| \lesssim K_0(v,w), \qquad \forall |v|, |w| \leqslant a\eta^{\sigma-\frac{1}{3}}. \tag{2.4.5}$$

Proof. Let $\sigma \in (0, \frac{1}{3})$ and $a > 0$. Denote $\phi_i^{\lambda,\eta} := \psi_i^{\lambda,\eta} - \psi_i$ for $i = 1, 2$. We have:

$$K_{\lambda,\eta}(v,w) - K_0(v,w) = \phi_1^{\lambda,\eta}(v)\psi_2^{\lambda,\eta}(w)\chi_{\{w<v\}} + \phi_1^{\lambda,\eta}(w)\psi_2^{\lambda,\eta}(v)\chi_{\{w>v\}} + \psi_1(v)\phi_2^{\lambda,\eta}(w)\chi_{\{w<v\}} + \psi_1(w)\phi_2^{\lambda,\eta}(v)\chi_{\{w>v\}}.$$

By (2.4.3), $|\phi_i^{\lambda,\eta}(v)| \lesssim \big(a^2|\lambda|\eta^{2\sigma} + a^3\eta^{3\sigma}\big)\psi_i(v)$ and by (2.4.1) and (2.4.2), $|\psi_i^{\lambda,\eta}(v)| \lesssim \psi_i(v)$ for all $|v| \leqslant a\eta^{\sigma-\frac{1}{3}}$ and $i = 1, 2$. Hence inequality (2.4.4) holds and for η small enough or a small enough with $\sigma \in [0, \frac{1}{3}[$ we get estimate (2.4.5). □

2.4.2 Continuity of the functional, introduction of the weighted spaces

The goal of this subsection is to prove a Proposition on the continuity of the operator $T_{\lambda,\eta}$ in a weighted functional space that we will define below. This continuity presents the key to the proof of the theorem on the existence of solutions. The proof is based on estimates of Green's function $K_{\lambda,\eta}$ with weights. We will start by defining the weights as well as the functional spaces on which we work, then we give two lemmas on which is based the proof of the proposition on continuity.

Let $\eta_0, \lambda_0 > 0$ small enough and let $\eta \in [0, \eta_0]$, $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$. We let

$$\langle v \rangle = \sqrt{1 + |v|^2}, \; v \in \mathbb{R}$$

and define the weights $p_i^{\lambda,\eta}$ for $i = 1, 2$ by:

For $\eta > 0$:

$$p_2^{\lambda,\eta}(v) := \begin{cases} \eta^{\frac{\gamma}{3}}|\mathfrak{a}_{\lambda,\eta}(v)|, & v \geqslant s_0\eta^{-\frac{1}{3}}, \\ \langle v \rangle^{-\gamma}, & |v| \leqslant s_0\eta^{-\frac{1}{3}}, \\ \eta^{\frac{\gamma}{3}}|\mathfrak{b}_{\lambda,\eta}(v)|, & v \leqslant -s_0\eta^{-\frac{1}{3}}. \end{cases} \tag{2.4.6}$$

For $\eta = 0$:

$$p_2^0(v) := \langle v \rangle^{-\gamma}, \quad \forall v \in \mathbb{R} \tag{2.4.7}$$

and

$$p_1^{\lambda,\eta}(v) := \frac{p_2^{\lambda,\eta}(v)}{\langle v \rangle^{2+\delta}}, \quad \forall v \in \mathbb{R}, \forall \eta \in [0, \eta_0],$$

for any $\delta \in (0, 2)$.

Note that $p_2^{\lambda,\eta}$ belongs to $L^2(\mathbb{R})$ since $2\gamma > 1$. We define the Banach space E_i^η, as the completion of $C_c^\infty(\mathbb{R}, \mathbb{C})$ for the norm: $\|\phi\|_{E_i^\eta} := \left\| \dfrac{\phi}{p_i^{\lambda,\eta}} \right\|_{L^\infty}$

$$E_i^\eta := \overline{\{\phi \in C_c^\infty(\mathbb{R}, \mathbb{C}) / \; \|\phi\|_{E_i^\eta} < +\infty\}}.$$

We have the embeddings
$$\|\cdot\|_{E_i^{\eta^*}} \leqslant \|\cdot\|_{E_i^{\eta}} \text{ for } \eta \leqslant \eta^*.$$

Lemma 2.4.3. *Let $\delta \in (0,2)$. Then, there exists a constant $C > 0$ such that:*
$$\int_{\mathbb{R}} |K_0(v,w)|\langle w\rangle^{-\gamma-\delta-2} \frac{\mathrm{d}w}{\langle v\rangle^{-\gamma}} \leqslant C, \quad \forall v \in \mathbb{R}. \tag{2.4.8}$$
thus T_0 in $\mathcal{L}(E_1^0, E_2^0)$ is continuous.

Proof. Thanks to the second point of the Remark 2.3.7, since the weights are symmetric, we establish the inequality (2.4.8) on $\mathbb{R}^+$. Let $\delta \in (0,2)$ and let $v_0 > 0$ big enough.

Step 1: $v \geqslant v_0$. We have
$$\int_{\mathbb{R}} |K_0(v,w)|\langle w\rangle^{-\gamma-\delta-2} \frac{\mathrm{d}w}{\langle v\rangle^{-\gamma}} \leqslant \langle v\rangle^{\gamma}\Bigg[\psi_1(v)\bigg(\int_{-\infty}^{-v_0} \frac{\psi_2(w)\mathrm{d}w}{|w|^{\gamma+\delta+2}} + \int_{-v_0}^{v_0} \frac{\psi_2(w)\mathrm{d}w}{\langle w\rangle^{\gamma+\delta+2}} + \int_{v_0}^{v} \frac{\psi_2(w)\mathrm{d}w}{w^{\gamma+\delta+2}}\bigg) + \psi_2(v)\int_v^{\infty} \frac{\psi_1(w)\mathrm{d}w}{\langle w\rangle^{\gamma+\delta+2}}\Bigg].$$
For $v \geqslant v_0$: $\psi_1(v) \lesssim v^{-\gamma}$ and $\psi_2(v) \lesssim v^{\gamma+1}$, on $[-v_0, v_0]$: ψ_1 and ψ_2 are bounded by a positive constant, and for $v \leqslant -v_0$ we have: $\psi_1(v) \lesssim |v|^{\gamma+1}$ and $\psi_2(v) \lesssim |v|^{-\gamma}$ since $\psi_1(-v) = \psi_2(v)$. Then,
$$\int_{\mathbb{R}} |K_0(v,w)|\langle w\rangle^{-\gamma-\delta-2} \frac{\mathrm{d}w}{\langle v\rangle^{-\gamma}} \lesssim \int_{-\infty}^{-v_0} \frac{\mathrm{d}w}{|w|^{2\gamma+\delta+2}} + 1 + \int_{v_0}^{v} \frac{\mathrm{d}w}{|w|^{\delta+1}} + v^{2\gamma+1}\int_v^{\infty} \frac{\mathrm{d}w}{w^{2\gamma+\delta+2}} \lesssim 1 + v^{-\delta} \lesssim 1.$$

Step 2: $v \in [0, v_0]$. In this case, we cut the integral as follows:
$$\int_{\mathbb{R}} |K_0(v,w)|\langle w\rangle^{-\gamma-\delta-2} \frac{\mathrm{d}w}{\langle v\rangle^{-\gamma}} \leqslant \langle v\rangle^{\gamma}\Bigg[\psi_1(v)\bigg(\int_{-\infty}^{-v_0} \frac{\psi_2(w)\mathrm{d}w}{|w|^{\gamma+\delta+2}} + \int_{-v_0}^{v} \frac{\psi_2(w)\mathrm{d}w}{\langle w\rangle^{\gamma+\delta+2}}\bigg) + \psi_2(v)\bigg(\int_v^{v_0} \frac{\psi_1(w)\mathrm{d}w}{\langle w\rangle^{\gamma+\delta+2}} + \int_{v_0}^{\infty} \frac{\psi_1(w)\mathrm{d}w}{w^{\gamma+\delta+2}}\bigg)\Bigg],$$
and as in the previous step: since $v \in [0, v_0]$ then, $\langle v\rangle^{\gamma}$, ψ_1 and ψ_2 are bounded by a positive constant. Also, we have for $w \leqslant -v_0$: $\psi_2(w) \lesssim |w|^{-\gamma}$, and for $w \geqslant v_0$: $\psi_1(w) \lesssim w^{-\gamma}$. Therefore,
$$\int_{\mathbb{R}} |K_0(v,w)|\langle w\rangle^{-\gamma-\delta-2} \frac{\mathrm{d}w}{\langle v\rangle^{-\gamma}} \lesssim \int_{-\infty}^{-v_0} \frac{\mathrm{d}w}{|w|^{2\gamma+\delta+2}} + 1 + \int_{v_0}^{\infty} \frac{\mathrm{d}w}{w^{2\gamma+\delta+2}} \lesssim 1.$$
□

Lemma 2.4.4. *Let $\eta_0, \lambda_0 > 0$ small enough. There exists a constant $C > 0$ independent of v, η and λ such that*

$$\int_{\mathbb{R}} |K_{\lambda,\eta}(v,w)| p_1^{\lambda,\eta}(w)\mathrm{d}w \leqslant C p_2^{\lambda,\eta}(v) \tag{2.4.9}$$

holds for all $v \in \mathbb{R}$, $\eta \in [0,\eta_0]$ and $|\lambda| \leqslant \lambda_0$.

Proof. The case $\eta = 0$ is treated in the previous Lemma so, let $\eta \neq 0$. We will proceed as in the previous Lemma and since $\bar{K}_{\bar{\lambda},\eta}(-v,-w) = K_{\lambda,\eta}(v,w)$ and $\bar{p}_i^{\bar{\lambda},\eta}(-v) = p_i^{\lambda,\eta}(v)$ for $i = 1, 2$ then, we show the inequality for $v \in \mathbb{R}^+$. Let denote by

$$E^{\lambda,\eta}(v) := \int_{\mathbb{R}} |K_{\lambda,\eta}(v,w)| \frac{p_1^{\lambda,\eta}(w)}{p_2^{\lambda,\eta}(v)} \mathrm{d}w.$$

Step 1: $v \geqslant s_0\eta^{-\frac{1}{3}}$. We will cut the integral into four parts, according to the behaviors of the $\psi_i^{\lambda,\eta}$ and the definition of the weight, as follows:

$$\begin{aligned} E^{\lambda,\eta}(v) \leqslant \frac{|\psi_1^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)} \Bigg[& \int_{-\infty}^{-s_0\eta^{-\frac{1}{3}}} |\psi_2^{\lambda,\eta}| p_1^{\lambda,\eta}(w)\mathrm{d}w + \int_{-s_0\eta^{-\frac{1}{3}}}^{s_0\eta^{-\frac{1}{3}}} |\psi_2^{\lambda,\eta}| p_1^{\lambda,\eta}(w)\mathrm{d}w \\ & + \int_{s_0\eta^{-\frac{1}{3}}}^{v} |\psi_2^{\lambda,\eta}| p_1^{\lambda,\eta}(w)\mathrm{d}w \Bigg] + \frac{|\psi_2^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)} \int_v^{\infty} |\psi_1^{\lambda,\eta}| p_1^{\lambda,\eta}(w)\mathrm{d}w. \end{aligned}$$

Thanks to the inequalities (2.4.1) and (2.4.2) of Lemma 2.4.1, we have for $v \geqslant s_0\eta^{-\frac{1}{3}}$,

$$\frac{|\psi_1^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)} \lesssim 1, \; |\psi_2^{\lambda,\eta}(v)| \lesssim \eta^{-\frac{\gamma+1}{3}} |\mathfrak{b}_{\lambda,\eta}(v)| \; \text{ and } \; \frac{|\psi_2^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)} \lesssim \eta^{-\frac{2\gamma+1}{3}} \frac{|\mathfrak{b}_{\lambda,\eta}(v)|}{|\mathfrak{a}_{\lambda,\eta}(v)|}.$$

For $|w| \leqslant s_0\eta^{-\frac{1}{3}}$, $|\psi_2^{\lambda,\eta}(w)| \lesssim \psi_2(w) \lesssim \langle w \rangle^{\gamma+1}$ and finally for $w \leqslant -s_0\eta^{-\frac{1}{3}}$,

$$|\psi_2^{\lambda,\eta}(w)| \lesssim p_2^{\lambda,\eta}(w),$$

with $p_2^{\lambda,\eta}(w) = \eta^{\frac{\gamma}{3}} |\mathfrak{a}_{\lambda,\eta}(w)|$ for this range of velocity. Therefore,

$$\begin{aligned} E^{\lambda,\eta}(v) &\lesssim \int_{-\infty}^{-s_0\eta^{-\frac{1}{3}}} \eta^{\frac{2\gamma}{3}} |\mathfrak{b}_{\lambda,\eta}(w)|^2 \frac{\mathrm{d}w}{\langle w \rangle^{2+\delta}} + \int_{-s_0\eta^{-\frac{1}{3}}}^{s_0\eta^{-\frac{1}{3}}} \frac{\mathrm{d}w}{\langle w \rangle^{1+\delta}} \\ &+ \int_{s_0\eta^{-\frac{1}{3}}}^{v} \eta^{-\frac{1}{3}} \frac{|\mathfrak{a}_{\lambda,\eta}(w)\mathfrak{b}_{\lambda,\eta}(w)|}{\langle w \rangle^{2+\delta}} \mathrm{d}w + \eta^{-\frac{1}{3}} \left| \frac{\mathfrak{b}_{\lambda,\eta}(v)}{\mathfrak{a}_{\lambda,\eta}(v)} \right| \int_v^{\infty} \frac{|\mathfrak{a}_{\lambda,\eta}(w)|^2}{\langle w \rangle^{2+\delta}} \mathrm{d}w \\ &= \sum_{i=1}^{4} I_i^{\lambda,\eta}. \end{aligned}$$

In order to estimate $I_1^{\lambda,\eta}$, $I_3^{\lambda,\eta}$ and $I_4^{\lambda,\eta}$, we will perform the changes of variables $w = \eta^{-\frac{1}{3}} t$ and $v = \eta^{-\frac{1}{3}} s$.

• **Estimation of** $I_1^{\lambda,\eta}$:

$$I_1^{\lambda,\eta} := \eta^{\frac{2\gamma}{3}} \int_{-\infty}^{-s_0\eta^{-\frac13}} \frac{|\mathfrak{b}_{\lambda,\eta}(w)|^2}{\langle w\rangle^{2+\delta}}\mathrm{d}w = \eta^{\frac{2\gamma-1}{3}} \int_{-\infty}^{-s_0} \frac{|\mathfrak{b}_\lambda(t)|^2}{\langle \eta^{-\frac13}t\rangle^{2+\delta}}\mathrm{d}t \lesssim \eta^{\frac{2\gamma+\delta+1}{3}} \int_{-\infty}^{-s_0} \frac{|\mathfrak{b}_\lambda(t)|^2}{|t|^{2+\delta}}\mathrm{d}t \lesssim 1.$$

• **Estimation of** $I_2^{\lambda,\eta}$:

$$I_2^{\lambda,\eta} := \int_{-s_0\eta^{-\frac13}}^{s_0\eta^{-\frac13}} \frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}} \leqslant \int_{\mathbb{R}} \frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}} \lesssim 1.$$

• **Estimation of** $I_3^{\lambda,\eta}(v)$: since $|\mathfrak{a}_\lambda(t)\mathfrak{b}_\lambda(t)| \lesssim t^{-\frac12}$ for $|\lambda| \leqslant \lambda_0$ and $t \geqslant s_0$, then

$$I_3^{\lambda,\eta}(v) := \eta^{-\frac13} \int_{s_0\eta^{-\frac13}}^{v} \frac{|\mathfrak{a}_{\lambda,\eta}(w)\mathfrak{b}_{\lambda,\eta}(w)|}{\langle w\rangle^{2+\delta}}\mathrm{d}w = \eta^{-\frac23} \int_{s_0}^{s} \frac{|\mathfrak{a}_\lambda(t)\mathfrak{b}_\lambda(t)|}{\langle \eta^{-\frac13}t\rangle^{2+\delta}}\mathrm{d}t \lesssim \eta^{\frac{\delta}{3}} \int_{s_0}^{s} \frac{\mathrm{d}t}{t^{\frac52+\delta}} \lesssim 1.$$

• **Estimation of** $I_4^{\lambda,\eta}(v)$: we have

$$I_4^{\lambda,\eta}(v) := \eta^{-\frac13}\left|\frac{\mathfrak{b}_{\lambda,\eta}(v)}{\mathfrak{a}_{\lambda,\eta}(v)}\right| \int_{v}^{\infty} \frac{|\mathfrak{a}_{\lambda,\eta}(w)|^2}{\langle w\rangle^{2+\delta}}\mathrm{d}w = \eta^{-\frac23}\left|\frac{\mathfrak{b}_\lambda(s)}{\mathfrak{a}_\lambda(s)}\right| \int_{s}^{\infty} \frac{|\mathfrak{a}_\lambda(t)|^2}{\langle \eta^{-\frac13}t\rangle^{2+\delta}}\mathrm{d}s \lesssim \frac{\eta^{\frac{\delta}{3}}}{s^{\frac12}} \int_{s_0}^{s} \frac{\mathrm{d}t}{t^{2+\delta}} \lesssim 1$$

since $\frac{|\mathfrak{a}_\lambda(t)|}{|\mathfrak{a}_\lambda(s)|} \leqslant 1$ and $|\mathfrak{a}_\lambda(s)\mathfrak{b}_\lambda(s)| \lesssim t^{-\frac12}$ for $t \geqslant s \geqslant s_0$ and $|\lambda| \leqslant \lambda_0$ (by the inequalities of (2.3.10)).

Step 2: $v \in [0, s_0\eta^{-\frac13}]$. We will proceed exactly as in the previous step by cutting the integral this time as follows

$$\begin{aligned} E^{\lambda,\eta}(v) \leqslant \frac{|\psi_1^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)}\Bigg[\int_{-\infty}^{-s_0\eta^{-\frac13}} |\psi_2^{\lambda,\eta}|p_1^{\lambda,\eta}(w)\mathrm{d}w + \int_{-s_0\eta^{-\frac13}}^{v} |\psi_2^{\lambda,\eta}|p_1^{\lambda,\eta}(w)\mathrm{d}w \\ + \int_{v}^{s_0\eta^{-\frac13}} |\psi_2^{\lambda,\eta}|p_1^{\lambda,\eta}(w)\mathrm{d}w\Bigg] + \frac{|\psi_2^{\lambda,\eta}(v)|}{p_2^{\lambda,\eta}(v)} \int_{s_0\eta^{-\frac13}}^{\infty} |\psi_1^{\lambda,\eta}|p_1^{\lambda,\eta}(w)\mathrm{d}w. \end{aligned}$$

We have in this case: $p_2^{\lambda,\eta}(v) = \langle v\rangle^{-\gamma}$ and $|\psi_i^{\lambda,\eta}(v)| \lesssim \psi_i(v)$ for $i = 1,2$ by the inequalities (2.4.1) and (2.4.2), with $\psi_1(v) \lesssim p_2^{\lambda,\eta}(v)$ and $\psi_2(v) \lesssim \langle v\rangle^{\gamma+1}$ by (2.2.6) and (2.2.7). Then,

$$\begin{aligned} E^{\lambda,\eta}(v) \lesssim \int_{-\infty}^{-s_0\eta^{-\frac13}} \eta^{\frac{2\gamma}{3}}|\mathfrak{b}_{\lambda,\eta}(w)|^2\frac{\mathrm{d}w}{\langle w\rangle^{2+\delta}} + \int_{-s_0\eta^{-\frac13}}^{v} \frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}} \\ + \langle v\rangle^{2\gamma+1} \int_{v}^{s_0\eta^{-\frac13}} \frac{\mathrm{d}w}{\langle w\rangle^{2\gamma+\delta+2}} + \eta^{\frac{\gamma}{3}}\langle v\rangle^{2\gamma+1} \int_{s_0\eta^{-\frac13}}^{\infty} \frac{|\mathfrak{a}_{\lambda,\eta}(w)|}{\langle w\rangle^{\gamma+\delta+2}}\mathrm{d}w. \end{aligned}$$

The first two integrals are bounded by $I_1^{\lambda,\eta} + I_2^{\lambda,\eta}$ which is uniformly bounded with respect to v by step 1. For the last two terms, we write

$$\langle v\rangle^{2\gamma+1} \int_{v}^{s_0\eta^{-\frac13}} \frac{\mathrm{d}w}{\langle w\rangle^{2\gamma+\delta+2}} = \int_{v}^{s_0\eta^{-\frac13}} \frac{\langle v\rangle^{2\gamma+1}}{\langle w\rangle^{2\gamma+1}}\frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}} \leqslant \int_{\mathbb{R}} \frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}} \lesssim 1$$

and

$$\eta^{\frac{\gamma}{3}}\langle v\rangle^{2\gamma+1}\int_{s_0\eta^{-\frac{1}{3}}}^{\infty}\frac{|\mathfrak{a}_{\lambda,\eta}(w)|}{\langle w\rangle^{\gamma+\delta+2}}\mathrm{d}w=\eta^{\frac{\gamma}{3}}\langle v\rangle^{\gamma}\int_{s_0\eta^{-\frac{1}{3}}}^{\infty}\frac{\langle v\rangle^{\gamma+1}}{\langle w\rangle^{\gamma+1}}\frac{|\mathfrak{a}_{\lambda,\eta}(w)|}{\langle w\rangle^{1+\delta}}\mathrm{d}w\lesssim\int_{\mathbb{R}}\frac{\mathrm{d}w}{\langle w\rangle^{1+\delta}}\lesssim 1$$

since $v\leqslant s_0\eta^{-\frac{1}{3}}$ and $|\mathfrak{a}_{\lambda,\eta}(w)|\leqslant 1$ for $w\geqslant s_0\eta^{-\frac{1}{3}}$. □

Proposition 2.4.5 (Continuity of $T_{\lambda,\eta}$)**.** *Let $\eta_0,\lambda_0>0$ small enough. Let $\eta\in[0,\eta_0]$ and $\lambda\in\mathbb{C}$ such that $|\lambda|\leqslant\lambda_0$. Then, the map $T_{\lambda,\eta}:E_1^\eta\longrightarrow E_2^\eta$ is linear continous. Moreover, there exists $C>0$ independent of η and λ such that*

$$\|T_{\lambda,\eta}(g)\|_{E_2^\eta}\leqslant C\|g\|_{E_1^\eta},\quad\forall g\in E_1^\eta. \tag{2.4.10}$$

Proof. The proof of this Proposition is a direct consequence of Lemma 2.4.4. □

2.5 Existence of the eigen-solution

2.5.1 Existence of solutions for the penalized equation

In this subsection, we use the "right inverse" operator $T_{\lambda,\eta}$ to rewrite once again the penalized equation (2.1.15) as a fixed point problem for the identity plus a compact map. Then, the Fredholm Alternative will allow us to apply the Implicit Function Theorem in order to get existence of solutions for this new problem, thus solutions for the equation (2.1.15).

Define $F:\{\lambda\in\mathbb{C};|\lambda|\leqslant\lambda_0\}\times[0,\eta_0]\times C_b(\mathbb{R},\mathbb{C})\longrightarrow C_b(\mathbb{R},\mathbb{C})$ by

$$F(\lambda,\eta,h):=h-\mathcal{T}_{\lambda,\eta}(h),$$

with

$$\mathcal{T}_{\lambda,\eta}(h):=\frac{1}{p_2^{\lambda,\eta}}T_{\lambda,\eta}\big[Vp_2^{\lambda,\eta}h-\langle p_2^{\lambda,\eta}h-M,\Phi\rangle\Phi\big].$$

Note that finding a solution $h(\lambda,\eta)$ solution to $F\big(\lambda,\eta,h(\lambda,\eta)\big)=0$ gives a solution to the penalized equation by taking $M_{\lambda,\eta}=h(\lambda,\eta)p_2^{\lambda,\eta}$.

The function Φ is a function satisfying the following assumptions:

1. For all v in $\mathbb{R}$, $\Phi(v)=\Phi(-v)>0$.
2. For all $\varepsilon>0$, there exists $g_1^\varepsilon\in C_c(\mathbb{R})$ such that $\|\Phi/p_1^{\lambda,\eta}-g_1^\varepsilon\|_\infty<\varepsilon$ with $\mathrm{supp}(g_1^\varepsilon)$ independent of λ and η.
3. Even if it means multiplying Φ by a constant, we can take it such that $\langle\Phi,M\rangle=1$.

Any continuous function with compact support and satisfying 1. and 3. is suitable. The function $\Phi := \Phi_{\lambda,\eta} = c_{\lambda,\eta}\langle v\rangle^{-\sigma} p_1^{\lambda,\eta}$ satisfies all the previous assumptions, where $\sigma > 0$ and $c_{\lambda,\eta}$ is a constant of "normalization" such that $\langle \Phi_{\lambda,\eta}, M\rangle = 1$.

Remark 2.5.1.

1. For the following, we fix a continuous function Φ, with compact support included in $[-R, R]$ with $R > 0$, and satisfying assumptions 1. and 3. above.

2. Note that $\mathcal{T}_{\lambda,0}$ does not depend on λ since $T_{\lambda,0}$ does not. Let's denote it by $\mathcal{T}_0$:
$$\mathcal{T}_0(h) := \mathcal{T}_{\lambda,0}(h) = \frac{1}{p_2^0} T_0\big[V p_2^0 h - \langle p_2^0 h - M, \Phi\rangle \Phi\big]$$

3. The map $\mathcal{T}_{\lambda,\eta}$ is affine with respect to h. We denote by $\mathcal{T}^l_{\lambda,\eta}$ its linear part:
$$\mathcal{T}^l_{\lambda,\eta}(h) := \frac{1}{p_2^{\lambda,\eta}} T_{\lambda,\eta}\big[V p_2^{\lambda,\eta} h - \langle p_2^{\lambda,\eta} h, \Phi\rangle \Phi\big].$$

Lemma 2.5.2 (Continuity, differentiability and compactness of $\mathcal{T}_{\lambda,\eta}$)**.**
Let $\eta_0, \lambda_0 > 0$ small enough. Let $\eta \in [0, \eta_0]$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$. Then, the map $\mathcal{T}_{\lambda,\eta} : C_b(\mathbb{R}, \mathbb{C}) \longrightarrow C_b(\mathbb{R}, \mathbb{C})$ is

1. *continuous with respect to λ and η, moreover*
$$\|\mathcal{T}_{\lambda,\eta} - \mathcal{T}_{\lambda,\eta'}\|_{\mathcal{L}(C_b)} \underset{\eta\to\eta'}{\longrightarrow} 0, \quad \forall \eta' \in [0, \eta_0] \tag{2.5.1}$$
and
$$\|\mathcal{T}_{\lambda,\eta} - \mathcal{T}_{\lambda',\eta}\|_{\mathcal{L}(C_b)} \underset{\lambda\to\lambda'}{\longrightarrow} 0, \quad \forall \lambda' \in \mathbb{C}; |\lambda'| \leqslant \lambda_0. \tag{2.5.2}$$

2. *differentiable in $C_b(\mathbb{R}, \mathbb{C})$ and its differential is*
$$\frac{\partial \mathcal{T}_{\lambda,\eta}}{\partial h} = \mathcal{T}^l_{\lambda,\eta} = \frac{1}{p_2^{\lambda,\eta}} T_{\lambda,\eta}\big[V p_2^{\lambda,\eta} \cdot - \langle p_2^{\lambda,\eta} \cdot, \Phi\rangle \Phi\big]. \tag{2.5.3}$$

3. *The map $\mathcal{T}^l_0$ is compact.*

where $\mathcal{L}(C_b)$ is the space of linear continuous operators from $C_b(\mathbb{R}, \mathbb{C})$ to itself.

Proof of Lemma 2.5.2.
1. Continuity of $\mathcal{T}_{\lambda,\eta}$ with respect to λ and η. It is sufficient to prove this continuity for the map $\mathcal{T}_{\lambda,\eta}$ composed with the characteristic function $\chi_{[-R,R]}$, since $\mathcal{T}^l_{\lambda,\eta}$ can be written as
$$\mathcal{T}^l_{\lambda,\eta}(h) = \frac{1}{p_2^{\lambda,\eta}} T_{\lambda,\eta}\big[g_1 p_1^{\lambda,\eta} h - \langle p_2^{\lambda,\eta} h, \Phi\rangle g_2 p_1^{\lambda,\eta}\big],$$

with $g_1 := V p_2^{\lambda,\eta}/p_1^{\lambda,\eta}$ and $g_2 := \Phi/p_1^{\lambda,\eta}$ belong to $C_0(\mathbb{R})$, the set of continuous functions converging to zero at the infinity. Indeed, let us denote by $\mathcal{T}_{\lambda,\eta}^R := \mathcal{T}_{\lambda,\eta}^l \circ \chi_{[-R,R]}$. Let $h \in C_b(\mathbb{R},\mathbb{C})$ and let $\varepsilon > 0$. Then, there exists $g_1^\varepsilon, g_2^\varepsilon \in C_c^\infty(\mathbb{R})$ such that $\|g_i^\varepsilon - g_i\|_\infty \leqslant \varepsilon/(2C)$ for $i = 1, 2$, where C is the constant of Lemma 2.4.4. Let $R_\varepsilon > 0$ such that $\mathrm{supp}(g_1^\varepsilon) \cup \mathrm{supp}(g_2^\varepsilon) \subset [-R_\varepsilon, R_\varepsilon]$. We have thanks to the Lemma 2.4.4 and since $p_2^{\lambda,\eta}(v) \leqslant p_2^0(v)$ with $\langle p_2^0, \Phi\rangle = 1$:

$$\begin{aligned}\|\mathcal{T}_{\lambda,\eta}^l(h) - \mathcal{T}_{\lambda,\eta}^{R_\varepsilon}(h)\|_\infty &\leqslant C\big[\|g_1^\varepsilon - g_1\|_\infty \|h\|_\infty + \|g_2^\varepsilon - g_2\|_\infty |\langle p_2^{\lambda,\eta} h, \Phi\rangle|\big] \\ &\leqslant C\big[\|g_1^\varepsilon - g_1\|_\infty + \|g_2^\varepsilon - g_2\|_\infty |\langle p_2^0, \Phi\rangle|\big] \|h\|_\infty.\end{aligned}$$

Hence,

$$\|\mathcal{T}_{\lambda,\eta}^l(h) - \mathcal{T}_{\lambda,\eta}^{R_\varepsilon}(h)\|_\infty \leqslant \varepsilon \|h\|_\infty, \quad \forall h \in C_b(\mathbb{R},\mathbb{C}). \tag{2.5.4}$$

Remark 2.5.3.

1. Note that $\mathrm{supp}(g_1^\varepsilon)$ does not depend on h, λ and η, since g_1 does not depend on the latter three: $g_1 = \langle v\rangle^{2+\delta} V = \frac{\gamma(\gamma+2)}{\langle v\rangle^{2-\delta}} \in C_0(\mathbb{R},\mathbb{C})$ for $\delta \in (0,2)$. Similarly for g_2, by assumption. Moreover, since Φ has compact support then, for s_0 large enough and η small enough, $\mathrm{supp}(\Phi) \subset [-s_0\eta^{-\frac{1}{3}}, s_0\eta^{-\frac{1}{3}}]$ with $p_1^{\lambda,\eta}(v) = p_1^0(v)$ on this last interval. Therefore, R_ε is independent of h, λ and η.

2. Since Φ has compact support then, $|\langle p_2^{\lambda,\eta}, \Phi\rangle| = |\langle p_2^0, \Phi\rangle| = 1$ for η small enough. Therefore,
$$\big\|\mathcal{T}_{\lambda,\eta}(\langle p_2^{\lambda,\eta} h, \Phi\rangle \Phi)\big\|_\infty \lesssim \|h\|_\infty.$$

Let us now show the continuity of $\mathcal{T}_{\lambda,\eta}^R$. Let $|v| \leqslant R$ and let η_0 small enough such that $\eta_0^{-\frac{1}{3}} > R$. Then, $p_2^{\lambda,\eta}(v) = p_2^0(v)$ for all $\eta \in [0,\eta_0]$ and for all $|v| \leqslant R$, and we have:

$$\mathcal{T}_{\lambda,\eta}^R(h)(v) - \mathcal{T}_{\lambda,\eta'}^R(h)(v) = \int_{-R}^R \big[K_{\lambda,\eta}(v,w) - K_{\lambda,\eta'}(v,w)\big]\big[g_1 h(w) - \langle p_2^0 h, \Phi\rangle g_2(w)\big] \frac{p_1^0(w)}{p_2^0(v)} \mathrm{d}w.$$

Therefore,

$$|\mathcal{T}_{\lambda,\eta}^R(h)(v) - \mathcal{T}_{\lambda,\eta'}^R(h)(v)| \leqslant \int_{-R}^R \big|K_{\lambda,\eta}(v,w) - K_{\lambda,\eta'}(v,w)\big| \frac{p_1^0(w)}{p_2^0(v)} \mathrm{d}w \big(\|g_1\|_\infty + \|g_2\|_\infty\big)\|h\|_\infty.$$

Similarly for λ,

$$|\mathcal{T}_{\lambda,\eta}^R(h)(v) - \mathcal{T}_{\lambda',\eta}^R(h)(v)| \leqslant \int_{-R}^R \big|K_{\lambda,\eta}(v,w) - K_{\lambda',\eta}(v,w)\big| \frac{p_1^0(w)}{p_2^0(v)} \mathrm{d}w \big(\|g_1\|_\infty + \|g_2\|_\infty\big)\|h\|_\infty.$$

We conclude with Lebesgue's theorem thanks to the continuity of $\|K_{\lambda,\eta}\|_{L^\infty([-R,R]\times[-R,R])}$ with respect to λ and η, and since $|K_{\lambda,\eta}|$ is dominated by K_0 on $[-R,R]\times[-R,R]$ thanks to (2.4.5). Hence the limits (2.5.1) and (2.5.2) hold.
2. The second point is immediate since $\mathcal{T}_{\lambda,\eta}$ is an affine map with respect to h.

3. Let us first show that $\mathcal{T}_0^R$ is compact by Ascoli-Arzéla theorem. Define $B_R(0,1) := \{h \in C([-R,R],\mathbb{C}); \|h\|_\infty \leqslant 1\}$ and introduce $\mathcal{F} := \mathcal{T}_0^R(B_R(0,1))$. The set $\mathcal{F}$ is bounded because $\mathcal{T}_0^R$ is bounded. The set $\mathcal{F}$ is equicontinuous since for $|v_1 - v_2| \leqslant \varepsilon / C_R$ we have: $|\mathcal{T}_0^R(h)(v_1) - \mathcal{T}_0^R(h)(v_2)| \leqslant \varepsilon$, $\forall h \in B_R(0,1)$, where and C_R is a constant that depends only on R. Indeed,

$$\begin{aligned}
\mathcal{T}_0^R(h)(v_1) - \mathcal{T}_0^R(h)(v_2) &= \int_{|w|\leqslant R} \left(\frac{K_0(v_1,w)}{p_2^0(v_1)} - \frac{K_0(v_2,w)}{p_2^0(v_2)} \right) \big[g_1 h - \langle p_2^0 h, \Phi\rangle g_2\big](w) p_1^0(w) \mathrm{d}w \\
&= \int_{|w|\leqslant R} \frac{p_2^0(v_2) - p_2^0(v_1)}{p_2^0(v_2)} K_0(v_1,w) \frac{p_1^0(w)}{p_2^0(v_1)} \big[g_1 h - \langle p_2^0 h, \Phi\rangle g_2\big](w) \mathrm{d}w \\
&+ \int_{|w|\leqslant R} \big[K_0(v_1,w) - K_0(v_2,w)\big] \frac{p_1^0(w)}{p_2^0(v_2)} \big[g_1 h - \langle p_2^0 h, \Phi\rangle g_2\big](w) \mathrm{d}w \\
&=: I_1 + I_2.
\end{aligned}$$

Since p_2^0 is Lipschitz on the compact $[R,R]$ then,

$$|I_1| \leqslant \tilde{C}_R |v_1 - v_2| \big(\|g_1\|_\infty + \|g_2\|_\infty\big) \|h\|_\infty \leqslant C'_R |v_1 - v_2|.$$

For I_2 we write:

$$\begin{aligned}
I_2 = &\int_{|w|\leqslant R} \Big[[\psi_1(v_1) - \psi_1(v_2)] \psi_2(w) \chi_{w<v_1} + \psi_1(w) [\psi_2(v_1) - \psi_2(v_2)] \chi_{w>v_1} \Big] \frac{p_1^0(w)}{p_2^0(v_2)} \\
&\times \big[g_1 h - \langle p_2^0 h, \Phi\rangle g_2\big](w) \mathrm{d}w + \int_{v_1}^{v_2} K_0(v_2,w) \frac{p_1^0(w)}{p_2^0(v_1)} \big[g_1 h - \langle p_2^0 h, \Phi\rangle g_2\big](w) \mathrm{d}w.
\end{aligned}$$

Thus, since ψ_1 and ψ_2 are Lipschitz on the compact $[R,R]$ then, $|I_2| \leqslant C''_R |v_1 - v_2|$. Hence,

$$\big|\mathcal{T}_0^R(h)(v_1) - \mathcal{T}_0^R(h)(v_2)\big| \leqslant C_R |v_1 - v_2|.$$

The compactness of $\mathcal{T}_0^l$ follows from the compactness of $\mathcal{T}_0^R$ and the inequality (2.5.4). Indeed, we have by (2.5.4), $\|\mathcal{T}_0^l(h) - \mathcal{T}_0^{R_\varepsilon}(h)\|_\infty \leqslant \varepsilon$, $\forall h \in C_b(\mathbb{R},\mathbb{C})$; $\|h\|_\infty \leqslant 1$, and since $\mathcal{T}_0^{R_\varepsilon}$ is compact with R_ε being fixed and independent of h, λ and η then, there exists $N_\varepsilon \in \mathbb{N}$, $\{h_i\}_{i=1}^{N_\varepsilon} \subset C_b(\mathbb{R},\mathbb{C})$ such that: $\mathcal{T}_0^{R_\varepsilon}(h) \in \bigcup_{i=1}^{N_\varepsilon} B(h_i,\varepsilon)$. Therefore, $\mathcal{T}_0^l(h) \in \bigcup_{i=1}^{N_\varepsilon} B(h_i, 2\varepsilon)$. Hence the compactness of $\mathcal{T}_0^l$ holds. □

Proposition 2.5.4 (Assumptions of the Implicit Function Theorem)**.**

1. *The map $F(\lambda,\eta,\cdot) = Id - \mathcal{T}_{\lambda,\eta}$ is continuous in $C_b(\mathbb{R},\mathbb{C})$ uniformly with respect to λ and η. Moreover, there exists $c > 0$, independent of λ and η such that*

$$\|F(\lambda,\eta,h_1) - F(\lambda,\eta,h_2)\|_\infty \leqslant c\|h_1 - h_2\|_\infty, \quad \forall h_1, h_2 \in C_b(\mathbb{R},\mathbb{C}), \forall \eta, \forall |\lambda| \leqslant \lambda_0.$$

2. *F is continuous with respect to λ and η and we have:*

$$\|F(\lambda,\eta,\cdot)-F(\lambda,\eta',\cdot)\|_{\mathcal{L}(C_b)} \underset{\eta\to\eta'}{\longrightarrow} 0 \quad \textit{and} \quad \|F(\lambda,\eta,\cdot)-F(\lambda',\eta,\cdot)\|_{\mathcal{L}(C_b)} \underset{\lambda\to\lambda'}{\longrightarrow} 0.$$

3. *$F(\lambda,\eta,\cdot)$ is differentiable in $C_b(\mathbb{R},\mathbb{C})$, moreover:*

$$\frac{\partial F}{\partial h}(\lambda,\eta,\cdot) = Id - \mathcal{T}^l_{\lambda,\eta}, \quad \forall|\lambda|\leqslant\lambda_0, \forall\eta\in[0,\eta_0].$$

4. *We have: $F(0,0,\frac{M}{p_2^0})=0$ and $\frac{\partial F}{\partial h}(0,0,\frac{M}{p_2^0})$ is invertible.*

Proof. 1. Let $h_1,h_2\in C_b$ and let $\eta>0$ and $\lambda\in\mathbb{C}$ such that $|\lambda|\leqslant\lambda_0$. Then,

$$\begin{aligned}\|F(\lambda,\eta,h_1)-F(\lambda,\eta,h_2)\|_\infty &\leqslant \|(h_1-h_2)+\mathcal{T}^l_{\lambda,\eta}(h_1-h_2)\|_\infty\\ &\leqslant \big(1+C[\|g_1\|_\infty+\|g_2\|_\infty]\big)\|h_1-h_2\|_\infty\\ &\leqslant c\|h_1-h_2\|_\infty.\end{aligned}$$

2. The proof of this point is a direct consequence of the first point of Lemma 2.5.2.
3. Follows from the second point of Lemma 2.5.2.
4. We have for $(\lambda,\eta,h)=(0,0,M/p_2^0)$, $\langle p_2^0h-M,\Phi\rangle=0$. Then,

$$F(0,0,M/p_2^0)=\frac{1}{p_2^0}\big(M-T_0[VM]\big).$$

Thus, multiplying this last equality by $p_2^0=M$ and applying the left inverse $L_0:=-\partial_v^2+\tilde{W}(v)$, we obtain:

$$L_0\big(M-T_0[VM]\big)=L_0(M)-VM=[-\partial_v^2+W(v)]M=0.$$

Hence, $F(0,0,\frac{M}{p_2^0})=0$ thanks to the injectivity of the left inverse L_0.
For the differential, we have $\frac{\partial F}{\partial h}(0,0,\frac{M}{p_2^0})=Id-\mathcal{T}^l_0$. By the Fredholm Alternative, this point is true if $\mathrm{Ker}(Id-\mathcal{T}^l_0)=\{0\}$. Let $h\in C_b(\mathbb{R},\mathbb{C})$ such that $h-\mathcal{T}^l_0(h)=0$. By multiplying this last equation by p_2^0 and applying the operator L_0, we obtain

$$[-\partial_v^2+W(v)](p_2^0h)=\langle p_2^0h,\Phi\rangle\Phi.$$

Now, integrating the previous equation against M and using the fact that $\langle\Phi,M\rangle=1$, we get

$$\langle p_2^0h,\Phi\rangle=0.$$

Therefore, p_2^0h is solution to $[-\partial_v^2+W(v)]f=0$. Then, there exists $c_1,c_2\in\mathbb{C}$ such that $p_2^0h=c_1M+c_2Z$, which implies that $h=c_1\frac{M}{p_2^0}+c_2\frac{Z}{p_2^0}$. Since $h\in C_b$ and $\frac{Z}{p_2^0}\notin C_b$ then, $c_2=0$ and $h=c_1\frac{M}{p_2^0}$. Thus, $\langle p_2^0h,\Phi\rangle=c_1=0$. Hence, $h=0$. This completes the proof of the Proposition. □

Theorem 2.5.5 (Existence of solutions with constraint)**.**

There is a unique function $M_{\lambda,\eta}$ *in* $E_2^\eta \subset L^2(\mathbb{R}, \mathbb{C})$ *solution to*

$$[-\partial_v^2 + W(v) + \mathrm{i}\eta v - \lambda\eta^{\frac{2}{3}}]M_{\lambda,\eta}(v) = b(\lambda,\eta)\Phi(v), \quad v \in \mathbb{R}. \tag{2.5.5}$$

Moreover,

$$\left\| \frac{M_{\lambda,\eta}}{p_2^{\lambda,\eta}} - \frac{M}{p_2^0} \right\|_\infty \underset{\eta\to 0}{\longrightarrow} 0, \tag{2.5.6}$$

where $b(\lambda,\eta) := \langle N_{\lambda,\eta}, \Phi\rangle$ *with* $N_{\lambda,\eta} := M_{\lambda,\eta} - M$*.*

Remark 2.5.6.

1. By construction, the solution $M_{\lambda,\eta}$ is symmetric and we have

$$\bar{M}_{\lambda,\eta}(-v) = M_{\lambda,\eta}(v), \quad \forall v \in \mathbb{R}. \tag{2.5.7}$$

2. By introducing the function $c(\lambda,\eta)$, satisfying $c(\lambda,\eta)M_{\lambda,\eta}(0) = 1$, we can always take $M_{\lambda,\eta}(0) = 1$ in order to simplify the notations. Such a function $c(\lambda,\eta)$ exists since the solution $M_{\lambda,\eta}$ given by the theorem 2.5.5 does not vanish at 0. Indeed, $M_{\lambda,\eta}(0) = 0$ leads us to the following contradiction

$$M(0) = 1 = \frac{M(0)}{p_2^0(0)} - \frac{M_{\lambda,\eta}(0)}{p_2^{\lambda,\eta}(0)} \leqslant \left\| \frac{M_{\lambda,\eta}}{p_2^{\lambda,\eta}} - \frac{M}{p_2^0} \right\|_\infty \underset{\eta\to 0}{\longrightarrow} 0.$$

Moreover, $c(\lambda,\eta)$ is holomorphic in $\{|\lambda| \leqslant \lambda_0\}$ and continuous in $\eta \in [0,\eta_0]$.

Proof. By Proposition 2.5.4, F satisfies the assumptions of the Implicit Function Theorem around the point $(0,0,\frac{M}{p_2^0})$. Then, there exists $\lambda_0,\eta_0 > 0$ small enough, there exists a unique function $h : \{|\lambda| \leqslant \lambda_0\} \times \{|\eta| \leqslant \eta_0\} \longrightarrow C_b(\mathbb{R},\mathbb{C})$, continuous with respect to λ and η such that

$$F(\lambda,\eta,h(\lambda,\eta)) = 0, \text{ for all } (\lambda,\eta) \in \{|\lambda| < \lambda_0\} \times \{|\eta| < \eta_0\}.$$

Let denote $h_{\lambda,\eta} := h(\lambda,\eta)$. Note that $h_{\lambda,0}$ does not depend on λ. The continuity of h with respect to η implies that

$$\lim_{\eta\to 0} \|h_{\lambda,\eta} - h_{\lambda,0}\|_\infty = \lim_{\eta\to 0} \|h_{\lambda,\eta} - h_{0,0}\|_\infty = 0. \tag{2.5.8}$$

Finally, we take $M_{\lambda,\eta} := p_2^{\lambda,\eta} h_{\lambda,\eta}$ and the proof of the theorem is complete. □

2.5.2 Properties of the solution to the penalized equation

Corollary 2.5.7 (Properties of $M_{\lambda,\eta}$).

1. *There exists a constant C such that, for all $v \in \mathbb{R}, |\lambda| \leqslant \lambda_0$ and $\eta \in [0, \eta_0]$*

$$|M_{\lambda,\eta}(v)| \leqslant CM(v). \tag{2.5.9}$$

2. *For all $v \in \mathbb{R}$ and $|\lambda| \leqslant \lambda_0$*

$$\lim_{\eta \to 0} M_{\lambda,\eta}(v) = M(v). \tag{2.5.10}$$

Therefore,

$$\lim_{\eta \to 0} \int_{\mathbb{R}} M_{\lambda,\eta}(v) M(v) \mathrm{d}v = \int_{\mathbb{R}} M^2(v) \mathrm{d}v \quad \textit{and} \quad M_{\lambda,\eta} \underset{\eta \to 0}{\longrightarrow} M \ \textit{in} \ L^2(\mathbb{R}). \tag{2.5.11}$$

3. *We have the following limit*

$$\lim_{\eta \to 0} \int_{\mathbb{R}} \eta^{\frac{1}{3}} v M_{\lambda,\eta}(v) M(v) \mathrm{d}v = 0. \tag{2.5.12}$$

Proof. 1. We have

$$|M_{\lambda,\eta}(v)| \leqslant p_2^{\lambda,\eta}(v) \|h_{\lambda,\eta}\|_\infty \leqslant C p_2^0(v) = CM(v)$$

since $M_{\lambda,\eta} = p_2^{\lambda,\eta} h_{\lambda,\eta}$, with $\|h_{\lambda,\eta}\|_\infty \leqslant C$ uniformly with respect to λ and η thanks to (2.5.8), and since $p_2^{\lambda,\eta}(v) \leqslant p_2^0(v)$. For this last inequality, we have $p_2^{\lambda,\eta}(v) = p_2^0(v)$ for $v \in [-s_0 \eta^{-\frac{1}{3}}, s_0 \eta^{-\frac{1}{3}}]$, and for $|v| \geqslant s_0 \eta^{-\frac{1}{3}}$ we have: $p_2^{\lambda,\eta}(v)/p_2^0(v) \leqslant (\eta^{-\frac{1}{3}} v)^{-\gamma} e^{-\frac{\sqrt{2}}{3}(\eta^{-\frac{1}{3}} v)^{\frac{3}{2}}} \leqslant 1$ since the function $t \mapsto t^{-\gamma} e^{-\frac{\sqrt{2}}{3} t^{\frac{3}{2}}}$ is decreasing for $t \in [s_0, +\infty)$ since $\gamma > 0$. Hence, the inequality (2.5.9) holds true.
2. We have

$$M_{\lambda,\eta}(v) - M(v) = \left(\frac{M_{\lambda,\eta}(v)}{p_2^{\lambda,\eta}(v)} - \frac{M(v)}{p_2^0(v)} \right) p_2^{\lambda,\eta}(v) + \frac{M(v)}{p_2^0(v)} \big(p_2^{\lambda,\eta}(v) - p_2^0(v) \big).$$

So, for $|v| \leqslant s_0 \eta^{-\frac{1}{3}}$, since $p_2^{\lambda,\eta}(v) = p_2^0(v)$ then,

$$|M_{\lambda,\eta}(v) - M(v)| \leqslant p_2^0(v) \left\| \frac{M_{\lambda,\eta}}{p_2^{\lambda,\eta}} - \frac{M}{p_2^0} \right\|_\infty \leqslant \left\| \frac{M_{\lambda,\eta}}{p_2^{\lambda,\eta}} - \frac{M}{p_2^0} \right\|_\infty \underset{\eta \to 0}{\longrightarrow} 0.$$

For $|v| \geqslant s_0 \eta^{-\frac{1}{3}}$, we have $p_2^{\lambda,\eta}(v) \leqslant p_2^0(v) \leqslant \eta^{\frac{\gamma}{3}}$, then

$$|M_{\lambda,\eta}(v) - M(v)| \leqslant \eta^{\frac{\gamma}{3}} \left(2 + \left\| \frac{M_{\lambda,\eta}}{p_2^{\lambda,\eta}} - \frac{M}{p_2^0} \right\|_\infty \right) \underset{\eta \to 0}{\longrightarrow} 0.$$

Then (2.5.11) is obtained by Lebesgue's theorem.
3. Let $\nu = \frac{1}{2}(\gamma - \frac{1}{2})$. We have $2\gamma - \nu - 1 > 0$. Then, $\langle v \rangle^{\nu - 2\gamma} \in L^1(\mathbb{R})$ and $|v| \langle v \rangle^{-\nu} \leqslant \eta^{\frac{\nu - 1}{3}}$

for $|v| \leqslant \eta^{-\frac{1}{3}}$. Now, since $|M_{\lambda,\eta}(v)| \lesssim p_2^{\lambda,\eta}(v)$ with $p_2^{\lambda,\eta}(v) = \langle v\rangle^{-\gamma}$ for $|v| \leqslant s_0\eta^{-\frac{1}{3}}$, then

$$\Big|\int_{\mathbb{R}} \eta^{\frac{1}{3}} v M_{\lambda,\eta}(v) M(v)\mathrm{d}v\Big| \lesssim \int_{|v|\leqslant s_0\eta^{-\frac{1}{3}}} \eta^{\frac{1}{3}}|v|\langle v\rangle^{-\nu+\nu-\gamma} M(v)\mathrm{d}v + \int_{|v|\geqslant s_0\eta^{-\frac{1}{3}}} \eta^{\frac{1}{3}}|v| p_2^{\lambda,\eta}(v) M(v)\mathrm{d}v.$$

Therefore, after making the change of variable $v = \eta^{-\frac{1}{3}}s$ in the second integral, we get

$$\Big|\int_{\mathbb{R}} \eta^{\frac{1}{3}} v M_{\lambda,\eta}(v) M(v)\mathrm{d}v\Big| \lesssim \Big[\eta^{\frac{\nu}{3}}\int_{\mathbb{R}}\langle v\rangle^{\nu-2\gamma}\mathrm{d}v + \eta^{\frac{2\gamma-1}{3}}\int_{|s|\geqslant s_0}|s|^{1-\gamma}|\mathfrak{a}_\lambda(|s|)|\mathrm{d}s\Big] \underset{\eta\to 0}{\longrightarrow} 0,$$

with $\mathfrak{a}_\lambda(s) = \mathfrak{a}_{\lambda,\eta}(v\eta^{-\frac{1}{3}})$ does not depend on η after rescaling. ☐

Corollary 2.5.8 (Rescaled solution)**.** *We define the function $H_{\lambda,\eta}$ for all s in $\mathbb{R}$ by*

$$H_{\lambda,\eta}(s) := \eta^{-\frac{\gamma}{3}} M_{\lambda,\eta}(\eta^{-\frac{1}{3}}s). \tag{2.5.13}$$

Then, $H_{\lambda,\eta}$ satisfies the rescaled equation

$$\big[-\partial_s^2 + \frac{\gamma(\gamma+1)}{|s|_\eta^2} + \mathrm{i}s - \lambda\big] H_{\lambda,\eta}(s) = -V_\eta(s) H_{\lambda,\eta}(s) - \eta^{-\frac{2+\gamma}{3}} b(\lambda,\eta)\Phi(\eta^{-\frac{1}{3}}s), \quad s\in\mathbb{R}.$$

Moreover, the following estimates hold

1. *For all $|s| \leqslant s_0$*
$$|H_{\lambda,\eta}(s)| \lesssim |s|_\eta^{-\gamma} \leqslant |s|^{-\gamma}. \tag{2.5.14}$$
2. *For all $|s| \geqslant s_0$*
$$|H_{\lambda,\eta}(s)| \lesssim \begin{cases} |\mathfrak{a}_\lambda(s)|, & s \geqslant s_0,\\ |\mathfrak{b}_\lambda(s)|, & s\leqslant -s_0,\end{cases} \tag{2.5.15}$$

where $|s|_\eta := (\eta^{\frac{2}{3}} + s^2)^{\frac{1}{2}}$ and $V_\eta(s) := \big[\gamma(\gamma+1)s^2 - \gamma\eta^{\frac{2}{3}}\big]|s|_\eta^{-4}$.

Proof. The proof of this Corollary is a direct consequence of the previous one, by making the change of variable $v = \eta^{-\frac{1}{3}}s$. ☐

2.5.3 Existence of an eigen couple $(\mu(\eta), M_{\mu,\eta})$ for the complete operator

The purpose of this section is, first, to find a solution for equation (2.1.5), which amounts to showing the existence of a λ, function of η, such that the additional term $\langle M_{\lambda,\eta} - M, \Phi\rangle = 0$. We prove it again using the implicit function theorem. In a second step, we will compute the eigenvalue $\mu = \lambda\eta^{\frac{2}{3}}$, and for this, we will establish some estimates on the solutions of the equations of $M_{\lambda,\eta}$ and $H_{\lambda,\eta}$ respectively. We summarize these two results in the following two propositions:

Proposition 2.5.9 (Constraint). *Define*

$$B(\lambda,\eta) := \eta^{-\frac{2}{3}} b(\lambda,\eta).$$

1. *The expression of $B(\lambda,\eta)$ is given by*
$$B(\lambda,\eta) = \eta^{-\frac{2}{3}} \langle N_{\lambda,\eta}, \Phi\rangle = \int_{\mathbb{R}} (\lambda - \mathrm{i}\eta^{\frac{1}{3}} v) M_{\lambda,\eta}(v) M(v) \mathrm{d}v. \tag{2.5.16}$$
2. *The η order of the coefficient in front of λ in the expansion on λ of $B(\lambda,\eta)$ is given by*
$$\lim_{\eta\to 0} B(\lambda,\eta) = \lambda \int_{\mathbb{R}} M^2(v)\mathrm{d}v. \tag{2.5.17}$$
3. *There exists $\tilde{\eta}_0, \tilde{\lambda}_0 > 0$ small enough, a function $\tilde{\mu} : \{|\eta| \leqslant \tilde{\eta}_0\} \longrightarrow \{|\lambda| \leqslant \tilde{\lambda}_0\}$ such that:*
for all $(\lambda,\eta) \in \{|\eta| < \tilde{\eta}_0\} \times \{|\lambda| < \tilde{\lambda}_0\}$, $\lambda = \tilde{\mu}(\eta)$ and the constraint is satisfied:
$$B(\lambda,\eta) = B(\tilde{\mu}(\eta),\eta) = 0.$$

Therefore, $\mu(\eta) = \eta^{\frac{2}{3}}\tilde{\mu}(\eta)$ is the eigenvalue associated to the eigenfunction $M_\eta := M_{\tilde{\mu}(\eta),\eta}$ for the operator $\mathcal{L}_\eta$, and the couple $\big(\mu(\eta), M_\eta\big)$ is a solution to the spectral problem (2.1.5).

Proposition 2.5.10 (Approximation of the eigenvalue). *Let $\alpha := \frac{2\gamma+1}{3}$ for all $\gamma \in]\frac{1}{2}, \frac{5}{2}[$. The eigenvalue $\mu(\eta)$ satisfies*

$$\mu(\eta) = \overline{\mu}(-\eta) = \kappa |\eta|^\alpha \big(1 + O(|\eta|^\alpha)\big), \tag{2.5.18}$$

where κ is a positive constant given by

$$\kappa := -2C_\beta^2 \int_0^\infty s^{1-\gamma} \mathrm{Im} H_0(s)\mathrm{d}s, \tag{2.5.19}$$

and where H_0 is the unique solution to

$$\big[-\partial_s^2 + \mathrm{i}s + \frac{\gamma(\gamma+1)}{s^2}\big] H_0(s) = 0, \ \ s \in \mathbb{R}^*, \tag{2.5.20}$$

satisfying

$$\int_{|s|\geqslant 1} |H_0(s)|^2 \mathrm{d}s < +\infty \ \ \textit{and} \ \ H_0(s) \underset{0}{\sim} |s|^{-\gamma}. \tag{2.5.21}$$

Proof of Proposition 2.5.9.
1. The first point is obtained by multiplying the equation (2.5.5) by M, integrating it twice by part and using the fact that $[-\partial_v^2 + W(v)]M = 0$ and $\langle \Phi, M\rangle = 1$.
2. We obtain the limit (2.5.17) by the last two points of the Corollary 2.5.7.
3. The proof of this point is an immediate consequence of the IFT applied to the function B around the point $(0,0)$. □

In order to get the Proposition 2.5.10, we need to prove the following two lemmas: The first one gives estimates on $M_{0,\eta}$ and $H_{0,\eta}$.

Lemma 2.5.11. *For all $\gamma > 1$ ons has*

$$|M_{0,\eta}(v) - M(v)| \lesssim \eta, \quad \forall v \in [-v_0, v_0]. \tag{2.5.22}$$

Moreover, for large velocities

$$|M_{0,\eta}(v) - M(v)| \lesssim \eta \langle v \rangle^{3-\gamma}, \ \forall |v| \in [v_0, s_0 \eta^{-\frac{1}{3}}]. \tag{2.5.23}$$

Therefore,

$$\left| H_{0,\eta}(s) - |s|_\eta^{-\gamma} \right| \lesssim \ |s|_\eta^{3-\gamma} \ \leqslant \langle s \rangle^{3-\gamma}, \ \ \forall |s| \in [0, s_0]. \tag{2.5.24}$$

The second one gives the formula of the diffusion coefficient.

Lemma 2.5.12.

1. *The small velocities don't participate to the limit in the approximation of $\mu(\eta)$:*

$$\lim_{\eta \to 0} \eta^{-\frac{2(\gamma-1)}{3}} \int_{|v| \leqslant v_0} v M_{0,\eta}(v) M(v) \mathrm{d}v = 0. \tag{2.5.25}$$

2. *For large velocities, we have the following limit:*

$$\lim_{\eta \to 0} \mathrm{i}\eta^{-\frac{2(\gamma-1)}{3}} \int_{|v| \geqslant v_0} v M_{0,\eta}(v) M(v) \mathrm{d}v = -2 \int_0^\infty s^{1-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s, \tag{2.5.26}$$

   where H_0 is the unique solution to (2.5.20) *satisfying the conditions* (2.5.21)*.*

Proof of Lemma 2.5.11. Recall that $N_{0,\eta} := M_{0,\eta} - M$ and it satisfies the equation

$$[-\partial_v^2 + W(v)] N_{0,\eta}(v) = -\mathrm{i}\eta v [N_{0,\eta}(v) + M(v)] - \langle N_{0,\eta}, \Phi \rangle \Phi(v), \ \forall v \in \mathbb{R}. \tag{2.5.27}$$

Thanks to the symmetry $\bar{N}_{0,\eta}(-v) = N_{0,\eta}(v)$, we establish the inequalities (2.5.22) and (2.5.23) on $[0, s_0 \eta^{-\frac{1}{3}}]$. By writing the solution of the equation (2.5.27) in the basis of solutions of $[-\partial_v^2 + W(v)] f = 0$, which is given by $\{M, Z\}$, we get:

$$\begin{aligned} N_{0,\eta}(v) = & \left(c_1 - \int_0^v [\mathrm{i}\eta w Z(N_{0,\eta} + M) + b(0,\eta)\Phi(w)] \mathrm{d}w \right) M(v) \\ & + \left(c_2 + \int_0^v [\mathrm{i}\eta w M(N_{0,\eta} + M) + b(0,\eta)\Phi(w)] \mathrm{d}w \right) Z(v), \end{aligned}$$

where c_1 and c_2 are two complex constants to determine and $b(0,\eta) := \langle N_{0,\eta}, \Phi \rangle$. Now, by using the condition $N_{0,\eta}(0) = 0$, we get $c_1 = 0$, and since $|N_{0,\eta}(\eta^{-\frac{1}{3}})| \lesssim M(\eta^{-\frac{1}{3}}) \sim \eta^{\frac{\gamma}{3}}$ (that we get from (2.5.9) for $v = \eta^{-\frac{1}{3}}$) then,

$$c_2 = -\int_0^\infty [\mathrm{i}\eta w M(N_{0,\eta} + M) + b(0,\eta)\Phi(w)] \mathrm{d}w,$$

otherwise we will have $|N_{0,\eta}(\eta^{-\frac{1}{3}})| \sim \eta^{-\frac{\gamma-2}{3}}$, which contradicts the fact that $|N_{0,\eta}(\eta^{-\frac{1}{3}})| \lesssim \eta^{\frac{\gamma}{3}}$ for all $\gamma \in (\frac{1}{2}, \frac{5}{2})$. Therefore,

$$\begin{aligned} N_{0,\eta}(v) = &- \mathrm{i}\eta\Big[M(v)\int_0^v wZ(N_{0,\eta}+M)\mathrm{d}w + Z(v)\int_v^\infty wM(N_{0,\eta}+M)\mathrm{d}w\Big] \\ &- b(0,\eta)\Big[M(v)\int_0^v \Phi Z\mathrm{d}w + Z(v)\int_v^\infty \Phi M\mathrm{d}w\Big]. \end{aligned}$$

By (2.5.16), $b(0,\eta) = -\mathrm{i}\eta\int_{\mathbb{R}} wM_{0,\eta}M\mathrm{d}w$, and since $|N_{0,\eta}(v)| \lesssim M(v)$ then, by dividing the last equality by $\eta\langle v\rangle^{3-\gamma}$, we get

$$\left|\frac{N_{0,\eta}(v)}{\eta\langle v\rangle^{3-\gamma}}\right| \lesssim \langle v\rangle^{-3}\int_0^v [wM+\Phi]Z\mathrm{d}w + \langle v\rangle^{\gamma-3}Z(v)\int_v^\infty [wM+\Phi]\mathrm{d}w.$$

We have $Z(v) = M(v)\int_0^v \frac{\mathrm{d}w}{M^2(w)} \lesssim \langle v\rangle^{\gamma+1}$ and $\Phi(v) \leqslant p_1^0(v) \leqslant \langle v\rangle^{-\gamma-2}$. Thus, $\left|\frac{N_{0,\eta}(v)}{\eta\langle v\rangle^{3-\gamma}}\right| \lesssim 1$ for all $v \in \mathbb{R}^+$, in particular for $v \in [0, v_0]$ we get (2.5.22) and for $v \in [0, s_0\eta^{-\frac{1}{3}}]$ we get (2.5.23). ☐

Proof of Lemma 2.5.12. First of all, since $\bar{M}_{0,\eta}(-v) = M_{0,\eta}(v)$ and $M(-v) = M(v)$ for all $v \in \mathbb{R}$, then

$$-\mathrm{i}\int_{\mathbb{R}} vM_{0,\eta}(v)M(v)\mathrm{d}v = 2\int_0^\infty v\mathrm{Im}M_{0,\eta}(v)M(v)\mathrm{d}v = 2\int_0^\infty v\mathrm{Im}\big(M_{0,\eta}(v) - M(v)\big)M(v)\mathrm{d}v.$$

1. Recall that $\alpha := \frac{2\gamma+1}{3}$. Then, $1-\alpha = \frac{2(\gamma-1)}{3}$. We have thanks to (2.5.9) and (2.5.22):

$$\eta^{1-\alpha}\left|\int_{|v|\leqslant v_0} vM_{0,\eta}(v)M(v)\mathrm{d}v\right| \lesssim \begin{cases} \eta^{\frac{2(1-\gamma)}{3}}\displaystyle\int_{|v|\leqslant v_0}\langle v\rangle^{1-2\gamma}\mathrm{d}v, & \gamma \in (\frac{1}{2}, 1), \\ \eta^{\frac{5-2\gamma}{3}}\displaystyle\int_{|v|\leqslant v_0}\langle v\rangle^{4-\gamma}\mathrm{d}v, & \gamma \in (1, \frac{5}{2}). \end{cases}$$

Hence (2.5.25) holds true for all $\gamma \in (\frac{1}{2}, \frac{5}{2})$, since $2-\alpha = (5-2\gamma)/3 > 0$ for $\gamma > 1$. The case $\gamma = 1$ is done by Lebesgue's theorem thanks to (2.5.10) and the domination of $v|M_{0,\eta}|M$ by $v\langle v\rangle^{-2\gamma} \in L^1(0, v_0)$ thanks to (2.5.9).
2. For the the integral on $[v_0, +\infty)$, we split it into two parts as follows:

$$2\eta^{1-\alpha}\Big[\int_{v_0}^{s_0\eta^{-\frac{1}{3}}} v\mathrm{Im}\big(M_{0,\eta}(v) - M(v)\big)M(v)\mathrm{d}v + \int_{s_0\eta^{-\frac{1}{3}}}^\infty v\mathrm{Im}M_{0,\eta}(v)M(v)\mathrm{d}v\Big].$$

In order to compute the limit of the previous expressions, we proceed to a change of variable $v = \eta^{-\frac{1}{3}} s$, which means that we need to compute

$$\lim_{\eta\to 0}\Big[\int_{\eta^{\frac{1}{3}}v_0}^{s_0} s|s|_\eta^{-\gamma}\mathrm{Im}\big[H_{0,\eta}(s)-|s|_\eta^{-\gamma}\big]\mathrm{d}s+\int_{s_0}^{\infty} s|s|_\eta^{-\gamma}\mathrm{Im}H_{0,\eta}(s)\mathrm{d}s\Big].$$

For that purpose, we will use the "weak-strong" convergence in the Hilbert space $L^2(0,\infty)$. First, recall the following estimates given by (2.5.14), (2.5.15) and (2.5.24):

• For $s \in (0, s_0)$,

$$\big|H_{0,\eta}(s)-|s|_\eta^{-\gamma}\big| \lesssim \begin{cases} |s|_\eta^{-\gamma}, & \gamma\in(\frac{1}{2},1] \\ |s|_\eta^{3-\gamma}, & \gamma\in(1,\frac{5}{2}) \end{cases} \lesssim \begin{cases} |s|^{-\gamma}, & \gamma\in(\frac{1}{2},1], \\ \langle s\rangle^{3-\gamma}, & \gamma\in(1,\frac{5}{2}). \end{cases}$$

• For $s \geqslant s_0$ and for all $\gamma \in (\frac{1}{2}, \frac{5}{2})$,

$$\big|H_{0,\eta}(s)\big| \lesssim \big|\mathfrak{a}_0(s)\big|.$$

Hence, the two sequences $(H_{0,\eta})_\eta$ and $(\mathsf{H}_\eta)_\eta$ are uniformly bounded in $L^1(0,\infty)$ and $L^2(0,\infty)$ respectively, where H_η is defined by

$$\mathsf{H}_\eta(s) := \begin{cases} s^{\frac{1}{2}}\mathrm{Im}H_{0,\eta}(s) = s^{\frac{1}{2}}\mathrm{Im}\big[H_{0,\eta}(s)-|s|_\eta^{-\gamma}\big], & \gamma\in(\frac{1}{2},1], 0<s\leqslant s_0, \\ s^{-1}\mathrm{Im}H_{0,\eta}(s) = s^{-1}\mathrm{Im}\big[H_{0,\eta}(s)-|s|_\eta^{-\gamma}\big], & \gamma\in(1,\frac{5}{2}), 0<s\leqslant s_0, \\ s\mathrm{Im}H_{0,\eta}(s) & \text{for all } \gamma\in(\frac{1}{2},\frac{5}{2}) \text{ and } s\geqslant s_0. \end{cases}$$

Now, since the sequence H_η is bounded in $L^2(0,\infty)$, uniformly with respect to η then, up to a subsequence, that we will again denote by H_η, we have H_η converges weakly in $L^2(0,\infty)$, thus converges in $\mathcal{D}'(0,\infty)$. Let's identify this limit that we denote by H_0. We have on the one hand, $H_{0,\eta}$ converges to H_0 in $\mathcal{D}'(0,\infty)$. Indeed, recall that $H_{0,\eta}$ satisfies the equation

$$\big[-\partial_s^2+\frac{\gamma(\gamma+1)}{|s|_\eta^2}+\mathrm{i}s\big]H_{0,\eta}(s)=\eta^{\frac{2}{3}}\frac{\gamma(\gamma+2)}{|s|_\eta^4}H_{0,\eta}(s)-\eta^{-\frac{2+\gamma}{3}}b(0,\eta)\Phi(\eta^{-\frac{1}{3}}s).$$

Let $\varphi \in \mathcal{D}(0,\infty)$. Then, by multiplying the previous equation by φ and by integrating it by parts, we obtain

$$\int_0^\infty\Big[-\partial_s^2+\frac{\gamma(\gamma+1)}{|s|_\eta^2}+\mathrm{i}s\Big]\varphi(s)H_{0,\eta}(s)\mathrm{d}s$$
$$=\int_0^\infty\Big[\eta^{\frac{2}{3}}\frac{\gamma(\gamma+2)}{|s|_\eta^4}H_{0,\eta}(s)-\eta^{-\frac{\gamma}{3}}B(0,\eta)\Phi(\eta^{-\frac{1}{3}}s)\Big]\varphi(s)\mathrm{d}s.$$

By (2.5.16) and (2.5.12), $B(0,\eta) \to 0$ when $\eta \to 0$ and since $\Phi(\eta^{-\frac{1}{3}}s) \lesssim p_1^{0,\eta}(\eta^{-\frac{1}{3}}s) \leqslant$

$\eta^{\frac{2+\gamma+\delta}{3}}|s|^{-2-\gamma-\delta}$ then, by Lebesgue's theorem $H_{0,\eta}$ converges to H_0, in $\mathcal{D}'(0,\infty)$, the unique solution to

$$\big[-\partial_s^2+\frac{\gamma(\gamma+1)}{s^2}+\mathrm{i}s\big]H_0(s)=0.$$

The uniqueness of H_0 comes from the fact that the previous equation admits a unique solution in $L^2(1,\infty)$, up to a multiplicative constant [LP19] and that $H_0(s)\underset{0}{\sim}s^{-\gamma}$ which is obtained by passing to the limit, in η, in the inequality $|H_{0,\eta}(s_1)/|s_1|_\eta^{-\gamma}-1|\leqslant\|M_{0,\eta}/p_2^{0,\eta}-M/p_2^0\|_\infty$ for $s_1>0$ small enough.
On the other hand, since $s^{-\frac12}$ and s are in $C^\infty(0,\infty)$ and $H_{0,\eta}\to H_0$ in $\mathcal{D}'(0,\infty)$ with H_0 unique. Then, H_η converges to H_0 in $\mathcal{D}'(0,\infty)$ and therefore $\mathsf{H}_\eta\rightharpoonup\mathsf{H}_0$ weakly in $L^2(0,\infty)$ with H_0 unique and given by

$$\mathsf{H}_0(s):=\begin{cases}s^{\frac12}\mathrm{Im}H_0(s), & \gamma\in(\frac12,1],0<s\leqslant s_0,\\[2ex] s^{-1}\mathrm{Im}H_0(s), & \gamma\in(1,\frac52),0<s\leqslant s_0,\\[2ex] s\mathrm{Im}H_0(s) & \text{for all }\gamma\in(\frac12,\frac52)\text{ and }s\geqslant s_0.\end{cases}$$

Moreover, thanks to the uniqueness of this limit, the whole sequence converges. Finally, we conclude by passing to the limit in the scalar product $\langle\mathsf{H}_\eta,\mathsf{I}_\eta\rangle$, where I_η definded by

$$\mathsf{I}_\eta:=\begin{cases}s^{\frac12}|s|_\eta^{-\gamma}, & \gamma\in(\frac12,1],0<s\leqslant s_0,\\[2ex] s^2|s|_\eta^{-\gamma}, & \gamma\in(1,\frac52),0<s\leqslant s_0,\\[2ex] |s|_\eta^{-\gamma}, & \gamma\in(\frac12,\frac52),s\geqslant s_0,\end{cases}$$

converges strongly in $L^2(0,\infty)$ to

$$\mathsf{I}_0=\begin{cases}s^{\frac12-\gamma}, & \gamma\in(\frac12,1],0<s\leqslant s_0,\\[2ex] s^{2-\gamma}, & \gamma\in(1,\frac52),0<s\leqslant s_0,\\[2ex] s^{-\gamma}, & \gamma\in(\frac12,\frac52),s\geqslant s_0.\end{cases}$$

Hence the limit (2.5.26) holds true. □

Proof of Proposition 2.5.10. By doing an expansion in λ for B and by Proposition 2.5.9, we get

$$B(\lambda,\eta)=\eta^{-\frac23}b(\lambda,\eta)=\eta^{-\frac23}b(0,\eta)+\lambda\int_{\mathbb{R}}M_{0,\eta}M\mathrm{d}v+O(\lambda^2).$$

Then, for $\lambda = \tilde{\mu}(\eta)$ and since $B(\tilde{\mu}(\eta), \eta) = 0$, we obtain

$$\tilde{\mu}(\eta) = -\eta^{-\frac{2}{3}} b(0,\eta) \Big(\int_{\mathbb{R}} M_{0,\eta} M \mathrm{d}v \Big)^{-1} + o\big(\eta^{-\alpha} b(0,\eta)\big),$$

which implies that

$$\eta^{-\alpha}\mu(\eta) = \eta^{\frac{2}{3}-\alpha}\tilde{\mu}(\eta) = -\eta^{-\alpha} b(0,\eta) \Big(\int_{\mathbb{R}} M_{0,\eta} M \mathrm{d}v \Big)^{-1}.$$

By (2.5.11) and (2.5.26),

$$\lim_{\eta \to 0} \int_{\mathbb{R}} M_{0,\eta}(v) M(v) \mathrm{d}v = \|M\|_2^2$$

and

$$\lim_{\eta \to 0} \eta^{-\alpha} b(0,\eta) = 2 \int_0^\infty s^{1-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s$$

respectively. Hence, $\lim_{\eta \to 0} \eta^{-\alpha}\mu(\eta) = \kappa$. For $\eta \in [-\eta_0, 0]$, the symmetry $\mu(\eta) = \overline{\mu}(-\eta)$ holds by complex conjugation on the equation. Thus, the proof of Proposition 2.5.10 is complete.

□

Remark 2.5.13. We do the same calculations as in [LP19] to compute the coefficient κ. Also, since it is given by the same integral formula (2.5.19) then, we have $\kappa > 0$.

Proof of the main Theorem 2.1.1. The existence and uniqueness of the eigen-solution $\big(\mu(\eta), M_\eta\big)$ is given by Proposition 2.5.9. The first point, (2.1.6), is given by the Corrolary 2.5.7 and finally, the second point is given by the Proposition 2.5.10. □

CHAPTER 3

Fractional diffusion for Fokker-Planck equation with heavy tail equilibrium: an à la Koch spectral method in any dimension

Résumé

Dans ce chapitre, nous étendons la méthode spectrale développée dans [DP23a] à toute dimension $d \geqslant 1$, afin de construire une solution propre pour l'opérateur de Fokker-Planck avec équilibres à queue lourde, de la forme $(1+|v|^2)^{\frac{\beta}{2}}$, dans l'intervalle $\beta \in]d, d+4[$. La méthode développée en dimension 1 a été inspirée des travaux de H. Koch sur l'équation de KdV non linéaire [Koc15]. Dans ce chapitre, la stratégie est la même qu'en dimension 1 mais les outils sont différents, puisque la dimension 1 était basée sur des méthodes d'EDO. Comme conséquence directe de notre construction, nous obtenons la limite de diffusion fractionnaire pour l'équation cinétique de Fokker-Planck, pour la densité $\rho := \int_{\mathbb{R}^d} f \mathrm{d}v$ avec un Laplacien fractionnaire $\kappa(-\Delta)^{\frac{\beta-d+2}{6}}$ et un coefficient de diffusion κ strictement positif.

Abstract

In this chapter, we extend the spectral method developed in [DP23a] to any dimension $d \geqslant 1$, in order to construct an eigen-solution for the Fokker-Planck operator with heavy tail equilibria, of the form $(1+|v|^2)^{-\frac{\beta}{2}}$, in the range $\beta \in]d, d+4[$. The method developed in dimension 1 was inspired by the work of H. Koch on nonlinear KdV equation [Koc15]. The strategy in this chapter is the same as in dimension 1 but the tools are different, since dimension 1 was based on ODE methods. As a direct consequence of our construction, we obtain the fractional diffusion limit for the kinetic Fokker-Planck equation, for the correct density $\rho := \int_{\mathbb{R}^d} f \mathrm{d}v$, with a fractional Laplacian $\kappa(-\Delta)^{\frac{\beta-d+2}{6}}$ and a positive diffusion coefficient κ.

Contents

3.1 Introduction

3.1.1 Setting of the problem

In this present chapter, we deal with the kinetic Fokker-Planck (FP) equation, which describes in a deterministic way the Brownian motion of a set of particles. It is given by the following form

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = Q(f), \quad t\geqslant 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \\ \\ f(0,x,v) = f_0(x,v), \qquad x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \end{cases} \tag{3.1.1}$$

where the collisional Fokker-Planck operator Q is given by

$$Q(f) = \nabla_v\cdot\left(F\nabla_v\left(\frac{f}{F}\right)\right), \tag{3.1.2}$$

and F is the equilibrium of Q, a fixed function which depends only on v and satisfying

$$Q(F) = 0 \quad \text{ and } \quad \int_{\mathbb{R}^d} F(v)\,\mathrm{d}v = 1.$$

Provided $f_0\geqslant 0$, the unknown $f(t,x,v)\geqslant 0$ can be interpreted as the density of particles occupying at time $t\geqslant 0$, the position $x\in\mathbb{R}^d$ with velocity $v\in\mathbb{R}^d$.

Recall that one of the motivations for studying the *classical* or *fractional diffusion limit* is to simplify the equations for some collisional kinetic models when the interaction between particles are the dominant phenomena and when the observation time is very large. For that purpose, we introduce a small parameter, $\varepsilon\ll 1$, the mean free path and we proceed to rescaling the distribution function $f(t,x,v)$ in time and space

$$t = \frac{t'}{\theta(\varepsilon)} \quad \text{and} \quad x = \frac{x'}{\varepsilon} \quad \text{with} \quad \theta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

which leads to the following rescaled equation (without primes)

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon), & t \geqslant 0, x \in \mathbb{R}^d, v \in \mathbb{R}^d, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x \in \mathbb{R}^d, v \in \mathbb{R}^d. \end{cases} \tag{3.1.3}$$

Note that initial condition written in non rescaled variable are well prepared conditions.

The goal is then to study the behavior of the solution f^ε as $\varepsilon \to 0$. Formally, passing to the limit when $\varepsilon \to 0$ in the equation (3.1.3), we obtain that the limit f^0 is in the kernel of Q which is spanned by the equilibrium F, which means that $f^0 = \rho(t,x)F(v)$. Thus, it amounts to find the equation satisfied by the density ρ. Note that this limit depends on the nature of the equilibrium F considered as well as on the chosen change of time scale $\theta(\varepsilon)$.

For Gaussian equilibria, it is classical (see [BSS84],[BLP79],[DGP00],[LK74],[DPR04] for Boltzmann and [DMG87] for Fokker Planck) that by taking the classical time scaling $\theta(\varepsilon) = \varepsilon^2$, we obtain a *diffusion equation*

$$\partial_t \rho - \nabla_x \cdot (D \nabla_x \rho) = 0, \tag{3.1.4}$$

where

$$D = \int v Q^{-1}(-vF) \mathrm{d}v. \tag{3.1.5}$$

For slowly decreasing equilibria, or so-called *heavy tail* equilibria of the form $F(v) \sim \langle v \rangle^{-\beta}$, it is more complicated, and this study has been the interest of many papers in the last few years, with different methods and for different collision operators. Fractional diffusion limit has been obtained in the case of the linear Boltzmann equation when the cross section is such that the operator has a spectral gap, see [MMM11] for the pioneer paper in the case of space independent cross section, where the authors used a method based on Fourier-Laplace transformation, and see [Mel10] for a weak convergence result obtained by the Moment method, which also applies to cross sections that depend on the position variable. See also [JKO09] for a probabilistic approach.

In the present work, we consider for any $\beta > d$, heavy tail equilibria

$$F(v) = \frac{C_\beta^2}{(1+|v|^2)^{\frac{\beta}{2}}},$$

where C_β is a normalization constant.

The diffusion limit for the FP equation seems more complicated then the linear Boltzmann one, and the main difficulty is due to the fact that the Fokker-Planck operator Q has no spectral gap. In addition, for this equation, all the terms of the operator participate in the limit, i.e. the collision and advection parts. In [NP15], the classical scaling is studied and it is proved in any dimension d that we obtain a diffusion equation (3.1.4), with diffusion coefficient (3.1.5) as soon as $\beta > d+4$. The critical case where $\beta = d+4$ is studied in [CNP19], where the expected result of classical diffusion with an anomalous time scaling is proved, $\theta(\varepsilon) = \varepsilon^2 \ln(\varepsilon^{-1})$. A unified presentation of the result for even more general cases of β can be found in recent papers where the result has been obtained, by probabilistic method in [FT21] and [FT20], and using a *quasi-spectral* problem in [BM22]. In this last paper, in addition to the diffusion limit results, an estimates on the fluid approximation error have been obtained. We refer also to [BDL22] for this last point, where the authors have developed an L^2-hypocoercivity approach and established an optimal decay rate, determined by a fractional Nash type inequality, compatible with the fractional diffusion limit.

In this chapter we focus on the the case $d < \beta < d+4$. By taking as test function the eigenvector of the whole Fokker-Planck operator (advection + collisions), which converges towards equilibrium F, we capture at the limit the "diffusion" equation for any $\beta > d$. The computation of the eigenvalue gives us the right scaling in time, $\theta(\varepsilon)$, and the diffusion coefficient κ at the same time. We are therefore interested in a new problem: *the construction of an eigen-solution for the whole Fokker-Planck operator*, which is the main subject of this chapter.

This spectral problem for the FP operator has already been obtained recently in dimension 1 [LP19] with a method based on the reconnection of two branches on $\mathbb{R}_+$ and $\mathbb{R}_-$, but this method of reconnection is difficult to adapt in dimension d. This led us to look for another strategy, which was the subject of [DP23a], a method inspired by the work of H. Koch on nonlinear KdV equation [Koc15], which allowed us to construct an eigen-solution for the spectral problem associated to the whole Fokker-Planck operator with ODE methods in dimension 1. The aim of this chapter is to develop PDE methods in order to obtain the result in any dimension. This method is interesting since it can be used for different potentials like convolution, or for nonlinear equations as well. Moreover, as in dimension 1, a splitting of the Fokker-Planck operator is involved, which recalls the *enlargement theory* for nonlinear Boltzmann operator when there are spectral gap issues. This theory was developed by Gualdani, Mischler and Mouhot in [GMM10] whose key idea was based on the decomposition of the operator into two parts, a dissipative part plus a regularizing part. See also [Ger20] and references therein.

Note that we don't look at the same spectral problem as in the paper by E. Bouin and C. Mouhot [BM22]. Indeed, in this paper we were interested in the improvement and generalization of the construction given in [LP19] to solve the problem

$$\big[Q + \mathrm{i}\varepsilon\xi\cdot v\big] M_{\mu,\varepsilon} = \mu\ M_{\mu,\varepsilon},$$

with ξ being the Fourier variable of x. While in [BM22] the authors considered the following quasi-spectral problem:

$$\big[Q + \mathrm{i}\varepsilon\xi\cdot v\big] \phi_{\mu,\varepsilon} = \mu\ \frac{\phi_{\mu,\varepsilon}}{\langle v\rangle^2}, \tag{3.1.6}$$

with $\phi_{\mu,\varepsilon} \in L^2\big(\mathbb{R}^d; \frac{\mathrm{d}v}{\langle v\rangle^2}\big)$ satisfying $\displaystyle\int_{\mathbb{R}^d} \phi_{\mu,\varepsilon}(v) M(v)\ \frac{\mathrm{d}v}{\langle v\rangle^2} = 1$. The key idea in (3.1.6) is the introduction of a weight that allowed to recover the spectral gap inequality for the latter operator thanks to the Hardy-Poincaré inequality

$$\int_{\mathbb{R}^d} fQ(f)\ \mathrm{d}v \geqslant C \int_{\mathbb{R}^d} |f - rM|^2\ \frac{\mathrm{d}v}{\langle v\rangle^2},$$

where r is a weighted density defined by

$$r(t,x) := \int_{\mathbb{R}^d} f\ \frac{\mathrm{d}v}{\langle v\rangle^2}. \tag{3.1.7}$$

Thus, by totally different techniques based on energy estimates and the study of the resolvent, E. Bouin and C. Mouhot showed the existence of a "fluid mode", a couple $\big(\mu(\varepsilon), \phi_{\mu,\varepsilon}\big)$ solution of the problem (3.1.6). Thanks to this construction, they obtain the convergence of f^ε/F towards $(\int_{\mathbb{R}^d} \frac{F}{\langle v\rangle^2}\mathrm{d}v)^{-1} r(t,x)$ in $L^2_t\big([0,T]; H_x^{-\frac{\beta-d+2}{3}} L^2_v(\frac{F}{\langle v\rangle^2})\big)$, when ε goes to 0, with r solution to a fractional diffusion equation. Finally, the diffusion limit with the *classical* density $\rho := \int f \mathrm{d}v$ is recovered, in a weak sense.

3.1.2 Setting of the result

Before stating our main result, let us give some notations that we will use along this chapter.

Notations. As in [LP19], in order to simplify the computation and work with a self-adjoint operator in L^2, we proceed to a change of unknown by writing

$$f = F^{\frac{1}{2}} g = C_\beta M g$$

with

$$M := C_\beta^{-1} F^{\frac{1}{2}} = \frac{1}{(1+|v|^2)^{\frac{\gamma}{2}}},$$

since we impose $\gamma := \frac{\beta}{2} > \frac{d}{2}$, $F \in L^1(\mathbb{R}^d)$ then, $M \in L^2(\mathbb{R}^d)$ and C_β is chosen such that

$$\int_{\mathbb{R}^d} F \mathrm{d}v = 1.$$

The equation (3.1.3) becomes

$$\theta(\varepsilon)\partial_t g^\varepsilon + \varepsilon v\cdot\nabla_x g^\varepsilon = \frac{1}{M}\nabla_v\cdot\left(M^2\nabla_v\left(\frac{g^\varepsilon}{M}\right)\right) = \Delta_v g^\varepsilon - W(v)g^\varepsilon,$$

with

$$W(v) = \frac{\Delta_v M}{M} = \frac{\gamma(\gamma-d+2)|v|^2-\gamma d}{(1+|v|^2)^2}.$$

We see the equation as

$$\theta(\varepsilon)\partial_t g^\varepsilon = -\mathcal{L}_\varepsilon g^\varepsilon,$$

where

$$\mathcal{L}_\varepsilon := -\Delta_v + W(v) + \varepsilon v\cdot\nabla_x = -(Q-\varepsilon v\cdot\nabla_x)$$

and

$$Q := -\Delta_v + W(v).$$

We operate a Fourier transform in x and since the operator Q has coefficient that do not depend on x, we get:

$$\theta(\varepsilon)\partial_t\hat{g}^\varepsilon = -\mathcal{L}_\eta\hat{g}^\varepsilon, \tag{3.1.8}$$

where

$$\mathcal{L}_\eta := -\Delta_v + W(v) + \mathrm{i}\eta v_1$$

with

$$\eta := \varepsilon|\xi| \quad \text{and} \quad v_1 := v\cdot\frac{\xi}{|\xi|},$$

where ξ being the space Fourier variable.

The operator $\mathcal{L}_\eta$ is an unbounded self-adjoint operator acting on L^2. Its domain is given by

$$D(\mathcal{L}_\eta) = \big\{g\in L^2(\mathbb{R}^d)\ ;\ \Delta_v g\in L^2(\mathbb{R}^d), v_1 g\in L^2(\mathbb{R}^d)\big\}.$$

Main results.

Theorem 3.1.1 (Eigen-solution for the Fokker-Planck operator)**.** *Assume that* $d<\beta<d+4$ *with* $\beta\neq d+1$*. Let* $\eta_0>0$ *and* $\lambda_0>0$ *small enough. Then, for all* $\eta\in[0,\eta_0]$*, there exists a unique eigen-couple* $\big(\mu(\eta),M_\eta\big)$ *in* $\{\mu\in\mathbb{C},|\mu|\leqslant\eta^{\frac{2}{3}}\lambda_0\}\times L^2(\mathbb{R}^d,\mathbb{C})$*, solution to the spectral problem*

$$\mathcal{L}_\eta(M_{\mu,\eta})(v) = \big[-\Delta_v + W(v) + \mathrm{i}\eta v_1\big]M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v),\ \ v\in\mathbb{R}^d. \tag{3.1.9}$$

Moreover,

1. The following convergence in the Sobolev space $H^1(\mathbb{R}^d)$ *holds:*

$$\big\|M_\eta - M\big\|_{H^1(\mathbb{R}^d)} \underset{\eta\to 0}{\longrightarrow} 0. \tag{3.1.10}$$

2. The eigenvalue $\mu(\eta)$ is given by

$$\mu(\eta) = \bar{\mu}(-\eta) = \kappa|\eta|^{\frac{\beta-d+2}{3}}\big(1 + O(|\eta|^{\frac{\beta-d+2}{3}})\big), \tag{3.1.11}$$

where κ is a positive constant given by

$$\kappa = -2C_\beta^2 \int_{\{s_1>0\}} s_1|s|^{-\gamma}\mathrm{Im}H_0(s)\mathrm{d}s, \tag{3.1.12}$$

and where H_0 is the unique solution to the equation

$$\big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|^2} + \mathrm{i}s_1\big]H_0(s) = 0, \quad \forall s \in \mathbb{R}^d \setminus \{0\}, \tag{3.1.13}$$

satisfying

$$\int_{\{|s_1|\geqslant 1\}} |H_0(s)|^2\mathrm{d}s < \infty \quad \text{and} \quad H_0(s) \underset{0}{\sim} |s|^{-\gamma}. \tag{3.1.14}$$

Introduce V, the space defined by

$$V := \left\{ f : \mathbb{R}^d \to \mathbb{R}, \int_{\mathbb{R}^d} \frac{|f|^2}{F}\,\mathrm{d}v < \infty \text{ and } \int_{\mathbb{R}^d} \left|\nabla_v\left(\frac{f}{F}\right)\right|^2 F\mathrm{d}v < \infty \right\},$$

V' being its dual, and

$$Y := \big\{f \in L^2\big([0,T]\times\mathbb{R}^d;V\big);\ \theta(\varepsilon)\partial_t f + \varepsilon v\cdot\nabla_x f \in L^2\big([0,T]\times\mathbb{R}^d;V'\big)\big\}.$$

Theorem 3.1.2 (Fractional diffusion limit for the Fokker-Planck equation)**.**

Assume that $d < \beta < d+4$ with $\beta \neq d+1$. Assume that $f_0 \in L^1(\mathbb{R}^{2d})$ is a non-negative function in $L^2_{F^{-1}}(\mathbb{R}^{2d}) \cap L^\infty_{F^{-1}}(\mathbb{R}^{2d})$. Let f^ε be the solution of (3.1.3) in Y with initial data f_0, with $\theta(\varepsilon) = \varepsilon^{\frac{\beta-d+2}{3}}$. Let κ be the constant given by (3.1.12). Then, f^ε converges weakly star in $L^\infty\big([0,T],\ L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ towards $\rho(t,x)F(v)$ where $\rho(t,x)$ is the solution to

$$\partial_t\rho + \kappa(-\Delta)^{\frac{\beta-d+2}{6}}\rho = 0, \quad \rho(0,x) = \int_{\mathbb{R}^d} f_0\mathrm{d}v. \tag{3.1.15}$$

Remark 3.1.3. The hypothesis $\beta \neq d+1$ is technical. It avoids to introduce logarithmic terms in the expression of $\mu(\eta)$.

Ideas of the proof and outline of the chapter.

The proof of Theorem 3.1.1 is done in two main steps, both based on the Implicit Function Theorem (IFT). First, we consider what we call a *penalized equation*, given by

$$\begin{cases} \big[-\Delta_v + W(v) + \mathrm{i}\eta v_1\big]M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v) - \langle M_{\mu,\eta} - M, \Phi\rangle\Phi(v),\ v \in \mathbb{R}^d, \\ M_{\mu,\eta} \in L^2(\mathbb{R}^d). \end{cases} \tag{3.1.16}$$

where Φ is a function, that satisfies some assumptions, that we will determine later. The additional term allows us to avoid the problem of reconnection by ensuring existence of a solution to equation (3.1.16) on the whole space $\mathbb{R}^d$ for any η and μ. This is one of the key points of this method. Also, note that the sign before the scalar product $\langle M_{\mu,\eta} - M, \Phi\rangle$ is important.

The aim of the first step is to show the existence of a unique solution for equation (3.1.16) for η and μ fixed, which is the purpose of Section 2. As we said above, we will decompose the operator "$-\Delta_v + W(v) + \mathrm{i}\eta v_1 - \mu$" in two parts. The first one is chosen such that it admits an inverse that is continuous as a linear operator between two suitable functional spaces, continuous with respect to the parameters η and μ and compact at $\eta = \mu = 0$. The second part of the operator is left in the right-hand side of the equation, i.e. is considered as a source term. The invertibility of the first part is the subject of the first subsection, and it is based on an elaborated version of the Lax-Milgram theorem. While the study of the inverse operator and its properties is the subject of the second subsection whose main result is the existence of solutions for equation (3.1.16).

In the second step, to ensure that the additional term vanishes, we have to chose $\mu(\eta)$ obtained via the Implicit Function Theorem around the point $(\mu, \eta) = (0, 0)$. The study of this constraint is the subject of a large part of section 3 which is composed of three subsections. The first one is dedicated to the L^2 estimates for the solution of the penalized equation (3.1.16). It consists in improving the space to which the solution found by Lax-Milgram belongs. It is the objective of the second subsection. The last subsection is dedicated to the approximation of the eigenvalue and the computation of the diffusion coefficient.

The last section is devoted to the proof of Theorem 3.1.2. It consists of two subsections, a priori estimates and limiting process in the weak formulation of equation (3.1.8).

3.2 Existence of solutions for the penalized equation

We start this section by some notations and definition of the considered operators. Let $\mu = \lambda\eta^{\frac{2}{3}}$ with $\lambda \in \mathbb{C}$ and let denote by $L_{\lambda,\eta}$ the operator

$$L_{\lambda,\eta} := -\Delta_v + \tilde{W}(v) + \mathrm{i}\eta v_1 - \lambda\eta^{\frac{2}{3}},$$

where

$$\tilde{W}(v) := \frac{\gamma(\gamma - d + 2)}{1 + |v|^2}.$$

Let denote by $V := \tilde{W} - W$. We have

$$V(v) = \frac{\gamma(\gamma + 2)}{(1 + |v|^2)^2}.$$

We will rewrite the equation (3.1.16) as follows

$$\begin{cases} L_{\lambda,\eta}(M_{\lambda,\eta}) = V(v)M_{\lambda,\eta} - \langle M_{\lambda,\eta} - M, \Phi\rangle\Phi, \ v \in \mathbb{R}^d, \\ \\ M_{\lambda,\eta} \in L^2(\mathbb{R}^d). \end{cases} \tag{3.2.1}$$

The two equations (3.1.16) and (3.2.1) are equivalent.

Remark 3.2.1.

1. Since $L_{\lambda,0}$ does not depend on λ, let's denote it by L_0, $L_0 := L_{\lambda,0}$.
2. If $\bar{\Phi}(-v) = \Phi(v)$ and $M_{\lambda,\eta}(v_1, v')$ satisfies the equation (3.2.1), then $\overline{M}_{\bar{\lambda},\eta}(-v_1, v')$ satisfies also (3.2.1), since the potential W is symmetric for a symmetric equilibrium M. Note that this is where the symmetry of the equilibrium M is used and therefore this is a "non-drift condition".
3. Note that the splitting of the potential W into $\tilde{W}$ and V is crucial in our study. It plays a very important role whether in the invertibility of the operator $L_{\lambda,\eta}$ or in the compactness of its inverse at the point $(\lambda, \eta) = (0, 0)$.

3.2.1 Coercivity and Lax-Milgram theorem

The purpose of this subsection is to show that the operator $L_{\lambda,\eta}$ defined above is invertible. For this, we are going to define a Hilbert space $\mathcal{H}_\eta$ as well as a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}_\eta}$ on which we apply a Lax-Milgram theorem.

Definition 3.2.2.

• We define the Hilbert space $\mathcal{H}_\eta$ as being the completion of the space $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ for the norm $\|\cdot\|_{\mathcal{H}_\eta}$ induced from the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}_\eta}$

$$\mathcal{H}_\eta := \overline{\left\{\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{C});\ \|\psi\|^2_{\tilde{\mathcal{H}}_\eta} := \langle\psi,\psi\rangle_{\tilde{\mathcal{H}}_\eta} < +\infty\right\}}, \tag{3.2.2}$$

where

$$\langle\psi,\phi\rangle_{\tilde{\mathcal{H}}_\eta} := \int_{\mathbb{R}^d} \nabla_v\left(\frac{\psi}{M}\right)\cdot\nabla_v\left(\frac{\bar{\phi}}{M}\right) M^2 \mathrm{d}v + \int_{\mathbb{R}^d} V\psi\bar{\phi}\,\mathrm{d}v + \eta\int_{\mathbb{R}^d} |v_1|\psi\bar{\phi}\,\mathrm{d}v,$$

and where $V(v) := \tilde{W}(v) - W(v) = \frac{\gamma(\gamma+2)}{(1+|v|^2)^2} > 0$ for all $v \in \mathbb{R}^d$.

We have the embeddings

$$\mathcal{H}_\eta \subseteq \mathcal{H}_{\eta^*} \subseteq \mathcal{H}_0, \quad \forall\ 0 \leqslant \eta^* \leqslant \eta$$

since $\|\cdot\|_{\tilde{\mathcal{H}}_0} \leqslant \|\cdot\|_{\tilde{\mathcal{H}}_{\eta^*}} \leqslant \|\cdot\|_{\tilde{\mathcal{H}}_\eta}$ for all $0 \leqslant \eta^* \leqslant \eta$.

• We define the sesquilinear form a on $\mathcal{H}_\eta \times \mathcal{H}_\eta$ by

$$a(\psi,\phi) := \int_{\mathbb{R}^d} \nabla_v\left(\frac{\psi}{M}\right)\cdot\nabla_v\left(\frac{\bar{\phi}}{M}\right) M^2 \mathrm{d}v + \int_{\mathbb{R}^d} V\psi\bar{\phi}\,\mathrm{d}v + \mathrm{i}\eta\int_{\mathbb{R}^d} v_1\psi\bar{\phi}\,\mathrm{d}v - \lambda\eta^{\frac{2}{3}}\int_{\mathbb{R}^d}\psi\bar{\phi}\,\mathrm{d}v.$$

Remark 3.2.3.

1. Note that $a(\psi,\psi) \neq \|\psi\|_{\tilde{\mathcal{H}}_\eta}$.

2. Note that the sesquilinear form a depends on λ and η and in order to simplify the notation, we omit the subscript when no confusion is possible.

3. Let us denote by $\tilde{Q}$ the operator $\tilde{Q} := -\Delta_v + \tilde{W}(v)$. We have $\tilde{Q} = Q + V$. Thus, the operator $\tilde{Q}$ is dissipative since

$$\int_{\mathbb{R}^d}\tilde{Q}(\psi)\psi\,\mathrm{d}v = \int_{\mathbb{R}^d} Q(\psi)\psi\,\mathrm{d}v + \int_{\mathbb{R}^d} V|\psi|^2\,\mathrm{d}v = \int_{\mathbb{R}^d}\left|\nabla_v\left(\frac{\psi}{M}\right)\right|^2 M^2 + V|\psi|^2\,\mathrm{d}v \geqslant 0.$$

   Note that we have also the equality

$$\int_{\mathbb{R}^d}\tilde{Q}(\psi)\psi\,\mathrm{d}v = \int_{\mathbb{R}^d}\left|\nabla_v\psi\right|^2\mathrm{d}v + c_{\gamma,d}\int_{\mathbb{R}^d}\frac{|\psi|^2}{\langle v\rangle^2}\,\mathrm{d}v,$$

   with $c_{\gamma,d} := \gamma(\gamma-d+2)$. Observe that $c_{\gamma,d} < 0$ for $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$ with $d > 4$.

4. Since $\tilde{Q} = Q + V$ then, the sesquilinear form a can be written as follows:

$$a(\psi,\phi) = \int_{\mathbb{R}^d}\nabla_v\psi\cdot\nabla_v\bar{\phi}\,\mathrm{d}v + c_{\gamma,d}\int_{\mathbb{R}^d}\frac{\psi\bar{\phi}}{\langle v\rangle^2}\,\mathrm{d}v + \mathrm{i}\eta\int_{\mathbb{R}^d} v_1\psi\bar{\phi}\,\mathrm{d}v - \lambda\eta^{\frac{2}{3}}\int_{\mathbb{R}^d}\psi\bar{\phi}\,\mathrm{d}v.$$

Lemma 3.2.4. *The norm defined by*

$$\|\psi\|^2_{\mathcal{H}_\eta} := \int_{\mathbb{R}^d}\left|\nabla_v\psi\right|^2\mathrm{d}v + \int_{\mathbb{R}^d}\frac{|\psi|^2}{\langle v\rangle^2}\,\mathrm{d}v + \eta\int_{\mathbb{R}^d}|v_1||\psi|^2\,\mathrm{d}v$$

is induced from the scalar product

$$\langle\psi,\phi\rangle_{\mathcal{H}_\eta} := \int_{\mathbb{R}^d}\nabla_v\psi\cdot\nabla_v\bar{\phi}\,\mathrm{d}v + \int_{\mathbb{R}^d}\frac{\psi\bar{\phi}}{\langle v\rangle^2}\,\mathrm{d}v + \eta\int_{\mathbb{R}^d}|v_1|\psi\bar{\phi}\,\mathrm{d}v,$$

and the two norms $\|\cdot\|_{\mathcal{H}_\eta}$ and $\|\cdot\|_{\tilde{\mathcal{H}}_\eta}$ are equivalent, i.e., there are two positive constants C_1 and C_2 such that

$$C_1\|\psi\|_{\tilde{\mathcal{H}}_\eta} \leqslant \|\psi\|_{\mathcal{H}_\eta} \leqslant C_2\|\psi\|_{\tilde{\mathcal{H}}_\eta}, \quad \forall\psi\in\mathcal{H}_\eta.$$

To prove this Lemma, we need the Hardy-Poincaré inequality that we recall in the following

Lemma 3.2.5 (Hardy-Poincaré inequality [BDGV10])**.** *Let $d \geqslant 1$ and $\alpha_* = \frac{2-d}{2}$. For any*

$\alpha < 0$, and $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$ for $d \geqslant 3$, there is a positive constant $\Lambda_{\alpha,d}$ such that

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 (D+|x|^2)^{\alpha-1} \mathrm{d}x \leqslant \int_{\mathbb{R}^d} |\nabla f|^2 (D+|x|^2)^{\alpha} \mathrm{d}x \tag{3.2.3}$$

holds for any function $f \in H^1\big((D+|x|^2)^\alpha \mathrm{d}x\big)$ and any $D \geqslant 0$, under the additional condition $\int_{\mathbb{R}^d} f (D+|x|^2)^{\alpha-1}\mathrm{d}x = 0$ and $D > 0$ if $\alpha < \alpha_*$.

Remark 3.2.6. For $f = \dfrac{g}{M}$, $D = 1$ and $\alpha = -\gamma$ in the previous lemma, the inequality becomes

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} \frac{|g|^2}{\langle v\rangle^2} \mathrm{d}v \leqslant \int_{\mathbb{R}^d} \left|\nabla_v \left(\frac{g}{M}\right)\right|^2 M^2 \,\mathrm{d}v, \tag{3.2.4}$$

and the orthogonality condition becomes

$$\int_{\mathbb{R}^d} \frac{gM}{\langle v\rangle^2} \mathrm{d}v = 0 \tag{3.2.5}$$

since $-\gamma < \frac{2-d}{2} =: \alpha_*$ for $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$.
If we denote by

$$\mathcal{P}(g) := \left(\int_{\mathbb{R}^d} \frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right)^{-1} \int_{\mathbb{R}^d} \frac{gM}{\langle v\rangle^2}\mathrm{d}v.$$

Then, the inequality (3.2.4) can be written for all $g \in \mathcal{H}_0$

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} \frac{\left|g - \mathcal{P}(g)M\right|^2}{\langle v\rangle^2} \mathrm{d}v \leqslant \int_{\mathbb{R}^d} \left|\nabla_v \left(\frac{g}{M}\right)\right|^2 M^2 \,\mathrm{d}v. \tag{3.2.6}$$

Proof of Lemma 3.2.4. Let's start with the right inequality: $\|\psi\|_{\mathcal{H}_\eta} \leqslant C_2 \|\psi\|_{\tilde{\mathcal{H}}_\eta}$. Let $\psi \in \mathcal{H}_\eta$. Then, since $M \in L^2(\mathbb{R}^d)$, by Cauchy-Schwarz inequality we get

$$\left|\int_{\mathbb{R}^d} \frac{\psi M}{\langle v\rangle^2}\mathrm{d}v\right| \leqslant \left(\int_{\mathbb{R}^d} \frac{|\psi|^2}{\langle v\rangle^4}\mathrm{d}v\right)^{\frac12} \left(\int_{\mathbb{R}^d} M^2 \mathrm{d}v\right)^{\frac12} \leqslant \frac{1}{\gamma(\gamma+2)} \|\psi\|_{\tilde{\mathcal{H}}_\eta}.$$

Now, since the function $\psi - \mathcal{P}(\psi)M$ satisfies the condition (3.2.5), $\mathcal{P}\big(\psi - \mathcal{P}(\psi)M\big) = 0$, then the inequality (3.2.4) can be used and therefore

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|\psi|^2}{\langle v\rangle^2}\mathrm{d}v &= \int_{\mathbb{R}^d} \frac{|\psi - \mathcal{P}(\psi)M + \mathcal{P}(\psi)M|^2}{\langle v\rangle^2}\mathrm{d}v \\
&\leqslant 2\left(\Lambda_{\alpha,d}^{-1} \int_{\mathbb{R}^d} \left|\nabla_v\left(\frac{\psi}{M}\right)\right|^2 M^2\,\mathrm{d}v + |\mathcal{P}(\psi)|^2 \int_{\mathbb{R}^d} \frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right) \\
&\leqslant 2\left(\Lambda_{\alpha,d}^{-1} \|\psi\|^2_{\tilde{\mathcal{H}}_\eta} + \frac{1}{\gamma^2(\gamma+2)^2}\left(\int_{\mathbb{R}^d} \frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right)^{-1} \|\psi\|^2_{\tilde{\mathcal{H}}_\eta}\right) \\
&\leqslant C_{\gamma,d} \|\psi\|^2_{\tilde{\mathcal{H}}_\eta}.
\end{aligned}$$

We have by the first point of Remark 3.2.3

$$\int_{\mathbb{R}^d} \tilde{Q}(\psi)\psi \, \mathrm{d}v = \int_{\mathbb{R}^d} \left|\nabla_v\left(\frac{\psi}{M}\right)\right|^2 M^2 + V|\psi|^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} \left|\nabla_v \psi\right|^2 \mathrm{d}v + c_{\gamma,d} \int_{\mathbb{R}^d} \frac{|\psi|^2}{\langle v\rangle^2} \, \mathrm{d}v.$$

From where we get

$$\int_{\mathbb{R}^d} \left|\nabla_v \psi\right|^2 \mathrm{d}v \leqslant \int_{\mathbb{R}^d} \left|\nabla_v\left(\frac{\psi}{M}\right)\right|^2 M^2 + V|\psi|^2 \, \mathrm{d}v + |c_{\gamma,d}| \int_{\mathbb{R}^d} \frac{|\psi|^2}{\langle v\rangle^2} \, \mathrm{d}v \leqslant \left(1+\tilde{C}_{\gamma,d}\right)\|\psi\|^2_{\tilde{\mathcal{H}}_\eta}.$$

Hence,

$$\|\psi\|_{\mathcal{H}_\eta} \leqslant C_2 \|\psi\|_{\tilde{\mathcal{H}}_\eta},$$

with $C_2 := \sqrt{2(1+\tilde{C}_{\gamma,d})}$ a positive constant which depends only on γ and d.
To get inequality $C_1\|\psi\|_{\tilde{\mathcal{H}}_\eta} \leqslant \|\psi\|_{\mathcal{H}_\eta}$, it is enough just to write

$$\int_{\mathbb{R}^d} \left|\nabla_v\left(\frac{\psi}{M}\right)\right|^2 M^2 + V|\psi|^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} \left|\nabla_v \psi\right|^2 \mathrm{d}v + c_{\gamma,d} \int_{\mathbb{R}^d} \frac{|\psi|^2}{\langle v\rangle^2} \, \mathrm{d}v \leqslant (1+|c_{\gamma,d}|)\|\psi\|^2_{\mathcal{H}_\eta}.$$

Hence,

$$C_1 \|\psi\|_{\tilde{\mathcal{H}}_\eta} \leqslant \|\psi\|_{\mathcal{H}_\eta},$$

with $C_1 := (2+|c_{\gamma,d}|)^{-\frac{1}{2}}$ a positive constant which depends only on γ and d. □

In the remainder of this section, we will work with the norm $\|\cdot\|_{\mathcal{H}_\eta}$.

Before moving on to the continuity of a, we will prove a Poincaré type inequality which we give in the following lemma:

Lemma 3.2.7. *Let $\eta > 0$ be fixed. Then, there exists a constant $C_0 > 0$, independent of η such that the following inequality holds true*

$$\|\psi\|_{L^2(\mathbb{R}^d)} \leqslant C_0 \eta^{-\frac{1}{3}} \|\psi\|_{\mathcal{H}_\eta}, \quad \forall \psi \in \mathcal{H}_\eta.$$

Proof. We will split the integral of $\|\psi\|^2_{L^2(\mathbb{R}^2)}$ into two parts $\{|v_1| \leqslant \eta^{-\frac{1}{3}}\}$ and $\{|v_1| \geqslant \eta^{-\frac{1}{3}}\}$.
• On $\{|v_1| \geqslant \eta^{-\frac{1}{3}}\}$, we simply have

$$\eta^{\frac{2}{3}} \int_{\{|v_1|\geqslant \eta^{-\frac{1}{3}}\}} |\psi|^2 \mathrm{d}v \leqslant \int_{\{|v_1|\geqslant \eta^{-\frac{1}{3}}\}} \eta |v_1| |\psi|^2 \mathrm{d}v \leqslant \|\psi\|^2_{\mathcal{H}_\eta}.$$

• While on $\{|v_1| \leqslant \eta^{-\frac{1}{3}}\}$, we introduce the function ζ_η defined by: $\zeta_\eta(v_1) := \zeta\big(\eta^{\frac{1}{3}} v_1\big)$, where $\zeta \in C^\infty(\mathbb{R})$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ on $B(0,1)$ and $\zeta \equiv 0$ outside of $B(0,2)$.

Then, one has

$$\begin{aligned}\eta^{\frac{2}{3}}\int_{\{|v_1|\leqslant\eta^{-\frac{1}{3}}\}}|\psi|^2\mathrm{d}v &\leqslant \eta^{\frac{2}{3}}\int_{\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\}}|\zeta_\eta\psi|^2\mathrm{d}v\\ &= \eta^{\frac{2}{3}}\int_{\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\}}\left|\int\int_{-2\eta^{-\frac{1}{3}}}^{v_1}\partial_{w_1}(\zeta_\eta\psi)\mathrm{d}w_1\right|^2\mathrm{d}v'\mathrm{d}v_1\\ &\leqslant \eta^{\frac{2}{3}}\int_{\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\}}\Big(\int_{-2\eta^{-\frac{1}{3}}}^{v_1}\mathrm{d}w_1\Big)\Big(\int_{-2\eta^{-\frac{1}{3}}}^{v_1}|\partial_{w_1}(\zeta_\eta\psi)|^2\mathrm{d}w_1\Big)\mathrm{d}v\\ &\leqslant 16\big\|\partial_{v_1}(\zeta_\eta\psi)\big\|^2_{L^2(\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\})}.\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\big|\partial_{v_1}(\zeta_\eta\psi)\big|^2 &= |\zeta_\eta'\psi|^2+|\zeta_\eta\partial_{v_1}\psi|^2+\zeta_\eta\zeta_\eta'\big(\bar\psi\partial_{v_1}\psi+\psi\overline{\partial_{v_1}\psi}\big)\\ &\leqslant \big(\eta^{-1}|v_1|^{-1}|\zeta_\eta'|^2\big)\eta|v_1||\psi|^2+|\partial_{v_1}\psi|^2+2\eta^{-\frac{1}{2}}|v_1|^{-\frac{1}{2}}|\zeta_\eta'|\big(\eta^{\frac{1}{2}}|v_1|^{\frac{1}{2}}|\psi|\big)|\partial_{v_1}\psi|,\end{aligned}$$

and since $\zeta_\eta'=0$ except on $\{\eta^{-\frac{1}{3}}\leqslant|v_1|\leqslant 2\eta^{-\frac{1}{3}}\}$, where $|\eta^{-\frac{1}{2}}|v_1|^{-\frac{1}{2}}\zeta_\eta'(v_1)|\leqslant C$. Then, by integrating the last inequality and using Cauchy-Schwarz for the last term we get

$$\big\|\partial_{v_1}(\zeta_\eta\psi)\big\|^2_{L^2(\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\})}\lesssim\int_{\{|v_1|\geqslant\eta^{-\frac{1}{3}}\}}\eta|v_1||\psi|^2\mathrm{d}v+\|\partial_{v_1}\psi\|^2_{L^2(\{|v_1|\leqslant 2\eta^{-\frac{1}{3}}\})}\lesssim\|\psi\|^2_{\mathcal{H}_\eta}.$$

Note that we used the inclusion $\{\eta^{-\frac{1}{3}}\leqslant|v_1|\leqslant 2\eta^{-\frac{1}{3}}\}\subset\{|v_1|\geqslant\eta^{-\frac{1}{3}}\}$. Hence, the inequality of Lemma 3.2.7 holds. ☐

Lemma 3.2.8. *The sesquilinear form a is continuous on $\mathcal{H}_\eta\times\mathcal{H}_\eta$. Moreover, there exists a constant $C>0$, independent of λ and η such that, for all $\psi,\phi\in\mathcal{H}_\eta$*

$$|a(\psi,\phi)|\leqslant C\|\psi\|_{\mathcal{H}_\eta}\|\phi\|_{\mathcal{H}_\eta}.$$

Proof. It follows from the previous lemma that allows to handle the term $\eta^{\frac{2}{3}}\lambda\int\psi\bar\phi$. ☐

Remark 3.2.9.

1. By application of Riesz's theorem to continuous sesquilinear forms, there exists a continuous linear map $A_{\lambda,\eta}\in\mathcal{L}(\mathcal{H}_\eta)$ such that $a(\psi,\phi)=\langle A_{\lambda,\eta}\psi,\phi\rangle_{\mathcal{H}_\eta}$ for all $\psi,\phi\in\mathcal{H}_\eta$.

2. Note that $A_{\lambda,\eta}$ depends on λ and η since the form a depends on these last parameters.

Lemma 3.2.10. *Let $\eta>0$ and $\lambda\in\mathbb{C}$ fixed, such that $|\lambda|\leqslant\lambda_0$ with λ_0 small enough. Let $A_{\lambda,\eta}$ be the linear operator representing the sesquilinear form a. Then, there exists a* *constant $C > 0$, independent of λ and η such that*

$$\|\psi\|_{\mathcal{H}_\eta} \leqslant C\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}, \quad \forall \psi \in \mathcal{H}_\eta. \tag{3.2.7}$$

Proof. We have for all $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$: $|a + \mathrm{i}b + z| \geqslant |a| - |z|$. Now, applying this inequality to $|a(\psi,\psi)|$ and using the Lemma 3.2.7 for the term which contains λ, we write

$$\begin{aligned} |a(\psi,\psi)| &= \left| \int_{\mathbb{R}^d} \big(|\nabla_v \psi|^2 + c_\gamma \frac{|\psi|^2}{\langle v\rangle^2} + \mathrm{i}\eta v_1 |\psi|^2 - \lambda \eta^{\frac{2}{3}} |\psi|^2 \big) \mathrm{d}v \right| \\ &\geqslant \left| \int_{\mathbb{R}^d} \big(|\nabla_v \psi|^2 + c_\gamma \frac{|\psi|^2}{\langle v\rangle^2} \big) \mathrm{d}v \right| - |\lambda| \eta^{\frac{2}{3}} \|\psi\|_2^2 \\ &\geqslant \|\psi\|_{\mathcal{H}_0}^2 - C_0 |\lambda| \|\psi\|_{\mathcal{H}_\eta}^2. \end{aligned}$$

Then, since $|a(\psi,\psi)| = |\langle A_{\lambda,\eta}\psi, \psi\rangle_{\mathcal{H}_\eta}| \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta} \|\psi\|_{\mathcal{H}_\eta}$, we get

$$\|\psi\|_{\mathcal{H}_0}^2 = \|\nabla_v \psi\|_2^2 + c_\gamma \left\| \frac{\psi}{\langle v \rangle} \right\|_2^2 \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta} \|\psi\|_{\mathcal{H}_\eta} + C_0 |\lambda| \|\psi\|_{\mathcal{H}_\eta}^2. \tag{3.2.8}$$

Let denote

$$I_1^\eta := \int_{\{|v_1| \leqslant \eta^{-\frac{1}{3}}\}} \eta |v_1| |\psi|^2 \mathrm{d}v \quad \text{and} \quad I_2^\eta := \int_{\{|v_1| \geqslant \eta^{-\frac{1}{3}}\}} \eta |v_1| |\psi|^2 \mathrm{d}v.$$

Note that $\|\psi\|_{\mathcal{H}_\eta}^2 = \|\psi\|_{\mathcal{H}_0}^2 + I_1^\eta + I_2^\eta$. To estimate I_1^η and I_2^η, we need the following two steps.

Step 1: Estimation of I_1^η. Let ζ_η be the function defined in the proof of Lemma 3.2.7. Then,

$$I_1^\eta \leqslant \int_{\{|v_1| \leqslant 2\eta^{-\frac{1}{3}}\}} \eta |v_1| |\zeta_\eta \psi|^2 \mathrm{d}v \leqslant \eta^{\frac{2}{3}} \int_{\{|v_1| \leqslant 2\eta^{-\frac{1}{3}}\}} |\zeta_\eta \psi|^2 \mathrm{d}v \leqslant 16 \big\| \partial_{v_1} (\zeta_\eta \psi) \big\|_{L^2(\{|v_1| \leqslant 2\eta^{-\frac{1}{3}}\})}^2.$$

By the same calculations as in the proof of Lemma 3.2.7 for $\big\| \partial_{v_1} (\zeta_\eta \psi) \big\|_{L^2(\{|v_1| \leqslant 2\eta^{-\frac{1}{3}}\})}^2$, we get

$$I_1^\eta \leqslant C_1 \Big(I_2^\eta + \|\nabla_v \psi\|_2^2 \Big). \tag{3.2.9}$$

Step 2: Estimation of I_2^η. Let χ_η this time be the function defined by $\chi_\eta(v_1) := \chi(\eta^{\frac{1}{3}} v_1)$ with $\chi \in C^\infty(\mathbb{R})$ such that: $-1 \leqslant \chi \leqslant 1$, $\chi \equiv -1$ on $]-\infty, -1]$, $\chi \equiv 1$ on $[1, +\infty[$ and $\chi \equiv 0$ on $B(0, \frac{1}{2})$. Then,

$$I_2^\eta := \int_{\{|v_1| \geqslant \eta^{-\frac{1}{3}}\}} \eta |v_1| |\psi|^2 \mathrm{d}v \leqslant \int_{\{|v_1| \geqslant \frac{1}{2}\eta^{-\frac{1}{3}}\}} \eta v_1 \chi_\eta \psi \bar{\psi} \mathrm{d}v.$$

By integrating the equation of ψ multiplied by $\chi_\eta \bar{\psi}$ over $\{|v_1| \geqslant \frac{1}{2}\eta^{-\frac{1}{3}}\}$ and taking the imaginary part, we obtain

$$\int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \eta v_1 \chi_\eta \psi\bar{\psi} \mathrm{d}v = \mathrm{Im}\bigg(a(\psi, \chi_\eta\psi) - \int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \big[\nabla_v\psi\cdot\nabla_v(\chi_\eta\bar{\psi}) - \lambda\eta^{\frac{2}{3}}\chi_\eta\psi\bar{\psi}\big]\mathrm{d}v\bigg).$$

For the first term, by Cauchy-Schwarz: $|\mathrm{Im}\, a(\psi,\chi_\eta\psi)| \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\chi_\eta\psi\|_{\mathcal{H}_\eta}$, and for the last term, by the Lemma 3.2.7:

$$\left|\mathrm{Im}\lambda\eta^{\frac{2}{3}}\int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \chi_\eta\psi\bar{\psi}\mathrm{d}v\right| \leqslant C_0|\lambda|\|\psi\|^2_{\mathcal{H}_\eta}.$$

Finally, for the second term, we write

$$\begin{aligned}
\left|\mathrm{Im}\int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \nabla_v\psi\cdot\nabla_v(\chi_\eta\bar{\psi})\mathrm{d}v\right| &= \left|\mathrm{Im}\int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \chi'_\eta\bar{\psi}\partial_{v_1}\psi\mathrm{d}v\right| \\
&= \left|\mathrm{Im}\int_{\{\frac{1}{2}\eta^{-\frac{1}{3}}\leqslant|v_1|\leqslant\eta^{-\frac{1}{3}}\}} \chi'_\eta\bar{\psi}\partial_{v_1}\psi\mathrm{d}v\right| \\
&\leqslant 2C_2\left|\int_{\{\frac{1}{2}\eta^{-\frac{1}{3}}\leqslant|v_1|\leqslant\eta^{-\frac{1}{3}}\}} \eta^{\frac{1}{2}}|v_1|^{\frac{1}{2}}|\psi||\partial_{v_1}\psi|\mathrm{d}v\right| \\
&\leqslant 2C_2\big(I_1^\eta\big)^{\frac{1}{2}}\big\|\nabla_v\psi\big\|_2 \qquad \text{(by Cauchy-Schwarz)} \\
&\leqslant C_3\bigg(I_2^\eta + \|\nabla_v\psi\|_2^2\bigg)^{\frac{1}{2}}\big\|\nabla_v\psi\big\|_2, \quad \text{(by the inequality (3.2.9))} \\
&\leqslant \frac{1}{4}I_2^\eta + C\|\nabla_v\psi\|_2^2,
\end{aligned}$$

where we used the inequality: $ab \leqslant C_3a^2+\frac{b^2}{4C_3}$ in the last line and where $C_2 = \sup\limits_{\frac{1}{2}\leqslant|t|\leqslant1} |t^{-\frac{1}{2}}\chi'(t)| = \|\eta^{-\frac{1}{2}}|v_1|^{-\frac{1}{2}}\chi'_\eta\|_{L^\infty(\{\frac{1}{2}\eta^{-\frac{1}{3}}\leqslant|v_1|\leqslant\eta^{-\frac{1}{3}}\})}$, $C_3 = 2\sqrt{C_1}C_2$ and $C = C_3 + \frac{1}{4}$. Therefore,

$$I_2^\eta \leqslant \int_{\{|v_1|\geqslant\frac{1}{2}\eta^{-\frac{1}{3}}\}} \eta v_1\chi_\eta\psi\bar{\psi}\mathrm{d}v \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\chi_\eta\psi\|_{\mathcal{H}_\eta} + \frac{1}{4}I_2^\eta + C\big\|\nabla_v\psi\big\|_2^2 + C_0|\lambda|\|\psi\|^2_{\mathcal{H}_\eta}.$$

Thus,

$$I_2^\eta \leqslant C\bigg(\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\chi_\eta\psi\|_{\mathcal{H}_\eta} + \big\|\nabla_v\psi\big\|_2^2 + |\lambda|\|\psi\|^2_{\mathcal{H}_\eta}\bigg).$$

Recall that we have $\|\nabla_v\psi\|_2^2 \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta}$ thanks to (3.2.8). Hence,

$$I_2^\eta \leqslant C\bigg(\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\chi_\eta\psi\|_{\mathcal{H}_\eta} + |\lambda|\|\psi\|^2_{\mathcal{H}_\eta}\bigg). \tag{3.2.10}$$

It only remains to handle the term $\|\chi_\eta\psi\|_{\mathcal{H}_\eta}$. We have, as in the proof of Lemma 3.2.7,

$$\begin{aligned}\left|\nabla_v(\chi_\eta\psi)\right|^2 &= |\chi_\eta'\psi|^2 + \left|\chi_\eta\nabla_v\psi\right|^2 + \chi_\eta\chi_\eta'\big(\bar{\psi}\partial_{v_1}\psi + \psi\overline{\partial_{v_1}\psi}\big)\\ &\leqslant \big(\eta^{-1}|v_1|^{-1}|\chi_\eta'|^2\big)\eta|v_1||\psi|^2 + \left|\nabla_v\psi\right|^2 + 2\eta^{-\frac{1}{2}}|v_1|^{-\frac{1}{2}}|\chi_\eta'|\big(\eta^{\frac{1}{2}}|v_1|^{\frac{1}{2}}|\psi|\big)|\partial_{v_1}\psi|.\end{aligned}$$

Then, $\|\nabla_v(\chi_\eta\psi)\|_2^2 \leqslant C\big(I_1^\eta + \|\nabla_v\psi\|_2^2\big) \leqslant C\|\psi\|^2_{\mathcal{H}_\eta}$ and therefore, $\|\chi_\eta\psi\|^2_{\mathcal{H}_\eta} \leqslant C\|\psi\|^2_{\mathcal{H}_\eta}$. By injecting this last inequality into (3.2.10), we get

$$I_2^\eta \leqslant C\bigg(\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta} + |\lambda|\|\psi\|^2_{\mathcal{H}_\eta}\bigg). \tag{3.2.11}$$

Thus, by summing (3.2.8), (3.2.9) and (3.2.11), we obtain

$$\|\psi\|^2_{\mathcal{H}_\eta} \leqslant C\bigg(\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta} + |\lambda|\|\psi\|^2_{\mathcal{H}_\eta}\bigg).$$

Finally, we obtain the inequality (3.2.7) by the inequality $ab \leqslant Ca^2 + \frac{b^2}{4C}$ applied to the term $\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta}$, and with λ small enough: $|\lambda| \leqslant \frac{1}{4C}$. □

Lemma 3.2.11 (Complementary Lemma)**.** *Let $\eta > 0$ fixed and let $\lambda_0 > 0$ small enough. Let $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$. Then, for all $\psi, F \in \mathcal{H}_\eta$ such that $|a(\psi,\psi)| \leqslant C\|F\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta}$, the following inequality holds*

$$\|\psi\|_{\mathcal{H}_\eta} \leqslant \tilde{C}\|F\|_{\mathcal{H}_\eta}, \tag{3.2.12}$$

where C and $\tilde{C}$ are two positive constants that do not depend on λ and η.

Proof. The proof is identical to that of the previous Lemma, just replace the inequality $|a(\psi,\psi)| = |\langle A_{\lambda,\eta}\psi,\psi\rangle_{\mathcal{H}_\eta}| \leqslant \|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta}$ by $|a(\psi,\psi)| \leqslant C\|F\|_{\mathcal{H}_\eta}\|\psi\|_{\mathcal{H}_\eta}$. □

Let denote by $\mathcal{H}_\eta'$ the topological dual of $\mathcal{H}_\eta$. By the Riesz Representation Theorem, for all $F \in \mathcal{H}_\eta'$, there exists a unique $f \in \mathcal{H}_\eta$ such that

$$(F,\phi) = \langle f,\phi\rangle_{\mathcal{H}_\eta}, \quad \forall\phi \in \mathcal{H}_\eta,$$

where (F,ϕ) denotes the value taken by $F \in \mathcal{H}_\eta'$ in $\phi \in \mathcal{H}_\eta$. Then, by Remark 3.2.9, the problem

$$a(\psi,\phi) = (F,\phi), \quad \forall\phi \in \mathcal{H}_\eta \tag{3.2.13}$$

is equivalent to the problem $A_{\lambda,\eta}\psi = f$, $f \in \mathcal{H}_\eta$. Therefore, equivalent to the invertibility of $A_{\lambda,\eta}$.

Proposition 3.2.12 (Existence of solution to the the variational problem)**.** *Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Let $\eta \in [0,\eta_0]$ and $\lambda \in \mathbb{C}$ fixed, with $|\lambda| \leqslant \lambda_0$. For all $F \in \mathcal{H}_\eta'$, equation (3.2.13) admits a unique solution $\psi^{\lambda,\eta} \in \mathcal{H}_\eta \subset \mathcal{H}_0$, satisfying the following* *estimate*

$$\|\psi^{\lambda,\eta}\|_{\mathcal{H}_0} \leqslant \|\psi^{\lambda,\eta}\|_{\mathcal{H}_\eta} \leqslant C\|F\|_{\mathcal{H}'_\eta}, \tag{3.2.14}$$

where C is a positive constant that does not depend on λ and η. Moreover, for $F \in L^2_{\langle v\rangle^2} \subset \mathcal{H}'_\eta$ we have

$$\|\psi^{\lambda,\eta}\|_{\mathcal{H}_0} \leqslant \|\psi^{\lambda,\eta}\|_{\mathcal{H}_\eta} \leqslant C\|F\|_{L^2_{\langle v\rangle^2}}, \tag{3.2.15}$$

where $L^2_{\langle v\rangle^2}$ denote the weighted L^2space: $L^2_{\langle v\rangle^2} := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{C};\ \int_{\mathbb{R}^d} |f|^2\langle v\rangle^2 \mathrm{d}v < \infty \right\}$.

Remark 3.2.13.

The sesquilinear form a depends continuously on η and holomorphically on λ.
The solution in the previous Proposition, is for λ and η fixed, and it depends on λ and η since a depends on these last parameters.

Proof of Proposition 3.2.12. This proof was taken from [Kho72] to prove the first statement of the Lax-Milgram lemma [page 235]. We want to prove that the linear map $A_{\lambda,\eta}$ representing the sesquilinear form a is invertible with continuous inverse, since it implies that for all $f \in \mathcal{H}_\eta$, the equation $A_{\lambda,\eta}\psi = f$ admits a unique solution $\psi^{\lambda,\eta} \in \mathcal{H}_\eta$.
First, the inequality (3.2.7) of Lemma 3.2.10, $\|\psi\|_{\mathcal{H}_\eta} \leqslant C\|A_{\lambda,\eta}\psi\|_{\mathcal{H}_\eta}$, shows that $A_{\lambda,\eta}$ is injective with continuous inverse, so it is a topological isomorphism from $\mathcal{H}_\eta$ to $\mathrm{R}(A_{\lambda,\eta})$; in particular $\mathrm{R}(A_{\lambda,\eta})$ is complete and therefore closed in $\mathcal{H}_\eta$, where we denote by $\mathrm{R}(A_{\lambda,\eta})$ the range of the operator $A_{\lambda,\eta}$, i.e., $\mathrm{R}(A_{\lambda,\eta}) := \{f \in \mathcal{H}_\eta;\ f = A_{\lambda,\eta}\psi,\ \psi \in \mathcal{H}_\eta\}$. To show that $A_{\lambda,\eta}$ is surjective, it is enough to prove that $\mathrm{R}(A_{\lambda,\eta})$ is dense; for this, let $\phi_0 \in \mathcal{H}_\eta$ such that $\langle A_{\lambda,\eta}\psi, \phi_0\rangle_{\mathcal{H}_\eta} = 0$ for all $\psi \in \mathcal{H}_\eta$; taking $\psi = \phi_0$ we get $a(\phi_0, \phi_0) = 0$, which gives $\phi_0 = 0$.
The inequality (3.2.14) comes from

$$\|\psi^{\lambda,\eta}\|_{\mathcal{H}_\eta} \leqslant C\|A_{\lambda,\eta}\psi^{\lambda,\eta}\|_{\mathcal{H}_\eta} \leqslant \|f\|_{\mathcal{H}_\eta} \leqslant \|F\|_{\mathcal{H}'_\eta}.$$

For the second one, it comes from the fact that the weighted space $L^2_{\langle v\rangle^2}$ is continuously embedded in $\mathcal{H}'_\eta$. □

We will denote by $T_{\lambda,\eta}$ the inverse operator of $L_{\lambda,\eta}$ for λ and η fixed, i.e., the operator which associates to F the solution $\psi^{\lambda,\eta} =: T_{\lambda,\eta}(F)$.

3.2.2 Implicit Function Theorem

In this subsection, we use the operator $T_{\lambda,\eta}$ to rewrite equation (3.2.1) as a fixed point problem for the identity plus a compact map. Then, the Fredholm Alternative will allow us to apply the Implicit Function Theorem in order to have the existence of solutions.

For this purpose, let's define $F : \{\lambda \in \mathbb{C}; |\lambda| \leqslant \lambda_0\} \times [0, \eta_0] \times \mathcal{H}_0 \longrightarrow \mathcal{H}_0$ by

$$F(\lambda, \eta, h) := h - \mathcal{T}_{\lambda,\eta}(h),$$

with

$$\mathcal{T}_{\lambda,\eta}(h) := T_{\lambda,\eta}\big[Vh - \langle h - M, \Phi\rangle\Phi\big].$$

Note that finding a solution $h(\lambda, \eta)$ solution to $F\big(\lambda, \eta, h(\lambda, \eta)\big) = 0$ gives a solution to the penalized equation by taking $M_{\lambda,\eta} = h(\lambda, \eta)$.

The function Φ satisfies the following assumptions:

1. For all v in $\mathbb{R}^d$, $\Phi(v) = \Phi(-v) > 0$.
2. The function Φ belongs to the weighted Sobolev space $H^1_{\langle v\rangle^2} := H^1(\mathbb{R}^d, \langle v\rangle^2 \mathrm{d}v)$, and for all v in $\mathbb{R}^d$, $\Phi(v) \leqslant \frac{M(v)}{\langle v\rangle^2}$.
3. Even if it means multiplying Φ by a constant, we can take it such that $\langle \Phi, M\rangle = 1$.

For the following, we will take the function $\Phi := c_{\gamma,d}\langle v\rangle^{-2-\gamma}$ which satisfies all the previous assumptions, where $c_{\gamma,d} = \big(\int_{\mathbb{R}^d}\langle v\rangle^{-2-2\gamma}\mathrm{d}v\big)^{-1}$.

Remark 3.2.14. Note that the operator $\mathcal{T}_{\lambda,0}$ does not depend on λ since $T_{\lambda,0}$ does not. Let's denote it by $\mathcal{T}_0$. Also, $\mathcal{T}_{\lambda,\eta}$ is affine with respect to h, we denote by $\mathcal{T}^l_{\lambda,\eta}$ its linear part.

Lemma 3.2.15 (Continuity of $\mathcal{T}_{\lambda,\eta}$)**.** *Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Let $\eta \in [0, \eta_0]$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$. Then,*

1. *The map $\mathcal{T}_{\lambda,\eta} : \mathcal{H}_0 \longrightarrow \mathcal{H}_\eta$ is continuous. Moreover, there exists a constant $C > 0$, independent of λ and η such that*
$$\|\mathcal{T}^l_{\lambda,\eta}(h)\|_{\mathcal{H}_\eta} \leqslant C\|h\|_{\mathcal{H}_0}, \quad \forall h \in \mathcal{H}_0, \tag{3.2.16}$$
and the embedding $\mathcal{T}^l_{\lambda,\eta}(\mathcal{H}_0) \subset \mathcal{H}_\eta \subset \mathcal{H}_0$ holds for all $\eta \in [0, \eta_0]$ and for all $\lambda \in \{|\lambda| \leqslant \lambda_0\}$. Hence the map $\mathcal{T}_{\lambda,\eta} : \mathcal{H}_0 \longrightarrow \mathcal{H}_0$ is continuous.

2. *The map $\mathcal{T}_{\lambda,\eta}$ is continuous with respect to λ and η. Moreover, there exists a constant $C > 0$, independent of λ and η such that, for all $\eta' \in [0, \eta_0]$ and for all $|\lambda'| \leqslant \lambda_0$*
$$\|\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_0} \leqslant C\left(\left|1 - \frac{\eta'}{\eta}\right| + \left|1 - \left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\right)\big(\|h\|_{\mathcal{H}_0} + \|\Phi\|_{L^2_{\langle v\rangle^2}}\big) \tag{3.2.17}$$
and
$$\|\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_0} \leqslant C|\lambda - \lambda'|\big(\|h\|_{\mathcal{H}_0} + \|\Phi\|_{L^2_{\langle v\rangle^2}}\big), \tag{3.2.18}$$
for all $h \in \mathcal{H}_0$.

Proof. **1.** The first point follows from the second inequality of the Proposition 3.2.15. Indeed, we have by (3.2.15), for all $F \in L^2_{\langle v\rangle^2}$

$$\|T_{\lambda,\eta}(F)\|_{\mathcal{H}_\eta} \leqslant C\|F\|_{L^2_{\langle v\rangle^2}}$$

For $h_1, h_2 \in \mathcal{H}_0$, we have $\mathcal{T}_{\lambda,\eta}(h_1) - \mathcal{T}_{\lambda,\eta}(h_2) = \mathcal{T}^l_{\lambda,\eta}(h_1 - h_2)$. Let denote $h := h_1 - h_2$ and $F := Vh - \langle h, \Phi\rangle\Phi \in L^2_{\langle v\rangle^2}$. We have $\mathcal{T}^l_{\lambda,\eta}(h) = T_{\lambda,\eta}(F)$. Thus, by the last inequality and by Cauchy-Schwarz for the term $|\langle h, \Phi\rangle|$, we obtain

$$\|\mathcal{T}^l_{\lambda,\eta}(h)\|_{\mathcal{H}_\eta} \leqslant C\left\|\langle v\rangle^2 V\frac{h}{\langle v\rangle} - \langle h, \Phi\rangle\langle v\rangle\Phi\right\|_2 \leqslant C\Big(\big\|\langle v\rangle^2 V\big\|_\infty + \big\|\langle v\rangle\Phi\big\|_2^2\Big)\left\|\frac{h}{\langle v\rangle}\right\|_{L^2} \leqslant \tilde{C}\|h\|_{\mathcal{H}_0}.$$

The embedding $\mathcal{T}^l_{\lambda,\eta}(\mathcal{H}_0) \subset \mathcal{H}_\eta \subset \mathcal{H}_0$ comes from the previous inequality and the fact that $\|\mathcal{T}^l_{\lambda,\eta}(h)\|_{\mathcal{H}_0} \leqslant \|\mathcal{T}^l_{\lambda,\eta}(h)\|_{\mathcal{H}_\eta}$ for all $h \in \mathcal{H}_0$.

2. Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Let $\eta \in [0, \eta_0]$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$. Recall that $T_{\lambda,\eta}$ is the inverse of $L_{\lambda,\eta} := \tilde{Q} + \mathrm{i}\eta v_1 - \lambda\eta^{\frac{2}{3}}$ with $\tilde{Q} := -\Delta_v + \tilde{W}(v)$.

Continuity of $\mathcal{T}_{\lambda,\eta}$ with respect to λ. Let $\lambda' \in \mathbb{C}$ such that $|\lambda'| \leqslant \lambda_0$. We have for $h \in \mathcal{H}_0$

$$\big[\tilde{Q} + \mathrm{i}\eta v_1 - \lambda\eta^{\frac{2}{3}}\big]\big(T_{\lambda,\eta}\big[Vh - \langle h - M, \Phi\rangle\Phi\big]\big) = Vh - \langle h - M, \Phi\rangle\Phi$$

and

$$\big[\tilde{Q} + \mathrm{i}\eta v_1 - \lambda'\eta^{\frac{2}{3}}\big]\big(T_{\lambda',\eta}\big[Vh - \langle h - M, \Phi\rangle\Phi\big]\big) = Vh - \langle h - M, \Phi\rangle\Phi.$$

Thus, the function $\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h) = (T_{\lambda,\eta} - T_{\lambda',\eta})\big[Vh - \langle h - M, \Phi\rangle\Phi\big]$ satisfies the equation

$$\tilde{Q}[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)] + \mathrm{i}\eta v_1[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)] - \lambda\eta^{\frac{2}{3}}[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)] = (\lambda - \lambda')\eta^{\frac{2}{3}}\mathcal{T}_{\lambda',\eta}(h).$$

Then, by integrating the previous equality multiplied by $\overline{[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)]}$, we obtain

$$a_{\lambda,\eta}\big(\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h), \mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)\big) = (\lambda - \lambda')\eta^{\frac{2}{3}}\int_{\mathbb{R}^d}\mathcal{T}_{\lambda',\eta}(h)\overline{[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)]}\mathrm{d}v.$$

Now, by Cauchy-Schwarz inequality

$$\left|(\lambda - \lambda')\eta^{\frac{2}{3}}\int_{\mathbb{R}^d}\mathcal{T}_{\lambda',\eta}(h)\overline{[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)]}\mathrm{d}v\right| \leqslant |\lambda - \lambda'|\eta^{\frac{2}{3}}\|\mathcal{T}_{\lambda',\eta}(h)\|_2\|\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)\|_2,$$

and by Lemma 3.2.7 we get

$$\left|(\lambda - \lambda')\eta^{\frac{2}{3}}\int_{\mathbb{R}^d}\mathcal{T}_{\lambda',\eta}(h)\overline{[\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)]}\mathrm{d}v\right| \leqslant C|\lambda - \lambda'|\|\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta}\|\mathcal{T}_{\lambda,\eta}(h) - \mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta}.$$

Therefore,

$$\left|a_{\lambda,\eta}\big(\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h),\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h)\big)\right| \leqslant C|\lambda-\lambda'|\|\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta}\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta}.$$

Hence, by the Complementary Lemma 3.2.11, we write

$$\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_0} \leqslant \|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta} \leqslant C|\lambda-\lambda'|\|\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{H}_\eta}.$$

That leads to

$$\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda',\eta}(h)\|_{\mathcal{L}(\mathcal{H}_0)} \leqslant C|\lambda-\lambda'|\big(\|h\|_{\mathcal{H}_0}+\|\langle v\rangle\Phi\|_2\big).$$

Continuity of $\mathcal{T}_{\lambda,\eta}$ with respect to η. Let $\eta' \in [0,\eta_0]$. Without loss of generality, we can assume that $\eta \leqslant \eta'$. Then, as before, we have for $h \in \mathcal{H}_0$

$$\big[\tilde{Q}+\mathrm{i}\eta v_1-\lambda\eta^{\frac{2}{3}}\big]\big(T_{\lambda,\eta}\big[Vh-\langle h-M,\Phi\rangle\Phi\big]\big) = Vh-\langle h-M,\Phi\rangle\Phi$$

and

$$\big[\tilde{Q}+\mathrm{i}\eta' v_1-\lambda\eta'^{\frac{2}{3}}\big]\big(T_{\lambda,\eta'}\big[Vh-\langle h-M,\Phi\rangle\Phi\big]\big) = Vh-\langle h-M,\Phi\rangle\Phi.$$

Thus, the function $\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h) = (T_{\lambda,\eta}-T_{\lambda,\eta'})\big[Vh-\langle h-M,\Phi\rangle\Phi\big]$ satisfies the equation

$$[\tilde{Q}+\mathrm{i}\eta v_1-\lambda\eta^{\frac{2}{3}}](\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)) = [\mathrm{i}(\eta-\eta')v_1-\lambda(\eta^{\frac{2}{3}}-\eta'^{\frac{2}{3}})]\mathcal{T}_{\lambda,\eta'}(h),$$

and integrating this equation against $\overline{[\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)]}$ we get

$$\begin{aligned}a\big(\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h),\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\big) &= \mathrm{i}(\eta-\eta')\int_{\mathbb{R}^d} v_1\mathcal{T}_{\lambda,\eta'}(h)\overline{[\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)]}\mathrm{d}v\\ &\quad-\lambda(\eta^{\frac{2}{3}}-\eta'^{\frac{2}{3}})\int_{\mathbb{R}^d}\mathcal{T}_{\lambda,\eta'}(h)\overline{[\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)]}\mathrm{d}v\\ &=: I_1^{\lambda,\eta,\eta'}+I_2^{\lambda,\eta,\eta'}.\end{aligned}$$

For $I_1^{\lambda,\eta,\eta'}$, we write

$$\begin{aligned}\big|I_1^{\lambda,\eta,\eta'}\big| &\leqslant \left|1-\frac{\eta'}{\eta}\right|\big\|\eta^{\frac{1}{2}}|v_1|^{\frac{1}{2}}\mathcal{T}_{\lambda,\eta'}(h)\big\|_2\big\|\eta^{\frac{1}{2}}|v_1|^{\frac{1}{2}}[\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)]\big\|_2\\ &\leqslant \left|1-\frac{\eta'}{\eta}\right|\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}.\end{aligned}$$

Now for $I_2^{\lambda,\eta,\eta'}$, by using Lemma 3.2.7, we write

$$\begin{aligned}\big|I_2^{\lambda,\eta,\eta'}\big| &\leqslant \eta^{\frac{2}{3}}|\lambda|\left|1-\left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\|\mathcal{T}_{\lambda,\eta'}(h)\|_2\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_2\\ &\leqslant C|\lambda|\left|1-\left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|a_{\lambda,\eta}\big(\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h),\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\big)\right|\\&\leqslant\left(\left|1-\frac{\eta'}{\eta}\right|+C|\lambda|\left|1-\left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\right)\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}.\end{aligned}$$

Which implies, by inequality (3.2.12) of the complementary lemma, that

$$\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}\leqslant\left(\left|1-\frac{\eta'}{\eta}\right|+C|\lambda|\left|1-\left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\right)\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}.$$

Then, since

$$\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_0}\leqslant\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}$$

and since $\eta\leqslant\eta'$ implies that

$$\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_\eta}\leqslant\|\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_{\eta'}}\leqslant C\big(\|h\|_{\mathcal{H}_0}+\|\langle v\rangle\Phi\|_2\big),$$

we get

$$\|\mathcal{T}_{\lambda,\eta}(h)-\mathcal{T}_{\lambda,\eta'}(h)\|_{\mathcal{H}_0}\leqslant C\left(\left|1-\frac{\eta'}{\eta}\right|+C\lambda_0\left|1-\left|\frac{\eta'}{\eta}\right|^{\frac{2}{3}}\right|\right)\big(\|h\|_{\mathcal{H}_0}+\|\langle v\rangle\Phi\|_2\big).$$

Which ends of the proof. ☐

Lemma 3.2.16. *The map $\mathcal{T}_0^l$ is compact.*

Proof. First, since the two functions $g_1:=\langle v\rangle^2V$ and $g_2:=\Phi$ belong to $C_0^1(\mathbb{R}^d,\mathbb{R})$ and $H^1_{\langle v\rangle^2}(\mathbb{R}^d,\mathbb{R})$ respectively, where $C_0^1(\mathbb{R}^d,\mathbb{R})$ denote the space of C^1 functions converging to 0 at infinity as well as their first derivatives, then for $\varepsilon>0$, there exists $g_1^\varepsilon,g_2^\varepsilon\in C_c^\infty(\mathbb{R}^d,\mathbb{R})$ such that $\|g_1^\varepsilon-g_1\|_{W^{1,\infty}}\leqslant\frac{\varepsilon}{2C}$ and $\|g_2^\varepsilon-g_2\|_{H^1_{\langle v\rangle^2}}\leqslant\frac{\varepsilon}{2C}$, where C is the constant of inequality (3.2.16). Now if we denote by $\mathcal{T}_0^\varepsilon$ the operator $\mathcal{T}_0^\varepsilon(h):=T_0\big[g_1^\varepsilon\frac{h}{\langle v\rangle^2}-\langle h,\Phi\rangle g_2^\varepsilon\big]$, then we can write:

$$\begin{aligned}\|\mathcal{T}_0^l(h)-\mathcal{T}_0^\varepsilon(h)\|_{\mathcal{H}_0}&=\big\|T_0\big[(g_1^\varepsilon-g_1)h/\langle v\rangle^2-\langle h,\Phi\rangle(g_2^\varepsilon-g_2)\big]\big\|_{\mathcal{H}_0}\\&\leqslant C\Big(\|g_1^\varepsilon-g_1\|_\infty+\big\|\langle v\rangle\Phi\big\|_2\|g_2^\varepsilon-g_2\|_{L^2_{\langle v\rangle^2}}\Big)\|h\|_{\mathcal{H}_0}\\&\leqslant\varepsilon\|h\|_{\mathcal{H}_0}.\end{aligned}$$

Hence, $\|\mathcal{T}_0^l-\mathcal{T}_0^\varepsilon\|_{\mathcal{L}(\mathcal{H}_0)}\leqslant\varepsilon$. Thus, the operator $\mathcal{T}_0^l$ can be seen as the limit of the operator $\mathcal{T}_0^\varepsilon$ when ε goes to 0. Indeed, for $(h_n)_n\subset\mathcal{H}_0$ such that $\|h_n\|_{\mathcal{H}_0}\leqslant1$ we have up to a subsequence, $h_n \rightharpoonup h$ in $\mathcal{H}_0$. Moreover, we have

$$\begin{aligned}\|\mathcal{T}_0^l(h_n) - \mathcal{T}_0^l(h)\|_{\mathcal{H}_0} &\leqslant \|\mathcal{T}_0^l(h_n) - \mathcal{T}_0^\varepsilon(h_n)\|_{\mathcal{H}_0} + \|\mathcal{T}_0^\varepsilon(h_n) - \mathcal{T}_0^\varepsilon(h)\|_{\mathcal{H}_0} + \|\mathcal{T}_0^\varepsilon(h) - \mathcal{T}_0^l(h)\|_{\mathcal{H}_0} \\ &\leqslant \varepsilon\|h_n\|_{\mathcal{H}_0} + \|\mathcal{T}_0^\varepsilon(h_n) - \mathcal{T}_0^\varepsilon(h)\|_{\mathcal{H}_0} + \varepsilon\|h\|_{\mathcal{H}_0} \\ &\leqslant 2\varepsilon + \|\mathcal{T}_0^\varepsilon(h_n) - \mathcal{T}_0^\varepsilon(h)\|_{\mathcal{H}_0}. \end{aligned} \tag{3.2.19}$$

Let us now prove that we have the strong convergence $\|\mathcal{T}_0^\varepsilon(h_n) - \mathcal{T}_0^\varepsilon(h)\|_{\mathcal{H}_0} \to 0$.
For that purpose, we will use Rellich's theorem for the sequence $\mathtt{H}_n^\varepsilon$ defined by $\mathtt{H}_n^\varepsilon := g_1^\varepsilon \frac{h_n}{\langle v\rangle^2} - \langle h_n, \Phi\rangle g_2^\varepsilon$. Indeed, it is uniformly bounded in $H^1_{\langle v\rangle^2}$ since we have:

$$\begin{aligned}\int_{\mathbb{R}^d} \langle v\rangle^2 |\mathtt{H}_n^\varepsilon|^2 \mathrm{d}v &\leqslant 2\int_{\mathbb{R}^d}\left(|g_1^\varepsilon|^2\frac{|h_n|^2}{\langle v\rangle^2} + \left\|\frac{h_n}{\langle v\rangle}\right\|_2^2 \|\langle v\rangle\Phi\|_2^2 \langle v\rangle^2 |g_2^\varepsilon|^2\right)\mathrm{d}v \\ &\leqslant 2\left(\|g_1^\varepsilon\|_\infty^2 + \|\Phi\|^2_{L^2_{\langle v\rangle^2}}\|g_2^\varepsilon\|^2_{L^2_{\langle v\rangle^2}}\right)\|h_n\|^2_{\mathcal{H}_0} \lesssim 1\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} \langle v\rangle^2 \left|\nabla_v \mathtt{H}_n^\varepsilon\right|^2 \mathrm{d}v &= \int_{\mathbb{R}^d} \langle v\rangle^2 \left|\nabla_v g_1^\varepsilon \frac{h_n}{\langle v\rangle^2} + \frac{g_1^\varepsilon}{\langle v\rangle^2}\nabla_v h_n - 2\frac{v}{\langle v\rangle^2} g_1^\varepsilon \frac{h_n}{\langle v\rangle^2} - \langle h_n, \Phi\rangle \nabla_v g_2^\varepsilon\right|^2 \mathrm{d}v \\ &\lesssim \left(\|g_1^\varepsilon\|^2_{W^{1,\infty}} + \|\Phi\|^2_{L^2_{\langle v\rangle^2}}\|g_2^\varepsilon\|^2_{H^1_{\langle v\rangle^2}}\right)\|h_n\|^2_{\mathcal{H}_0} \lesssim 1,\end{aligned}$$

where g_1^ε and g_2^ε are uniformly bounded in $W^{1,\infty}$ and $H^1_{\langle v\rangle^2}$ respectively, and since $\|\Phi\|_{L^2_{\langle v\rangle^2}} \leqslant 1$ and $\|h_n\|_{\mathcal{H}_0} \leqslant 1$. Then, there exists $\mathtt{H}^\varepsilon \in H^1_{\langle v\rangle^2}$ such that $\langle v\rangle \mathtt{H}_n^\varepsilon \longrightarrow \langle v\rangle \mathtt{H}^\varepsilon$ in $L^2(K)$, up to a subsequence, for all $K \subset \mathbb{R}^d$ bounded, in particular for $K = B(0, R_\varepsilon)$, where $R_\varepsilon > 0$ is such that

$$\mathrm{supp}(g_1^\varepsilon) \cup \mathrm{supp}(g_2^\varepsilon) \subset B(0, R_\varepsilon).$$

The limit $\mathtt{H}^\varepsilon$ can be identified as the unique limit in $\mathcal{D}'(\mathbb{R}^d)$, $\mathtt{H}^\varepsilon = g_1^\varepsilon \frac{h}{\langle v\rangle^2} - \langle h, \Phi\rangle\Phi$. So for all $\varepsilon' > 0$, there exists $N_{\varepsilon'} \in \mathbb{N}$ such that, for all $n \geqslant N_{\varepsilon'}$ we have: $\|\mathtt{H}_n^\varepsilon - \mathtt{H}^\varepsilon\|_{L^2_{\langle v\rangle^2}} \leqslant \frac{\varepsilon'}{3C}$. Therefore, for $\varepsilon < \frac{\varepsilon'}{3}$ and $n \geqslant N_{\varepsilon'}$ we obtain, thanks to (3.2.19) and the inequality $\|\mathcal{T}_0^l(h)\|_{\mathcal{H}_0} \leqslant C\|\mathtt{H}\|_{L^2_{\langle v\rangle^2}}$, that:

$$\|\mathcal{T}_0^l(h_n) - \mathcal{T}_0^l(h)\|_{\mathcal{H}_0} \leqslant 2\varepsilon + C\|\mathtt{H}_n^\varepsilon - \mathtt{H}^\varepsilon\|_{L^2_{\langle v\rangle^2}} \leqslant \varepsilon'.$$

Hence the compactness of $\mathcal{T}_0^l$ holds. □

Proposition 3.2.17 (Assumptions of the Implicit Function Theorem)**.**

1. *The map $F(\lambda, \eta, \cdot) = Id - \mathcal{T}_{\lambda,\eta}$ is continuous in $\mathcal{H}_0$ uniformly with respect to λ and η. Moreover, there exists $c > 0$, independent of λ and η such that*

$$\|F(\lambda, \eta, h_1) - F(\lambda, \eta, h_2)\|_{\mathcal{H}_0} \leqslant c\|h_1 - h_2\|_{\mathcal{H}_0}, \quad \forall h_1, h_2 \in \mathcal{H}_0, \forall \eta \in [0, \eta_0], \forall |\lambda| \leqslant \lambda_0.$$

2. The map F is continuous with respect to λ and η and we have

$$\lim_{\eta\to\eta'}\|F(\lambda,\eta,h)-F(\lambda,\eta',h)\|_{\mathcal{H}_0}=\lim_{\lambda\to\lambda'}\|F(\lambda,\eta,h)-F(\lambda',\eta,h)\|_{\mathcal{H}_0}=0,\quad \forall h\in\mathcal{H}_0.$$

3. The map $F(\lambda,\eta,\cdot)$ is differentiable in $\mathcal{H}_0$. Moreover,

$$\frac{\partial F}{\partial h}(\lambda,\eta,\cdot)=Id-\mathcal{T}^l_{\lambda,\eta},\quad \forall|\lambda|\leqslant\lambda_0,\forall\eta\in[0,\eta_0].$$

4. We have $F(0,0,M)=0$ and $\frac{\partial F}{\partial h}(0,0,M)$ is invertible.

Proof. 1. Let $h_1,h_2\in\mathcal{H}_0$. Let $\eta\in[0,\eta_0]$ and $|\lambda|\leqslant\lambda_0$ with η_0 and λ_0 small enough. Then,

$$\begin{aligned}\|F(\lambda,\eta,h_1)-F(\lambda,\eta,h_2)\|_{\mathcal{H}_0}&\leqslant\|(h_1-h_2)+\mathcal{T}^l_{\lambda,\eta}(h_1-h_2)\|_{\mathcal{H}_0}\\&\leqslant\big(1+C\big)\|h_1-h_2\|_{\mathcal{H}_0}.\end{aligned}$$

2. The proof of this point is a direct consequence of the second point of Lemma 3.2.15.
3. The third point is immediate since $\mathcal{T}_{\lambda,\eta}$ is an affine map with respect to h.
4. Recall that $L_0=\tilde{Q}$ is the inverse of T_0 and $V:=\tilde{W}-W$. Thus, since we have

$$L_0\big(F(0,0,M)\big)=L_0\big(M-T_0[VM]\big)=[\tilde{Q}-V](M)=Q(M)=0.$$

Then, we obtain $F(0,0,M)=0$, thanks to the injectivity of L_0.
For the differential, we have $\frac{\partial F}{\partial h}(0,0,M)=Id-\mathcal{T}^l_0$. By the Fredholm Alternative, this point is true if $\mathrm{Ker}(Id-\mathcal{T}^l_0)=\{0\}$. Let $h\in\mathcal{H}_0$ such that $h-\mathcal{T}^l_0(h):=h-T_0\big[Vh-\langle h,\Phi\rangle\Phi\big]=0$. Applying the operator $L_0=\tilde{Q}$ to this last equality we obtain

$$\tilde{Q}(h)-Vh+\langle h,\Phi\rangle\Phi=Q(h)+\langle h,\Phi\rangle\Phi=0.$$

Integrating this last equation against M and using the fact that $\langle\Phi,M\rangle=1$, we get

$$0=\big\langle Q(h)+\langle h,\Phi\rangle\Phi,M\big\rangle=\langle h,\Phi\rangle\langle\Phi,M\rangle=\langle h,\Phi\rangle.$$

Therefore, h is solution to $Q(h)=0$. Then, there exists $c_1,c_2\in\mathbb{C}$ such that $h=c_1M+c_2Z$. Since $h\in\mathcal{H}_0$ and $Z\notin\mathcal{H}_0$ then, $c_2=0$ and $h=c_1M$. Thus, $\langle h,\Phi\rangle=c_1=0$. Hence, $h=0$. This completes the proof of the Proposition. ☐

Theorem 3.2.18 (Existence of solutions with constraint)**.** *There is a unique function $M_{\lambda,\eta}$ in $\mathcal{H}_0$ solution to the penalized equation*

$$[-\Delta_v+W(v)+\mathrm{i}\eta v_1-\lambda\eta^{\frac{2}{3}}]M_{\lambda,\eta}(v)=b(\lambda,\eta)\Phi(v),\quad v\in\mathbb{R}^d. \tag{3.2.20}$$

where $b(\lambda,\eta) := \langle N_{\lambda,\eta}, \Phi\rangle$ *with* $N_{\lambda,\eta} := M_{\lambda,\eta} - M$. *Moreover,*

$$\|N_{\lambda,\eta}\|_{\mathcal{H}_0} = \|M_{\lambda,\eta} - M\|_{\mathcal{H}_0} \underset{\eta\to 0}{\longrightarrow} 0. \tag{3.2.21}$$

Proof. By Proposition 3.2.17, F satisfies the assumptions of the Implicit Function Theorem around the point $(0,0,M)$. Then, there exists $\lambda_0, \eta_0 > 0$ small enough, there exists a unique function $\mathcal{M} : \{|\lambda| \leqslant \lambda_0\} \times [0,\eta_0] \longrightarrow \mathcal{H}_0$, continuous with respect to λ and η such that

$$F(\lambda,\eta,\mathcal{M}(\lambda,\eta)) = 0, \text{ for all } (\lambda,\eta) \in \{|\lambda| < \lambda_0\} \times [0,\eta_0[.$$

Let's denote $M_{\lambda,\eta} := \mathcal{M}(\lambda,\eta)$. The function $M_{\lambda,0}$ does not depend on λ and the continuity of $\mathcal{M}$ with respect to η implies that

$$\lim_{\eta\to 0}\|M_{\lambda,\eta} - M_{\lambda,0}\|_{\mathcal{H}_0} = \lim_{\eta\to 0}\|M_{\lambda,\eta} - M\|_{\mathcal{H}_0} = 0.$$

□

Remark 3.2.19.

1. Since $\Phi(-v) = \Phi(v)$ for all $v \in \mathbb{R}^d$ and the function $\overline{M}_{\bar\lambda,\eta}(-v_1,v')$ satisfies the equation (3.2.20) then, by uniqueness, $\overline{M}_{\bar\lambda,\eta}(-v_1,v')$ is solution to (3.2.20) and the following symmetry
$$\overline{M}_{\bar\lambda,\eta}(-v_1,v') = M_{\lambda,\eta}(v_1,v') \tag{3.2.22}$$
holds for all $(v_1,v') \in \mathbb{R}\times\mathbb{R}^{d-1}$, $\eta \in [0,\eta_0]$ and $|\lambda| \leqslant \lambda_0$.

2. The sequence $|b(\lambda,\eta)|$ is uniformly bounded with respect to λ and η since $|b(\lambda,\eta)| \underset{\eta\to 0}{\longrightarrow} 0$, which we obtain by the Cauchy-Schwarz inequality and the limit (3.2.21):
$$|b(\lambda,\eta)| = |\langle N_{\lambda,\eta},\Phi\rangle| \leqslant \left\|\frac{N_{\lambda,\eta}}{\langle v\rangle}\right\|_2 \|\langle v\rangle\Phi\|_2 \leqslant \|N_{\lambda,\eta}\|_{\mathcal{H}_0}\|\langle v\rangle\Phi\|_2 \underset{\eta\to 0}{\longrightarrow} 0. \tag{3.2.23}$$

3.3 Existence of the eigen-solution $(\mu(\eta), M_{\mu,\eta})$

The aim of this section is to prove Theorem 3.1.1. It is composed of three subsections. In the first one, we establish some L^2 estimates. The second one is devoted to the study of the constraint and the existence of the eigen-solution $(\mu(\eta), M_\eta)$. Finally, in the last subsection, we give an approximation of the eigenvalue and its relation with the diffusion coefficient.

3.3.1 L^2 estimates for the solution $M_{\lambda,\eta}$

In this subsection, we will establish some L^2 estimates for the solution of the penalised equation (3.2.1).

Proposition 3.3.1. *Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Let $M_{\lambda,\eta}$ be the solution of the penalised equation (3.2.20). Then, for all $\eta \in [0, \eta_0]$ and for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$, one has*

1. *For all $\gamma > \frac{d}{2}$, the function $M_{\lambda,\eta}$ is uniformly bounded, with respect to λ and η, in $L^2(\mathbb{R}^d, \mathbb{C})$. Moreover, the following estimate holds*
$$\|N_{\lambda,\eta}\|^2_{L^2(\mathbb{R}^d)} = \|M_{\lambda,\eta} - M\|^2_{L^2(\mathbb{R}^d)} \lesssim |\lambda| + \nu_\eta, \tag{3.3.1}$$
where $\nu_\eta \underset{\eta\to 0}{\longrightarrow} 0$.

2. *For all $\gamma > \frac{d+1}{2}$, the function $|v_1|^{\frac{1}{2}} M_{\lambda,\eta}$ is uniformly bounded, with respect to λ and η, in $L^2(\mathbb{R}^d, \mathbb{C})$.*

Proof. We are going to prove the first point, the second is done in a similar way. Let denote $v := (v_1, v') \in \mathbb{R} \times \mathbb{R}^{d-1}$. The proof of this Proposition is given in four steps and the idea is as follows: first, we decompose $\mathbb{R}^d$ into two parts, $\mathbb{R}^d = \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\} \cup \{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\}$, small/medium and large velocities. In the first step, using the equation of $M_{\lambda,\eta}$, we estimate the norm of $M_{\lambda,\eta}$ for large velocities to get

$$\|M_{\lambda,\eta}\|^2_{L^2(\{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\})} \leqslant \nu_1 \|M_{\lambda,\eta}\|^2_{L^2(\{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\})} + c_1,$$

where ν_1 and c_1 depend on s_0, λ and η. To estimate $\|M_{\lambda,\eta}\|_{L^2(\{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\})}$, it is enough to estimate $\|N_{\lambda,\eta}\|_{L^2(\{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\})}$ since M belongs to L^2, which is the purpose of steps two and three. In step 2, using a Poincaré type inequality, we show that

$$\|N_{\lambda,\eta}\|^2_{L^2(\{|v_1| \leqslant s_0\eta^{-\frac{1}{3}} \leqslant |v'|\})} \leqslant C_1 \|M_{\lambda,\eta}\|^2_{L^2(\{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\})} + c_2,$$

where C_1 is a positive constant and c_2 depends on s_0, λ and η. Then, in the third step, using the Hardy-Poincaré inequality, we prove that

$$\|N_{\lambda,\eta}\|^2_{L^2(\{|v| \leqslant s_0\eta^{-\frac{1}{3}}\})} \leqslant \nu_2 \|N_{\lambda,\eta}\|^2_{L^2(\{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\})} + \nu_3 \|M_{\lambda,\eta}\|^2_{L^2(\{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\})} + c_3,$$

with ν_2, ν_3 and c_3 depend on s_0, λ and η. The last step is left for the conclusion: we first fix s_0 large enough, then $|\lambda|$ small enough, then η small enough, we obtain $\nu_2 \leqslant \frac{1}{4}$, $\nu_3 \leqslant \frac{1}{4}$ and $\nu_1\big(C_1 + \frac{\nu_3}{1-\nu_2}\big) \leqslant \frac{1}{2}$, which allows us to conclude.

Before starting the proof, we will define some sets to simplify the notations and avoid long expressions. We set: $A_\eta := \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\}$ (resp., $\tilde{A}_\eta := \{|v_1| \leqslant 2s_0\eta^{-\frac{1}{3}}\}$), $B_\eta := \{|v| \leqslant s_0\eta^{-\frac{1}{3}}\}$, $C_\eta := \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}} \leqslant |v'|\}$ (resp., $\tilde{C}_\eta := \{|v_1| \leqslant 2s_0\eta^{-\frac{1}{3}} \leqslant 2|v'|\}$) and $D_\eta := \{|v_1| \geqslant \frac{s_0}{2}\eta^{-\frac{1}{3}}\}$.

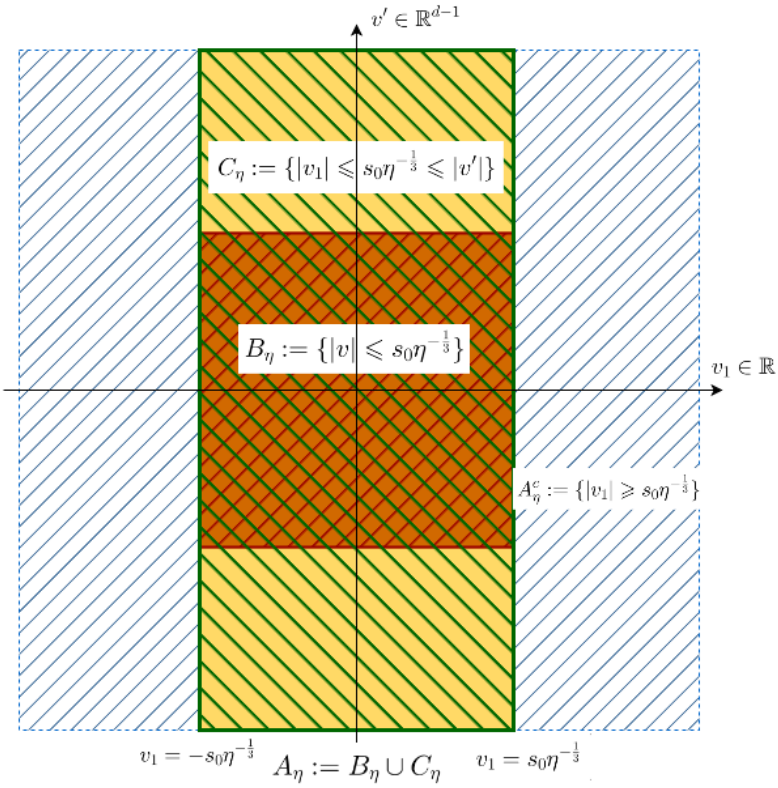


Figure 3.3.1: Decomposition of $\mathbb{R}^d$ into A_η and A_η^c.

The part A_η^c is represented by the blue zone, while the part A_η, in green stripes, is broken down into two other parts: the brown zone B_η for $|v'|$ small, and the yellow zone C_η for $|v'|$ large. The parts $\tilde{A}_\eta$, $\tilde{C}_\eta$ and D_η are an extensions "in the direction of v_1" of the parts A_η, C_η and A_η^c respectively, and are not shown in the figure above.

Step 1: Estimation of $\|M_{\lambda,\eta}\|_{L^2(A_\eta^c)}$. We summarize this step in the following inequality:

$$\|M_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} \leqslant \frac{1}{s_0^2}\left\|\eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}\right\|^2_{L^2(A_\eta^c)} \lesssim \frac{1}{s_0^3}\Big(\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + c_1^\eta\Big), \tag{3.3.2}$$

where $c_1^\eta = c_1(\lambda, \eta, s_0) = s_0^{-2\delta}\eta^{\frac{2\delta}{3}}(|b(\lambda,\eta)|+1)\||v_1|^\delta M\|^2_{L^2(\mathbb{R}^d)}$ where δ can be chosen as follows $\delta := \frac{1}{2}(\gamma - \frac{d}{2})$ to ensure that $|v|^\delta M$ belongs to L^2.

• **Estimation of** $\|\eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}\|_{L^2(A_\eta^c)}$. In order to localize the velocities on the part A_η^c and to be able to use the equation of $M_{\lambda,\eta}$ and make integrations by part, we introduce the function χ_η defined by: $\chi_\eta(v_1) := \chi\big(\frac{v_1}{s_0\eta^{-1/3}}\big)$, where $\chi \in C^\infty(\mathbb{R})$ is such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 0$ on $B(0, \frac{1}{2})$ and $\chi \equiv 1$ outside of $B(0,1)$. Then, one has: $\|\eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}\|_{L^2(A_\eta^c)} \leqslant \|\eta^{\frac{1}{3}} v_1 \chi_\eta M_{\lambda,\eta}\|_{L^2(D_\eta)}$. Now, multiplying the equation of $M_{\lambda,\eta}$ by $v_1 \bar{M}_{\lambda,\eta}\chi_\eta^2$, integrating it over D_η and taking the imaginary part, we get:

$$\begin{aligned}\left\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\right\|^2_{L^2(D_\eta)} &= -\eta^{-\frac{1}{3}}\mathrm{Im}\left(\int_{D_\eta} Q(M_{\lambda,\eta})v_1\bar{M}_{\lambda,\eta}\chi_\eta^2\mathrm{d}v\right) + \mathrm{Im}\left(\lambda\int_{D_\eta}\eta^{\frac{1}{3}}v_1|M_{\lambda,\eta}\chi_\eta|^2\mathrm{d}v\right)\\ &\quad - \eta^{-\frac{1}{3}}\mathrm{Im}\left(b(\lambda,\eta)\int_{D_\eta}\Phi v_1\bar{M}_{\lambda,\eta}\chi_\eta^2\mathrm{d}v\right)\\ &=: -E_1^\eta + E_2^\eta + E_3^\eta.\end{aligned}$$

Let's start with E_2^η and E_3^η which are simpler.

• **Estimation of** E_2^η: For this term, we just use the fact that on D_η: $\frac{s_0}{2} \leqslant \eta^{\frac{1}{3}}|v_1|$. Thus,

$$|E_2^\eta| := \left|\mathrm{Im}\left(\lambda\int_{D_\eta}\eta^{\frac{1}{3}}v_1|M_{\lambda,\eta}\chi_\eta|^2\mathrm{d}v\right)\right| \leqslant \frac{2|\lambda|}{s_0}\left\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\right\|^2_{L^2(D_\eta)}. \tag{3.3.3}$$

• **Estimation of** E_3^η: First of all, since $\Phi(v) := \left(\int_{\mathbb{R}^d}\langle v\rangle^{-2-2\gamma}\mathrm{d}v\right)^{-1}\frac{M(v)}{\langle v\rangle^2}$ then:

$$\Phi(v)\chi_{\{|v_i|\geqslant s_0\eta^{-\frac{1}{3}}\}}(v) \leqslant Cs_0^{-2}\eta^{\frac{2}{3}}M(v). \tag{3.3.4}$$

In particular,

$$\Phi(v)\chi_\eta(v_1) \leqslant Cs_0^{-2}\eta^{\frac{2}{3}}M(v). \tag{3.3.5}$$

Similarly, we have:

$$\|M\|_{L^2(\{|v_i|\geqslant s_0\eta^{-\frac{1}{3}}\})} \leqslant s_0^{-\delta}\eta^{\frac{\delta}{3}}\left\||v|^\delta M\right\|_{L^2(\mathbb{R}^d)}. \tag{3.3.6}$$

Then, using (3.3.5), we get

$$|E_3^\eta| := \left|\eta^{-\frac{1}{3}}\mathrm{Im}\left(b(\lambda,\eta)\int_{D_\eta}\Phi v_1\bar{M}_{\lambda,\eta}\chi_\eta^2\mathrm{d}v\right)\right| \leqslant 4\frac{|b(\lambda,\eta)|}{s_0^2}\|\chi_\eta M\|_{L^2(D_\eta)}\left\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\right\|_{L^2(D_\eta)}.$$

Finally, by inequality (3.3.6)

$$|E_3^\eta| \leqslant 2\frac{|b(\lambda,\eta)|}{s_0^2}\left(\left\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\right\|^2_{L^2(D_\eta)} + 4s_0^{-2\delta}\eta^{\frac{2\delta}{3}}\left\||v|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}\right). \tag{3.3.7}$$

• **Estimation of** E_1^η: By an integration by parts, we write

$$\begin{aligned}E_1^\eta &:= \eta^{-\frac{1}{3}}\mathrm{Im}\int_{D_\eta} Q(M_{\lambda,\eta})v_1\bar{M}_{\lambda,\eta}\chi_\eta^2\mathrm{d}v\\ &= \eta^{-\frac{1}{3}}\mathrm{Im}\int_{D_\eta}\left[\chi_\eta\bar{M}_{\lambda,\eta} + 2v_1\chi_\eta'\bar{M}_{\lambda,\eta}\right]\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\mathrm{d}v.\end{aligned}$$

Thus, by Cauchy-Schwarz

$$|E_1^\eta| \leqslant \eta^{-\frac{1}{3}}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|_{L^2(D_\eta)}\left(\|\chi_\eta M_{\lambda,\eta}\|_{L^2(D_\eta)} + 2\|v_1\chi_\eta' M_{\lambda,\eta}\|_{L^2(D_\eta)}\right).$$

Since $\chi_\eta' \equiv 0$ except on: $D_\eta \setminus A_\eta^c = \{\frac{s_0}{2}\eta^{-\frac{1}{3}} \leqslant |v_1| \leqslant s_0\eta^{-\frac{1}{3}}\} \subset A_\eta := \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\}$. Then,

$$\|v_1\chi_\eta' M_{\lambda,\eta}\|_{L^2(D_\eta)} = \|v_1\chi_\eta' M_{\lambda,\eta}\|_{L^2(D_\eta\setminus A_\eta^c)} \leqslant C\|M_{\lambda,\eta}\|_{L^2(D_\eta\setminus A_\eta^c)},$$

where $C = \sup\limits_{\frac{1}{2}\leqslant|t|\leqslant 1} |t\chi'(t)|$. Also, we have: $\|\chi_\eta M_{\lambda,\eta}\|_{L^2(D_\eta)} \leqslant \frac{1}{s_0}\|\eta^{\frac{1}{3}} v_1\chi_\eta M_{\lambda,\eta}\|_{L^2(D_\eta)}$. Thus,

$$|E_1^\eta| \leqslant \eta^{-\frac{1}{3}}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|_{L^2(D_\eta)}\left(\frac{1}{s_0}\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|_{L^2(D_\eta)} + C\|M_{\lambda,\eta}\|_{L^2(D_\eta\setminus A_\eta^c)}\right).$$

Finally, by Young's inequality:

$$|E_1^\eta| \lesssim s_0\eta^{-\frac{2}{3}}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)} + \frac{1}{s_0^3}\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} + \frac{1}{s_0}\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)}. \tag{3.3.8}$$

It remains to estimate $\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|_{L^2(D_\eta)}$. For this, one has

$$\begin{aligned}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)} &\leqslant \left\|\nabla_v\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)}\\ &= \mathrm{Re}\int_{D_\eta}\left[Q(M_{\lambda,\eta})\bar M_{\lambda,\eta}\chi_\eta^2 + 2\zeta_\eta'\zeta_\eta\frac{\bar M_{\lambda,\eta}}{M}\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M^2\right]\mathrm{d}v\\ &=: F_1^\eta + F_2^\eta.\end{aligned} \tag{3.3.9}$$

By integrating the equation of $M_{\lambda,\eta}$, multiplied by $\bar M_{\lambda,\eta}\chi_\eta^2$, over D_η, and using (3.3.4), we obtain

$$|F_1^\eta| \lesssim \frac{\eta^{\frac{2}{3}}}{s_0^2}\left[\left(|\lambda| + \frac{|b(\lambda,\eta)|}{s_0^2}\right)\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} + |b(\lambda,\eta)|\|M\|^2_{L^2(D_\eta)}\right]. \tag{3.3.10}$$

For F_2^η, by inequality ($2Cab \leqslant 2C^2a^2 + \frac{b^2}{2}$):

$$\begin{aligned}|F_2^\eta| &\leqslant 2\|\chi_\eta' M_{\lambda,\eta}\|_{L^2(D_\eta)}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|_{L^2(D_\eta)}\\ &\leqslant 2C\frac{\eta^{\frac{1}{3}}}{s_0}\|M_{\lambda,\eta}\|_{L^2(D_\eta\setminus A_\eta^c)}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|_{L^2(D_\eta)}\\ &\leqslant C'\frac{\eta^{\frac{2}{3}}}{s_0^2}\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + \frac{1}{2}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)},\end{aligned} \tag{3.3.11}$$

where $C = \sup\limits_{\frac{1}{2}\leqslant|t|\leqslant 1}|t\chi'(t)|$ and $C' = 2C^2$. Then, we obtain by returning to (3.3.9)

$$\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)} \leqslant |F_1^\eta| + C'\frac{\eta^{\frac{2}{3}}}{s_0^2}\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + \frac{1}{2}\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)}.$$

Therefore,

$$\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)} \lesssim |F_1^\eta| + \frac{\eta^{\frac{2}{3}}}{s_0^2}\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)}.$$

Hence, from (3.3.10), (3.3.6) and the last inequality

$$\left\|\partial_{v_1}\left(\frac{M_{\lambda,\eta}}{M}\right)M\chi_\eta\right\|^2_{L^2(D_\eta)} \lesssim \frac{\eta^{\frac{2}{3}}}{s_0^2}\Bigg[\left(|\lambda| + \frac{|b(\lambda,\eta)|}{s_0^2}\right)\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} + \|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + s_0^{-2\delta}\eta^{\frac{2\delta}{3}}|b(\lambda,\eta)|\left\||v_1|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}\Bigg].$$

Which implies, by inequality (3.3.8), that

$$|E_1^\eta| \lesssim \frac{1}{s_0}\Bigg[\left(|\lambda| + \frac{1+|b(\lambda,\eta)|}{s_0^2}\right)\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} + \|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + s_0^{-2\delta}\eta^{\frac{2\delta}{3}}|b(\lambda,\eta)|\left\||v_1|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}\Bigg]. \tag{3.3.12}$$

Thus, by summing the inequalities obtained from E_1^η, E_2^η and E_3^η, namely (3.3.12), (3.3.3) and (3.3.7) respectively, we obtain

$$\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} \lesssim \frac{1}{s_0}\Bigg[\left(|\lambda| + \frac{1+|b(\lambda,\eta)|}{s_0^2}\right)\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} + \|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + s_0^{-2\delta}\eta^{\frac{2\delta}{3}}|b(\lambda,\eta)|\left\||v_1|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}\Bigg].$$

Hence the following estimate

$$\|\eta^{\frac{1}{3}}v_1\chi_\eta M_{\lambda,\eta}\|^2_{L^2(D_\eta)} \lesssim \frac{1}{s_0}\left(\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} + s_0^{-2\delta}\eta^{\frac{2\delta}{3}}|b(\lambda,\eta)|\left\||v_1|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}\right) \tag{3.3.13}$$

holds true for $s_0 > 0$ large enough and for all $|\lambda| \leqslant \lambda_0$ and $\eta \in [0,\eta_0]$, with λ_0 and η_0 small enough. Finally, (3.3.2) comes from the previous inequality (3.3.13), and since $D_\eta \setminus A_\eta^c \subset A_\eta$ and $D_\eta \setminus A_\eta^c \subset D_\eta$ implies that,

$$\|M_{\lambda,\eta}\|^2_{L^2(D_\eta\setminus A_\eta^c)} \leqslant \|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + \|M\|^2_{L^2(D_\eta)} \leqslant \|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + 4s_0^{-2\delta}\eta^{\frac{2\delta}{3}}\left\||v_1|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}.$$

Step 2: Estimation of $\|N_{\lambda,\eta}\|_{L^2(C_\eta)}$. In this step, we will establish the following inequality:

$$\|N_{\lambda,\eta}\|^2_{L^2(C_\eta)} \lesssim \|N_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} + c_2^\eta, \tag{3.3.14}$$

where $c_2^\eta := s_0^{-\delta}\eta^{\frac{\delta}{3}}\left(s_0^2|\lambda| + s_0^3 + |b(\lambda,\eta)|\right)\||v|^\delta M\|^2_{L^2(\mathbb{R}^d)}$, and where we recall that $\delta := \frac{1}{2}(\gamma - \frac{d}{2})$, $C_\eta := \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}} \leqslant |v'|\}$ and $A_\eta^c := \{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\}$. We start with the following Lemma:

Lemma 3.3.2 (Poincaré-type inequality)**.** *Let $R>0$ be fixed and let C_R be the set defined by: $C_R := \{v\in\mathbb{R}^d;\ |v_1|\leqslant R\leqslant |v'|\}$. Then, there exists a constant $C>0$ such that, for any function ψ in the space $\mathcal{H} := \big\{\int_{C_R}\big|\partial_{v_1}(\frac{\psi}{M})\big|^2M^2\mathrm{d}v<\infty;\ \psi(-R,\cdot)=\psi(R,\cdot)=0\big\}$, the inequality*

$$\|\psi\|^2_{L^2(C_R)}\leqslant CR^2\big\|\partial_{v_1}(\frac{\psi}{M})M\big\|^2_{L^2(C_R)} \tag{3.3.15}$$

holds true.

Proof of Lemma 3.3.2. We have for $\psi\in\mathcal{H}$: $\dfrac{\psi}{M}=\displaystyle\int_{-R}^{v_1}\partial_{v_1}(\frac{\psi}{M})\mathrm{d}w.$
Then, by taking the square and applying the Cauchy-Schwarz inequality, we get:

$$|\psi|^2\leqslant M^2(v_1,v')\Big(\int_{-R}^{v_1}\partial_{v_1}\Big(\frac{\psi}{M}\Big)\mathrm{d}w_1\Big)^2\leqslant\int_{-R}^{R}\frac{M^2(v_1,v')}{M^2(w_1,v')}\mathrm{d}w_1\int_{-R}^{R}\Big|\partial_{v_1}\Big(\frac{\psi}{M}\Big)M\Big|^2\mathrm{d}w_1.$$

Now, we have for $v_1,w_1\in[-R,R]$ and $|v'|\geqslant R$, $\frac{M^2(v_1,v')}{M^2(w_1,v')}\lesssim 1$. Therefore,

$$|\psi|^2\lesssim R\int_{-R}^{R}\Big|\partial_{v_1}\Big(\frac{\psi}{M}\Big)M\Big|^2\mathrm{d}w_1.$$

Thus, we obtain the inequality (3.3.15) by integrating the last one over C_R. □

Now back to the estimate of $\|N_{\lambda,\eta}\|_{L^2(C_\eta)}$. Let $\zeta\in C^\infty(\mathbb{R})$ such that $0\leqslant\zeta\leqslant 1$, $\zeta\equiv 1$ on $B(0,1)$ and $\zeta\equiv 0$ outside of $B(0,2)$. We define ζ_η by: $\zeta_\eta(v_1):=\zeta\big(\frac{v_1}{s_0\eta^{-1/3}}\big)$. Then, for $\eta>0$ and $s_0>0$ fixed, by applying Lemma 3.3.2 for $R=s_0\eta^{-\frac13}$, we obtain:

$$\|N_{\lambda,\eta}\|^2_{L^2(C_\eta)}\leqslant\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde C_\eta)}\lesssim\ s_0^2\eta^{-\frac23}\Big\|\partial_{v_1}\Big(\frac{\zeta_\eta N_{\lambda,\eta}}{M}\Big)M\Big\|^2_{L^2(\tilde C_\eta)}, \tag{3.3.16}$$

recalling that $\tilde C_\eta:=\{|v_1|\leqslant 2s_0\eta^{-\frac13}\leqslant 2|v'|\}$. Furthermore,

$$\Big\|\partial_{v_1}\Big(\frac{\zeta_\eta N_{\lambda,\eta}}{M}\Big)M\Big\|^2_{L^2(\tilde C_\eta)}\leqslant\Big\|\nabla_v\Big(\frac{\zeta_\eta N_{\lambda,\eta}}{M}\Big)M\Big\|^2_{L^2(\tilde C_\eta)}=\mathrm{Re}\int_{\tilde C_\eta}Q(\zeta_\eta N_{\lambda,\eta})\zeta_\eta\bar N_{\lambda,\eta}\mathrm{d}v.$$

However,

$$\begin{aligned}\mathrm{Re}\int_{\tilde C_\eta}Q(\zeta_\eta N_{\lambda,\eta})\zeta_\eta\bar N_{\lambda,\eta}\mathrm{d}v&=\mathrm{Re}\int_{\tilde C_\eta}\big[Q(N_{\lambda,\eta})\bar N_{\lambda,\eta}\zeta_\eta^2-\zeta_\eta\zeta_\eta''|N_{\lambda,\eta}|^2-2\zeta_\eta\zeta_\eta'\bar N_{\lambda,\eta}\partial_{v_1}N_{\lambda,\eta}\big]\mathrm{d}v\\&=\mathrm{Re}\int_{\tilde C_\eta}Q(N_{\lambda,\eta})\bar N_{\lambda,\eta}\zeta_\eta^2\mathrm{d}v+\int_{\tilde C_\eta}|\zeta_\eta' N_{\lambda,\eta}|^2\mathrm{d}v,\end{aligned} \tag{3.3.17}$$

where we used the fact that $Q(\zeta_\eta N_{\lambda,\eta})=Q(N_{\lambda,\eta})\zeta_\eta-\zeta_\eta''N_{\lambda,\eta}-2\zeta_\eta'\partial_{v_1}N_{\lambda,\eta}$ in the first line, since $Q:=-\frac{1}{M}\nabla_v\big(M^2\nabla\big(\frac{\cdot}{M}\big)\big)=-\Delta_v+W(v)$, and did an integration by parts for the term $\int_{\tilde C_\eta}\zeta_\eta\zeta_\eta''|N_{\lambda,\eta}|^2\mathrm{d}v$, and used the identity: $\mathrm{Re}\big(\bar f\partial_{v_1}f\big)=\frac12\partial_{v_1}|f|^2$ in the second line.

To handle $\int_{\tilde{C}_\eta} |\zeta'_\eta N_{\lambda,\eta}|^2 \mathrm{d}v$, we have:

$$\int_{\tilde{C}_\eta} |\zeta'_\eta N_{\lambda,\eta}|^2 \mathrm{d}v = \int_{\tilde{C}_\eta \setminus B^c_\eta} |\zeta'_\eta N_{\lambda,\eta}|^2 \mathrm{d}v \leqslant \|\zeta'\|^2_{L^\infty(\tilde{C}_\eta \setminus B^c_\eta)} \|N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta \setminus B^c_\eta)},$$

since $\zeta'_\eta \equiv 0$ except on: $\tilde{C}_\eta \setminus B^c_\eta = \{s_0 \eta^{-\frac{1}{3}} \leqslant |v_1| \leqslant 2 s_0 \eta^{-\frac{1}{3}} \leqslant 2|v'|\} \subset A^c_\eta$, and that on $\tilde{C}_\eta \setminus B^c_\eta$ we have: $|\zeta'_\eta(v_1)| \lesssim \frac{\eta^{\frac{1}{3}}}{s_0}$. Then,

$$\int_{\tilde{C}_\eta} |\zeta'_\eta N_{\lambda,\eta}|^2 \mathrm{d}v \lesssim \frac{\eta^{\frac{2}{3}}}{s_0^2} \|N_{\lambda,\eta}\|^2_{L^2(A^c_\eta)}. \tag{3.3.18}$$

To handle $\mathrm{Re} \int_{\tilde{C}_\eta} Q(N_{\lambda,\eta}) \bar{N}_{\lambda,\eta} \zeta^2_\eta \mathrm{d}v$, we will proceed as in E^η_1. Indeed, recall that $N_{\lambda,\eta}$ satisfies the equation:

$$Q(N_{\lambda,\eta}) = (\lambda \eta^{\frac{2}{3}} - \mathrm{i}\eta v_1) N_{\lambda,\eta} + (\lambda \eta^{\frac{2}{3}} - \mathrm{i}\eta v_1) M - b(\lambda,\eta)\Phi.$$

Then, multiplying this equation by $\bar{N}_{\lambda,\eta}\zeta^2_\eta$ and integrating it over $\tilde{C}_\eta$, we get

$$\begin{aligned} \left| \mathrm{Re} \int_{\tilde{C}_\eta} Q(N_{\lambda,\eta}) \bar{N}_{\lambda,\eta} \zeta^2_\eta \mathrm{d}v \right| \lesssim |\lambda| \eta^{\frac{2}{3}} \left(\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} + \|M\|^2_{L^2(\tilde{C}_\eta)} \right) + \int_{\tilde{C}_\eta} \left| \eta v_1 M \bar{N}_{\lambda,\eta} \zeta^2_\eta \right| \mathrm{d}v \\ + |b(\lambda,\eta)| \int_{\tilde{C}_\eta} \left| \Phi \bar{N}_{\lambda,\eta} \zeta^2_\eta \right| \mathrm{d}v. \end{aligned} \tag{3.3.19}$$

Note that: $\mathrm{Re} \displaystyle\int_{\tilde{C}_\eta} \mathrm{i}\eta v_1 |N_{\lambda,\eta}\zeta_\eta|^2 \mathrm{d}v = 0$. Since inequality (3.3.4) remains true on $\tilde{C}_\eta$: $|v'| \geqslant s_0 \eta^{-\frac{1}{3}}$, we have:

$$\begin{aligned} \int_{\tilde{C}_\eta} \left| \Phi \bar{N}_{\lambda,\eta} \zeta^2_\eta \right| \mathrm{d}v &\lesssim s_0^{-2} \eta^{\frac{2}{3}} \|M\|_{L^2(\tilde{C}_\eta)} \|N_{\lambda,\eta}\zeta_\eta\|_{L^2(\tilde{C}_\eta)} && (3.3.20) \\ &\lesssim s_0^{-2-\delta} \eta^{\frac{2+\delta}{3}} \left\| |v'|^\delta M \right\|_{L^2(\mathbb{R}^d)} \|N_{\lambda,\eta}\zeta_\eta\|_{L^2(\tilde{C}_\eta)} \\ &\lesssim s_0^{-2-\delta} \eta^{\frac{2+\delta}{3}} \left(\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} + \left\| |v|^\delta M \right\|^2_{L^2(\mathbb{R}^d)} \right). && (3.3.21) \end{aligned}$$

Now, the right hand term of the first line in (3.3.19) is treated as follows:

$$\begin{aligned} \int_{\tilde{C}_\eta} \left| \eta v_1 M \bar{N}_{\lambda,\eta} \zeta^2_\eta \right| \mathrm{d}v &\leqslant 2 s_0^{1-\delta} \eta^{\frac{2+\delta}{3}} \|\zeta_\eta N_{\lambda,\eta}\|_{L^2(\tilde{C}_\eta)} \left\| |v|^\delta \zeta_\eta M \right\|_{L^2(\tilde{C}_\eta)} \\ &\leqslant s_0^{1-\delta} \eta^{\frac{2+\delta}{3}} \left(\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} + \left\| |v|^\delta M \right\|^2_{L^2(\mathbb{R}^d)} \right). \end{aligned} \tag{3.3.22}$$

Hence, from (3.3.16), (3.3.18) and the estimates obtained for the terms of (3.3.19) we obtain

$$\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} \lesssim \left(s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}} + s_0^{-\delta}\eta^{\frac{\delta}{3}}|b(\lambda,\eta)|\right)\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} + \|N_{\lambda,\eta}\|^2_{L^2(A^c_\eta)} + \eta^{\frac{\delta}{3}}\left(s_0^{2-\delta}|\lambda| + s_0^{3-\delta} + s_0^{-\delta}|b(\lambda,\eta)|\right)\left\||v|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}.$$

So, for s_0 fixed and $|\lambda|$ and η small enough, $\left(s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}} + s_0^{-\delta}\eta^{\frac{\delta}{3}}|b(\lambda,\eta)|\right) \leqslant \frac{1}{2}$ and the term $\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)}$ in the right side of the previous inequality is absorbed. Thus,

$$\|\zeta_\eta N_{\lambda,\eta}\|^2_{L^2(\tilde{C}_\eta)} \lesssim \|N_{\lambda,\eta}\|^2_{L^2(A^c_\eta)} + \eta^{\frac{\delta}{3}}\left(s_0^{2-\delta}|\lambda| + s_0^{3-\delta} + s_0^{-\delta}|b(\lambda,\eta)|\right)\left\||v|^\delta M\right\|^2_{L^2(\mathbb{R}^d)}.$$

Hence the inequality (3.3.14) holds.

Step 3: Estimation of $\|N_{\lambda,\eta}\|_{L^2(B_\eta)}$. Recall that $B_\eta := \{|v| \leqslant s_0\eta^{-\frac{1}{3}}\}$.We claim that:

$$\|N_{\lambda,\eta}\|^2_{L^2(B_\eta)} \lesssim \nu_1\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + s_0^2|\lambda|\|N_{\lambda,\eta}\|^2_{L^2(A^c_\eta)} + s_0^{2-\delta}\eta^{\frac{\delta}{3}}\left\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\right\|^2_{L^2(A^c_\eta)} + (s_0^2|\lambda| + |c_\eta - 1|)\|M\|_2^2 + s_0^{3-\delta}\eta^{\frac{\delta}{3}}\left\||v_1|^\delta M\right\|_2^2 \tag{3.3.23}$$

where $\nu_1 := \nu_1(\lambda,\eta,s_0) = s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}}$ and $c_\eta := \left(\int_{\mathbb{R}^d}\frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right)^{-1}\int_{\mathbb{R}^d}\frac{MM_{\lambda,\eta}}{\langle v\rangle^2}\mathrm{d}v.$

Let us denote $\tilde{N}_{\lambda,\eta} := M_{\lambda,\eta} - c_\eta M$ the orthogonal projection of $M_{\lambda,\eta}$ to M for the weighted scalar product $\int\frac{\cdot}{\langle v\rangle^2}$. On the one hand, we have:

$$\|N_{\lambda,\eta}\|^2_{L^2(B_\eta)} \lesssim \|\tilde{N}_{\lambda,\eta}\|^2_{L^2(B_\eta)} + |c_\eta - 1|\|M\|^2_{L^2(B_\eta)}$$

and

$$\|\tilde{N}_{\lambda,\eta}\|^2_{L^2(B_\eta)} \lesssim s_0^2\eta^{-\frac{2}{3}}\left\|\frac{\tilde{N}_{\lambda,\eta}}{\langle v\rangle}\right\|^2_{L^2(\mathbb{R}^d)},$$

since $\langle v\rangle \lesssim s_0\eta^{-\frac{1}{3}}$ on B_η. On the other hand, applying the inequality (3.2.4) to $\tilde{N}_{\lambda,\eta}$ which satisfies condition (3.2.5), we obtain:

$$\int_{\mathbb{R}^d}\frac{|\tilde{N}_{\lambda,\eta}|^2}{\langle v\rangle^2}\mathrm{d}v \leqslant C_{\gamma,d}\int_{\mathbb{R}^d}\left|\nabla_v\left(\frac{N_{\lambda,\eta}}{M}\right)\right|^2 M^2\mathrm{d}v.$$

Therefore,

$$\|N_{\lambda,\eta}\|^2_{L^2(B_\eta)} \lesssim s_0^2\eta^{-\frac{2}{3}}\left\|\nabla_v\left(\frac{N_{\lambda,\eta}}{M}\right)M\right\|^2_{L^2(\mathbb{R}^d)} + |c_\eta - 1|\|M\|^2_{L^2(\mathbb{R}^d)}. \tag{3.3.24}$$

We have moreover,

$$\left\|\nabla_v\left(\frac{N_{\lambda,\eta}}{M}\right)M\right\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}Q(N_{\lambda,\eta})\bar{N}_{\lambda,\eta}\mathrm{d}v.$$

Then, by integrating the equation of $N_{\lambda,\eta}$ multiplied by $\bar{N}_{\lambda,\eta}$, we obtain

$$\int_{\mathbb{R}^d} Q(N_{\lambda,\eta})\bar{N}_{\lambda,\eta}\mathrm{d}v + |\langle N_{\lambda,\eta}, \Phi\rangle|^2 = \mathrm{Re}\int_{\mathbb{R}^d}\Big(\lambda\eta^{\frac{2}{3}}\big(|N_{\lambda,\eta}|^2 + M\bar{N}_{\lambda,\eta}\big) - \mathrm{i}\eta v_1 M\bar{N}_{\lambda,\eta}\Big)\mathrm{d}v.$$

From where,

$$\left\|\nabla_v\left(\frac{N_{\lambda,\eta}}{M}\right)M\right\|^2_{L^2(\mathbb{R}^d)} \lesssim |\lambda|\eta^{\frac{2}{3}}\big(\|N_{\lambda,\eta}\|_2^2 + \|M\|_2^2\big) + \eta\int_{\mathbb{R}^d}|v_1 M \mathrm{Im} N_{\lambda,\eta}|\mathrm{d}v. \tag{3.3.25}$$

For the last term, using the fact that $\mathrm{Im}N_{\lambda,\eta} = \mathrm{Im}M_{\lambda,\eta}$, we write

$$\eta^{\frac{1}{3}}\int_{\mathbb{R}^d}|v_1 M\mathrm{Im}N_{\lambda,\eta}|\mathrm{d}v = \int_{A_\eta}\eta^{\frac{1}{3}}|v_1|^{1-\delta}|v_1|^\delta M|\mathrm{Im}N_{\lambda,\eta}|\mathrm{d}v + \int_{A_\eta^c}\eta^{\frac{1}{3}}|v_1\mathrm{Im}M_{\lambda,\eta}|M\mathrm{d}v, \tag{3.3.26}$$

and since on $A_\eta := \{|v_1| \leqslant s_0\eta^{-\frac{1}{3}}\}$ we have: $|v_1|^{1-\delta} \leqslant s_0^{1-\delta}\eta^{\frac{\delta-1}{3}}$, and on $A_\eta^c := \{|v_1| \geqslant s_0\eta^{-\frac{1}{3}}\}$, $M(v) \leqslant s_0^{-\delta}\eta^{\frac{\delta}{3}}|v_1|^\delta M(v)$ then, we obtain

$$\eta^{\frac{1}{3}}\int_{\mathbb{R}^d}|v_1 M\mathrm{Im}N_{\lambda,\eta}|\mathrm{d}v \tag{3.3.27}$$
$$\leqslant \frac{\eta^{\frac{\delta}{3}}}{s_0^\delta}\big\||v_1|^\delta M\big\|_2\Big(s_0\|N_{\lambda,\eta}\|_{L^2(A_\eta)} + \big\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\big\|_{L^2(A_\eta^c)}\Big)$$
$$\leqslant \frac{1}{2}\frac{\eta^{\frac{\delta}{3}}}{s_0^\delta}\Big(s_0\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + \big\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\big\|^2_{L^2(A_\eta^c)} + 2s_0\big\||v_1|^\delta M\big\|_2^2\Big). \tag{3.3.28}$$

Thus, returning to (3.3.25) we get:

$$\begin{aligned}\left\|\nabla_v\left(\frac{N_{\lambda,\eta}}{M}\right)M\right\|^2_{L^2(\mathbb{R}^d)} &\lesssim \eta^{\frac{2}{3}}(|\lambda| + s_0^{1-\delta}\eta^{\frac{\delta}{3}})\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + \eta^{\frac{2}{3}}|\lambda|\|N_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} \\ &\quad + s_0^{-\delta}\eta^{\frac{2+\delta}{3}}\big\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\big\|^2_{L^2(A_\eta^c)} + \eta^{\frac{2}{3}}|\lambda|\|M\|_2^2 + s_0^{1-\delta}\eta^{\frac{2+\delta}{3}}\big\||v_1|^\delta M\big\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Hence the inequality (3.3.23) holds by multiplying the previous one by $s_0^2\eta^{-\frac{2}{3}}$ and adding the term $|c_\eta - 1|\|M\|^2_{L^2(\mathbb{R}^d)}$.

Step 4: Conclusion. In this step, we will combine all the estimates obtained in the previous steps in order to conclude. First, by summing the inequalities (3.3.14) and (3.3.23) obtained in steps 2 and 3 respectively, and since $A_\eta = B_\eta \cup C_\eta$, we obtain:

$$\begin{aligned}\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} &\lesssim \nu_1\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + (s_0^2|\lambda| + 1)\|N_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} + \frac{\eta^{\frac{\delta}{3}}}{s_0^\delta}\big\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\big\|^2_{L^2(A_\eta^c)} \\ &\quad + (s_0^2|\lambda| + |c_\eta - 1|)\|M\|^2_{L^2(\mathbb{R}^d)} + c_2^\eta,\end{aligned} \tag{3.3.29}$$

where $\nu_1 := s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}}$ and $c_2^\eta := s_0^{-\delta}\eta^{\frac{\delta}{3}}\big(s_0^3 + s_0^2|\lambda| + |b(\lambda,\eta)|\big)\big\||v|^\delta M\big\|^2_{L^2(\mathbb{R}^d)}$. Now, since

$$\|N_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} \lesssim \|M_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} + s_0^{-2\delta}\eta^{\frac{2\delta}{3}}\left\||v_1|^\delta M\right\|^2_{L^2(A_\eta^c)},$$

then, using inequality (3.3.2) for the two terms $\|M_{\lambda,\eta}\|^2_{L^2(A_\eta^c)}$ (in the previous inequality) and $\left\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\right\|^2_{L^2(A_\eta^c)}$ (in (3.3.29)), returning to inequality (3.3.29) we obtain:

$$\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} \lesssim \big(\nu_1 + \frac{1}{s_0^3}\big)\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} + (s_0^2|\lambda| + |c_\eta - 1|)\|M\|^2_{L^2(\mathbb{R}^d)} + c_2^\eta.$$

Therefore, we first set s_0 large enough so that $\frac{1}{s_0^3} \leqslant \frac{1}{4}$, then for $|\lambda|$ and η small enough so that $\nu_1 := s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}} \leqslant \frac{1}{4}$, we get:

$$\|N_{\lambda,\eta}\|^2_{L^2(A_\eta)} \lesssim (s_0^2|\lambda| + |c_\eta - 1|)\|M\|^2_{L^2(\mathbb{R}^d)} + c_2^\eta \lesssim 1. \tag{3.3.30}$$

The right-hand side of the inequality above is uniformly bounded since $s_0^2|\lambda| \leqslant \frac{1}{4}$, $|c_\eta - 1| \to 0$ and $c_2^\eta \to 0$ when η goes 0. Indeed, we have

$$\begin{aligned} |c_\eta - 1| &= \left(\int_{\mathbb{R}^d} \frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right)^{-1}\left|\int_{\mathbb{R}^d} \frac{M(M_{\lambda,\eta} - M)}{\langle v\rangle^2}\mathrm{d}v\right| \\ &\leqslant \left\|\frac{M}{\langle v\rangle}\right\|_2^{-1}\left\|\frac{N_{\lambda,\eta}}{\langle v\rangle}\right\|_2 \leqslant C\|N_{\lambda,\eta}\|_{\mathcal{H}_0} \underset{\eta\to 0}{\longrightarrow} 0. \end{aligned} \tag{3.3.31}$$

For c_2^η, we have $c_2^\eta \lesssim \eta^{\frac{\delta}{3}}$ since $|v|^\delta M \in L^2$ for all $\gamma > \frac{d}{2}$ and since $|b(\lambda,\eta)| \lesssim 1$ thanks to the second point of Remark 3.2.19.
Now, we resume all the assumptions we did on s_0, λ et η:

$$\frac{C_1}{s_0}\left(|\lambda| + \frac{1 + |b(\lambda,\eta)|}{s_0^2}\right) \leqslant \frac{1}{2}, \quad \frac{C_2}{s_0^3} \leqslant \frac{1}{4}, \quad C_3\big(s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}}\big) \leqslant \frac{1}{4}$$

and

$$C_4\big(s_0^2|\lambda| + s_0^{3-\delta}\eta^{\frac{\delta}{3}} + s_0^{-\delta}\eta^{\frac{\delta}{3}}|b(\lambda,\eta)|\big) \leqslant \frac{1}{2}.$$

Recall that $\delta := \frac{1}{2}(\gamma - \frac{d}{2}) > 0$ for all $\gamma > \frac{d}{2}$. So, if we start by setting s_0 large enough, then λ small enough, then η small enough, we recover all the previous inequalities.

Finally, by injecting the inequality (3.3.29) into (3.3.2), we obtain:

$$\|M_{\lambda,\eta}\|^2_{L^2(A_\eta^c)} \leqslant \frac{1}{s_0^2}\left\|\eta^{\frac{1}{3}}v_1 M_{\lambda,\eta}\right\|_{L^2(A_\eta^c)} \lesssim \frac{1}{s_0^3}\big(\nu_1 + c_2^\eta\big) \lesssim 1. \tag{3.3.32}$$

Hence, $N_{\lambda,\eta}$ as well as $M_{\lambda,\eta}$ are uniformly bounded in $L^2(\mathbb{R}^d)$. Now, from (3.3.29) and (3.3.32) we obtain:

$$\|N_{\lambda,\eta}\|^2_{L^2(\mathbb{R}^d)} \lesssim |\lambda| + |c_\eta - 1| + c_2^\eta.$$

Hence the inequality (3.3.1) holds with $\nu_\eta := |c_\eta - 1| + c_2^\eta \underset{\eta\to 0}{\longrightarrow} 0$. ☐

3.3.2 Study of the constraint

In this subsection, we will show the existence of a μ, a function of η, such that the constraint $\langle M_{\mu(\eta),\eta} - M, \Phi \rangle = 0$ is satisfied. Let us start by giving the following result, which is a Corollary of Proposition 3.3.1.

Corollary 3.3.3. *Let $M_{\lambda,\eta}$ be the solution to equation (3.2.20). Then, for all $\lambda \in \mathbb{C}$ such that, $|\lambda| \leqslant \lambda_0$ with λ_0 small enough, the following limit holds:*

$$\lim_{\eta \to 0} \int_{\mathbb{R}^d} \eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}(v) M(v) \mathrm{d}v = 0. \tag{3.3.33}$$

For $\lambda = 0$, one has

$$\lim_{\eta \to 0} \int_{\mathbb{R}^d} M_{0,\eta}(v) M(v) \mathrm{d}v = \int_{\mathbb{R}^d} M^2(v) \mathrm{d}v. \tag{3.3.34}$$

Proof. For the first point, we proceed exactly as in (3.3.26), i.e. cutting the integral into two parts $A_\eta := \{|v_1| \leqslant s_0 \eta^{-\frac{1}{3}}\}$ and A_η^c, we write:

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}(v) M(v) \mathrm{d}v \right| \leqslant \eta^{\frac{1}{3}} & \int_{A_\eta} |v_1|^{1-\delta} |v_1|^{\delta} M(v) |M_{\lambda,\eta}(v)| \mathrm{d}v \\ & + \int_{A_\eta^c} |v_1|^{-\delta} |v_1|^{\delta} M(v) |\eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}(v)| \mathrm{d}v \\ & \leqslant s_0^{-\delta} \eta^{\frac{\delta}{3}} \big\| |v_1|^{\delta} M \big\|_2 \Big(s_0 \| M_{\lambda,\eta} \|_{L^2(A_\eta)} + \big\| \eta^{\frac{1}{3}} v_1 M_{\lambda,\eta} \big\|_{L^2(A_\eta^c)} \Big). \end{aligned}$$

Then, thanks to (3.3.30) and (3.3.32), we get

$$\left| \int_{\mathbb{R}^d} \eta^{\frac{1}{3}} v_1 M_{\lambda,\eta}(v) M(v) \mathrm{d}v \right| \lesssim \eta^{\frac{\delta}{3}} \underset{\eta \to 0}{\longrightarrow} 0.$$

For the second point, for $\lambda = 0$, we write

$$\left| \int_{\mathbb{R}^d} \big[M_{0,\eta}(v) - M(v) \big] M(v) \mathrm{d}v \right| \leqslant \| N_{0,\eta} \|_{L^2(\mathbb{R}^d)} \| M \|_{L^2(\mathbb{R}^d)},$$

and the limit (3.3.34) holds true thanks to the inequality (3.3.1) of Proposition 3.3.1. $\square$

Proposition 3.3.4 (Constraint)**.** *Define*

$$B(\lambda, \eta) := \eta^{-\frac{2}{3}} b(\lambda, \eta).$$

1. *The expression of $B(\lambda, \eta)$ is given by*

$$B(\lambda, \eta) = \eta^{-\frac{2}{3}} \langle N_{\lambda,\eta}, \Phi \rangle = \int_{\mathbb{R}^d} (\lambda - \mathrm{i} \eta^{\frac{1}{3}} v) M_{\lambda,\eta}(v) M(v) \mathrm{d}v. \tag{3.3.35}$$

2. *The η order of $B(\lambda,\eta)$ in its expansion with respect to λ is given by*

$$\lim_{\eta\to 0} \frac{\partial B}{\partial \lambda}(0,\eta) = \int_{\mathbb{R}^d} M^2(v)\mathrm{d}v. \tag{3.3.36}$$

3. *There exists $\tilde{\eta}_0, \tilde{\lambda}_0 > 0$ small enough, a function $\tilde{\lambda} : \{|\eta| \leqslant \tilde{\eta}_0\} \longrightarrow \{|\lambda| \leqslant \tilde{\lambda}_0\}$ such that,*
for all $(\lambda,\eta) \in [0,\tilde{\eta}_0[\times\{|\lambda| < \tilde{\lambda}_0\}$, $\lambda = \tilde{\lambda}(\eta)$ and the constraint is satisfied:

$$B(\lambda,\eta) = B(\tilde{\lambda}(\eta),\eta) = 0.$$

Consequently, $\mu(\eta) = \eta^{\frac{2}{3}}\tilde{\lambda}(\eta)$ is the eigenvalue associated to the eigenfunction $M_\eta := M_{\tilde{\lambda}(\eta),\eta}$ for the operator $\mathcal{L}_\eta$, and the couple $\big(\mu(\eta), M_\eta\big)$ is solution to the spectral problem (3.1.9).

Proof. 1. The first point is obtained by integrating the equation of $M_{\lambda,\eta}$ multiplied by M, and using the assumption $\langle M, \Phi\rangle = 1$.
2. This point is exactly the limit (3.3.34) of Corollary 3.3.3.
3. The third point follows from the Implicit Function Theorem applied to the function B around the point $(\lambda,\eta) = (0,0)$. □

3.3.3 Approximation of the eigenvalue

In this subsection, we will give an approximation for the eigenvalue $\mu(\eta)$ given in Proposition 3.3.4. The study of this limit is based on some estimates on $M_{0,\eta}$, the solution of equation (3.2.1) for $\lambda = 0$, as well as the solution of the rescaled equation.

Before giving the Proposition which summarizes the essential points of this subsection, we will first start by introducing the rescaled function of $M_{0,\eta}$ as well as the equation satisfied by this function. Recall that $M_{0,\eta}$ satisfies the equation:

$$\big[Q + \mathrm{i}\eta v_1\big] M_{0,\eta}(v) = -b(0,\eta)\Phi(v), \quad v \in \mathbb{R}^d,$$

with $Q = -\frac{1}{M}\nabla_v \cdot \big(M^2 \nabla_v\big(\frac{\cdot}{M}\big)\big)$ and $b(0,\eta) = \langle M_{0,\eta} - M, \Phi\rangle$. Then, the rescaled function H_η defined by $H_\eta(s) := \eta^{-\frac{\gamma}{3}} M_{0,\eta}(\eta^{-\frac{1}{3}}s)$ is solution to the rescaled equation

$$\big[Q_\eta + \mathrm{i}s_1\big] H_\eta(s) = -\eta^{-\frac{\gamma+2}{3}} b(0,\eta)\Phi_\eta(s), \quad s \in \mathbb{R}^d, \tag{3.3.37}$$

where

$$Q_\eta := -\frac{1}{|s|_\eta^{-\gamma}} \nabla_s \cdot \left(|s|_\eta^{-2\gamma} \nabla_s \left(\frac{\cdot}{|s|_\eta^{-\gamma}} \right) \right), \quad |s|_\eta^{-\gamma} := \eta^{-\frac{\gamma}{3}} M(\eta^{-\frac{1}{3}}s) = \big(\eta^{\frac{2}{3}} + |s|^2\big)^{-\frac{\gamma}{2}}$$

and

$$\Phi_\eta(s) := \Phi(\eta^{-\frac{1}{3}}s) = c_{\gamma,d}\, \eta^{\frac{\gamma+2}{3}} |s|_\eta^{-\gamma-2}. \tag{3.3.38}$$

Note that: $Q(M) = 0$ implies that $Q_\eta(|s|_\eta^{-\gamma}) = 0$.

Proposition 3.3.5 (Approximation of the eigenvalue). *Let* $\alpha := \frac{2\gamma-d+2}{3}$ *for all* $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$. *The eigenvalue* $\mu(\eta)$ *satisfies*

$$\mu(\eta) = \overline{\mu}(-\eta) = \kappa|\eta|^\alpha\big(1 + O(|\eta|^\alpha)\big), \tag{3.3.39}$$

where κ *is a positive constant given by*

$$\kappa := -2C_\beta^2 \int_{\{s_1>0\}} s_1|s|^{-\gamma}\mathrm{Im}H_0(s)\mathrm{d}s, \tag{3.3.40}$$

and where H_0 *is the unique solution to*

$$\Big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|^2} + \mathrm{i}s_1\Big]H_0(s) = 0,\ \ s \in \mathbb{R}^d \setminus \{0\}, \tag{3.3.41}$$

satisfying

$$\int_{\{|s_1|\geqslant 1\}} |H_0(s)|^2\mathrm{d}s < +\infty \ \ \text{and} \ \ H_0(s) \underset{0}{\sim} |s|^{-\gamma}. \tag{3.3.42}$$

Remark 3.3.6. Note that the existence of solutions for equation (3.3.41) is obtained by passing to the limit in the rescaled equation (3.3.37), while the uniqueness is obtained by an integration by part on $\mathbb{R}^d \setminus \{0\}$, using the two conditions of (3.3.42).

In order to get the Proposition 3.3.5, we need to prove the following series of lemmas:

The first one show that the small velocities in the first direction do not participate in the limit of the diffusion coefficient.

Lemma 3.3.7 (Small velocities).

1. *For all* $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$, *one has*

$$\int_{\{|v_1|\leqslant R\}} \left|\frac{\mathrm{Im}M_{0,\eta}(v)}{\langle v\rangle}\right|^2 \mathrm{d}v \lesssim \eta. \tag{3.3.43}$$

2. *For all* $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$

$$\lim_{\eta\to 0} \eta^{1-\alpha} \int_{\{|v_1|\leqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v = 0. \tag{3.3.44}$$

The second one contains some important estimates on the rescaled solution for large velocities.

Lemma 3.3.8 (Large velocities). *Let* $s_0 > 0$ *be fixed, large enough. We have the following estimates, uniform with respect to* η, *for the rescaled solution:*

1. *For all* $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$, *one has*

$$\big\||s_1|^{\frac{1}{2}}\mathrm{Im}H_\eta\big\|_{L^2(\{|s_1|\leqslant s_0\})} + \|s_1\mathrm{Im}H_\eta\|_{L^2(\{|s_1|\geqslant s_0\})} \lesssim 1. \tag{3.3.45}$$

2. For all $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$, *one has*

$$\left\| \frac{\mathrm{Im} H_\eta}{|s|_\eta} \right\|_{L^2(\{|s_1| \leqslant s_0\})} + \|s_1 \mathrm{Im} H_\eta\|_{L^2(\{|s_1| \geqslant s_0\})} \lesssim 1. \tag{3.3.46}$$

Proof of Lemma 3.3.7.

1. By Remark 3.2.19, since $M_{0,\eta}$ is symmetric with respect to v_1 in the following sense: $\bar{M}_{\bar{\lambda},\eta}(-v_1, v') = M_{\lambda,\eta}(v_1, v')$ then, $\mathrm{Im} M_{0,\eta}$ is odd with respect to v_1. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{M \mathrm{Im} M_{0,\eta}}{\langle v \rangle^2} \mathrm{d}v &= \int_{\mathbb{R}^{d-1}} \left[\int_{-\infty}^0 M \mathrm{Im} M_{0,\eta}(v_1, v') \frac{\mathrm{d}v_1}{\langle v \rangle^2} + \int_0^\infty M \mathrm{Im} M_{0,\eta}(v_1, v') \frac{\mathrm{d}v_1}{\langle v \rangle^2} \right] \mathrm{d}v' \\ &= \int_{\mathbb{R}^{d-1}} \int_0^\infty \left[\mathrm{Im} \bar{M}_{0,\eta}(v_1, v') + \mathrm{Im} M_{0,\eta}(v_1, v') \right] M \frac{\mathrm{d}v_1}{\langle v \rangle^2} \mathrm{d}v' \\ &= 0.\end{aligned}$$

Note that we used the *symmetry* of M in the previous equalities. Thus, the function $\mathrm{Im} M_{0,\eta}$ satisfies the condition (3.2.5). Then, by the Hardy-Poincaré inequality (3.2.4), there exists a positive constant $C_{\gamma,d}$ such that:

$$\int_{\{|v_1| \leqslant R\}} \left| \frac{\mathrm{Im} M_{0,\eta}(v)}{\langle v \rangle} \right|^2 \mathrm{d}v \leqslant \left\| \frac{\mathrm{Im} M_{0,\eta}}{\langle v \rangle} \right\|_2^2 \leqslant C_{\gamma,d} \left\| \nabla_v \left(\frac{\mathrm{Im} M_{0,\eta}}{M} \right) M \right\|_{L^2(\mathbb{R}^d)}^2.$$

Now, as in step 3 of the proof of Proposition 3.3.1, we have on the one hand,

$$\left\| \nabla_v \left(\frac{\mathrm{Im} M_{0,\eta}}{M} \right) M \right\|_2^2 = \int_{\mathbb{R}^d} Q(\mathrm{Im} M_{0,\eta}) \mathrm{Im} M_{0,\eta} \mathrm{d}v.$$

On the other hand,

$$Q(\mathrm{Im} M_{0,\eta}) = \eta v_1 \mathrm{Re} M_{0,\eta} - \eta \left(\int_{\mathbb{R}^d} v_1 M \mathrm{Re} M_{0,\eta} \mathrm{d}v \right) \Phi.$$

Which implies that,

$$\left| \int_{\mathbb{R}^d} Q(\mathrm{Im} M_{0,\eta}) \mathrm{Im} M_{0,\eta} \mathrm{d}v \right| \leqslant \eta \left\| |v_1|^{\frac{1}{2}} M_{0,\eta} \right\|_2 \left(\left\| |v_1|^{\frac{1}{2}} M_{0,\eta} \right\|_2 + \left\| |v_1|^{\frac{1}{2}} M \right\|_2 \|M_{0,\eta}\|_2 \|\Phi\|_2 \right).$$

Hence the inequality (3.3.43) holds thanks to $\left(1 + |v_1|^{\frac{1}{2}}\right) M_{0,\eta} \in L^2(\mathbb{R}^d)$ for $\gamma > \frac{d+1}{2}$ (Proposition 3.3.1).

2. First, since $v_1 \mathrm{Im} M_{0,\eta}$ and M are even functions with respect to v_1, then

$$\int_{\{|v_1| \leqslant R\}} v_1 M_{0,\eta}(v) M(v) \mathrm{d}v = 2 \int_0^R \int_{\mathbb{R}^{d-1}} v_1 \mathrm{Im} M_{0,\eta}(v) M(v) \mathrm{d}v' \mathrm{d}v_1. \tag{3.3.47}$$

Case 1: $\gamma \in]\frac{d}{2}, \frac{d+1}{2}]$. We have by Cauchy-Schwarz,

$$\eta^{1-\alpha}\left|\int_0^R\int_{\mathbb{R}^{d-1}} v_1 \mathrm{Im} M_{0,\eta}(v)M(v)\mathrm{d}v\right| \leqslant R\eta^{1-\alpha}\|\mathrm{Im}M_{0,\eta}\|_2\|M\|_2 \leqslant R\eta^{1-\alpha}\|N_{0,\eta}\|_2\|M\|_2 \underset{\eta\to 0}{\longrightarrow} 0,$$

since $1-\alpha = \frac{1+d-2\gamma}{3} \geqslant 0$ for all $\gamma \leqslant \frac{d+1}{2}$ and $\|\mathrm{Im}M_{0,\eta}\|_2 = \|\mathrm{Im}N_{0,\eta}\|_2 \leqslant \|N_{0,\eta}\|_2 \underset{\eta\to 0}{\longrightarrow} 0$.
Case 2: $\gamma \in]\frac{d+1}{2}, \frac{d+4}{2}[$. Similary, we have by Cauchy-Schwarz,

$$\begin{aligned}\eta^{1-\alpha}\left|\int_{\{|v_1|\leqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v\right| &\leqslant \eta^{1-\alpha}\left\|v_1\langle v\rangle M\right\|_{L^2(\{|v_1|\leqslant R\})}\left\|\frac{\mathrm{Im}M_{0,\eta}}{\langle v\rangle}\right\|_{L^2(\{|v_1|\leqslant R\})}\\ &\lesssim \eta^{2-\alpha} \underset{\eta\to 0}{\longrightarrow} 0,\end{aligned}$$

thanks to the inequality (3.3.43) and since $\alpha < 2$ for $\gamma < \frac{d+4}{2}$ and $v_1\langle v\rangle M \in L^2(\{|v_1| \leqslant R\})$ for $\gamma < \frac{d+4}{2}$. □

Proof of Lemma 3.3.8. We will establish estimates on different ranges of (rescaled) velocities, and in order to avoid long expressions in the proof, we will fix some notations of "sets" as in the proof of Proposition 3.3.1. Let denote $s := (s_1, s') \in \mathbb{R}\times\mathbb{R}^{d-1}$. Let $s_0 > 0$. We set: $A := \{|s_1| \leqslant s_0\}$ (resp., $\tilde{A} := \{|s_1| \leqslant 2s_0\}$), $B := \{|s| \leqslant s_0\}$, $C := \{|s_1| \leqslant s_0 \leqslant |s'|\}$ (resp., $\tilde{C} := \{|s_1| \leqslant 2s_0 \leqslant 2|s'|\}$) and finally $D := \{|s_1| \geqslant \frac{s_0}{2}\}$. Also, for $\eta > 0$, we denote by $\tilde{K}_\eta$ the function defined by $\tilde{K}_\eta := H_\eta - c_\eta|s|_\eta^{-\gamma}$, with c_η given by

$$c_\eta := \left(\int_{\mathbb{R}^d}|s|_\eta^{-2\gamma-2}\mathrm{d}s\right)^{-1}\int_{\mathbb{R}^d}\frac{|s|_\eta^{-\gamma}H_\eta(s)}{|s|_\eta^2}\mathrm{d}s, \quad \forall\eta > 0.$$

Note that

$$\int_{\mathbb{R}^d}\frac{|s|_\eta^{-\gamma}\tilde{K}_\eta(s)}{|s|_\eta^2}\mathrm{d}s = 0.$$

Thus, $\tilde{K}_\eta$ satisfies the orthogonality condition (3.2.5) of the Hardy-Poincaré Lemma 3.2.5.

Remark 3.3.9.

- Observe that
$$c_\eta := \left(\int_{\mathbb{R}^d}\frac{M^2}{\langle v\rangle^2}\mathrm{d}v\right)^{-1}\int_{\mathbb{R}^d}\frac{M(v)M_{0,\eta}(v)}{\langle v\rangle^2}\mathrm{d}v.$$
Hence, $c_\eta \underset{\eta\to 0}{\longrightarrow} 1$ by (3.3.31).

- Since $\bar{M}_{0,\eta}(-v_1, v') = M_{0,\eta}(v_1, v')$ for all $(v_1, v') \in \mathbb{R}\times\mathbb{R}^{d-1}$, then
$$\mathrm{Im}H_\eta(-s_1, s') = -\mathrm{Im}H_\eta(s_1, s'), \quad \forall(s_1, s') \in \mathbb{R}\times\mathbb{R}^{d-1}.$$
Therefore, Im $c_\eta = 0$.

1. Let $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$. To prove the first point of this lemma, we will proceed exactly as in the proof of Proposition 3.3.1. We estimate $\big\||s_1|^{\frac12}K_\eta\big\|_{L^2(B)}$ using the Hardy-Poincaré inequality, $\big\||s_1|^{\frac12}K_\eta\big\|_{L^2(C)}$ using the weighted Poincaré inequality, Lemma 3.3.2, and estimate $\|s_1H_\eta\|_{L^2(A^c)}$ using the equation of H_η. Thus, we obtain the inequality (3.3.45) by combining these estimates and since $|s_1|^{\frac12}|s|_\eta^{-\gamma}\leqslant |s_1|^{\frac12}|s|^{-\gamma}\in L^2(A)$ for $\gamma<\frac{d+1}{2}$.

Estimation of $\big\||s_1|^{\frac12}K_\eta\big\|_{L^2(B)}$. Recall that $B:=\{|s|\leqslant s_0\}$. On the one hand, we have:

$$\big\||s_1|^{\frac12}K_\eta\big\|^2_{L^2(B)}\lesssim \big\||s_1|^{\frac12}\tilde K_\eta\big\|^2_{L^2(B)}+|c_\eta-1|\big\||s_1|^{\frac12}|s|_\eta^{-\gamma}\big\|^2_{L^2(B)},$$

and by the Hardy-Poincaré inequality (3.2.4) we get

$$\big\||s_1|^{\frac12}\tilde K_\eta\big\|^2_{L^2(B)}\leqslant s_0^3\left\|\frac{\tilde K_\eta}{|s|_\eta}\right\|^2_{L^2(B)}\lesssim_{\gamma,d} s_0^3\left\|\nabla_s\left(\frac{\tilde K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_2^2.$$

Therefore,

$$\big\||s_1|^{\frac12}K_\eta\big\|^2_{L^2(B)}\lesssim s_0^3\left\|\nabla_s\left(\frac{\tilde K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_2^2+|c_\eta-1|\big\||s_1|^{\frac12}|s|_\eta^{-\gamma}\big\|^2_{L^2(B)}. \tag{3.3.48}$$

On the other hand, since $\nabla_s\big(\frac{\tilde K_\eta}{|s|_\eta^{-\gamma}}\big)=\nabla_s\big(\frac{K_\eta}{|s|_\eta^{-\gamma}}\big)$, then

$$\left\|\nabla_s\left(\frac{\tilde K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_2^2=\mathrm{Re}\int_{\mathbb{R}^d}Q_\eta(K_\eta)\bar K_\eta \mathrm{d}s\leqslant\left|\mathrm{Re}\int_{\mathbb{R}^d}\mathrm{i}s_1|s|_\eta^{-\gamma}\bar K_\eta\mathrm{d}s\right|=\left|\mathrm{Re}\int_{\mathbb{R}^d}s_1|s|_\eta^{-\gamma}\mathrm{Im}K_\eta\mathrm{d}s\right|.$$

Now, since Im $c_\eta=0$, by the second item of Remark 3.3.9, we write $|\mathrm{Im}K_\eta|=|\mathrm{Im}H_\eta|\leqslant|H_\eta|$. Thus, by splitting the integral above into two parts, on $A:=\{|s_1|\leqslant s_0\}$ and on A^c, we obtain:

$$\left\|\nabla_s\left(\frac{\tilde K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_2^2\leqslant\big\||s_1|^{\frac12}|s|_\eta^{-\gamma}\big\|_{L^2(A)}\big\||s_1|^{\frac12}K_\eta\big\|_{L^2(A)}+\big\||s|_\eta^{-\gamma}\big\|_{L^2(A^c)}\big\|s_1H_\eta\big\|_{L^2(A^c)}.$$

Hence, returning to (3.3.48), we get:

$$\begin{aligned}\big\||s_1|^{\frac12}K_\eta\big\|^2_{L^2(B)}\leqslant{}&\frac14\big\||s_1|^{\frac12}K_\eta\big\|^2_{L^2(A)}+Cs_0^3\big\||s|^{-\gamma}\big\|_{L^2(A^c)}\big\|s_1H_\eta\big\|_{L^2(A^c)}\\&+C\left(s_0^6\big\||s_1|^{\frac12}|s|^{-\gamma}\big\|^2_{L^2(A)}+|c_\eta-1|\big\||s_1|^{\frac12}|s|^{-\gamma}\big\|^2_{L^2(B)}\right)\end{aligned} \tag{3.3.49}$$

Estimation of $\big\||s_1|^{\frac12}K_\eta\big\|_{L^2(C)}$. Recall that $C:=\{|s_1|\leqslant s_0\leqslant|s'|\}$. This step is identical to step 2 of the proof of Proposition 3.3.1. We start with estimate on $\big\|\zeta_{s_0}K_\eta\big\|^2_{L^2(\tilde C)}$. We have by using the inequality (3.3.15):

$$\big\|\zeta_{s_0}K_\eta\big\|^2_{L^2(\tilde C)}\lesssim s_0^2\left\|\partial_{s_1}\left(\frac{\zeta_{s_0}K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_{L^2(\tilde C)}, \tag{3.3.50}$$

where $\zeta_{s_0}(s_1) := \zeta\big(\frac{s_1}{s_0}\big)$, with $\zeta \in C^\infty(\mathbb{R})$ such that: $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ on $B(0,1)$ and $\zeta \equiv 0$ outside of $B(0,2)$, and where $\tilde{C} := \{|s_1| \leqslant 2s_0 \leqslant 2|s'|\}$. We have

$$\left\|\partial_{s_1}\left(\frac{\zeta_{s_0} K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_{L^2(\tilde{C})} \leqslant \left\|\nabla_s\left(\frac{\zeta_{s_0} K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_{L^2(\tilde{C})} = \mathrm{Re}\int_{\tilde{C}} Q_\eta(\zeta_{s_0}K_\eta)\zeta_{s_0}\bar{K}_\eta \mathrm{d}s.$$

On the other hand, as in (3.3.17):

$$\mathrm{Re}\int_{\tilde{C}} Q_\eta(\zeta_{s_0}K_\eta)\zeta_{s_0}\bar{K}_\eta \mathrm{d}s = \mathrm{Re}\int_{\tilde{C}} Q_\eta(K_\eta)\bar{K}_\eta\zeta_{s_0}^2 \mathrm{d}s + \int_{\tilde{C}} |\zeta'_{s_0}\bar{K}_\eta|^2 \mathrm{d}s.$$

Then,

$$\begin{aligned}\mathrm{Re}\int_{\tilde{C}} Q_\eta(\zeta_{s_0}K_\eta)\zeta_{s_0}\bar{K}_\eta \mathrm{d}s \leqslant \left|\mathrm{Re}\int_{\tilde{C}} -\mathrm{i}s_1|s|_\eta^{-\gamma}\bar{K}_\eta\zeta_{s_0}^2\mathrm{d}s\right| &+ \eta^{-\frac{2+\gamma}{3}}|b(0,\eta)|\int_{\tilde{C}}\left|\Phi_\eta\bar{K}_\eta\zeta_{s_0}^2\right|\mathrm{d}s \\ &+ \int_{\tilde{C}} |\zeta'_{s_0}K_\eta|^2\mathrm{d}s.\end{aligned}$$

For the first term, we get

$$\left|\mathrm{Re}\int_{\tilde{C}} -\mathrm{i}s_1|s|_\eta^{-\gamma}\bar{K}_\eta\zeta_{s_0}^2\mathrm{d}s\right| \leqslant \left\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\right\|_{L^2(\tilde{C})}\left\||s_1|^{\frac{1}{2}}\zeta_{s_0}K_\eta\right\|_{L^2(\tilde{C})}.$$

For the second term, recall that $\Phi_\eta(s) := c_{\gamma,d}\,\eta^{\frac{\gamma+2}{3}}|s|_\eta^{-\gamma-2}$, (3.3.38), we get:

$$\begin{aligned}\eta^{-\frac{2+\gamma}{3}}|b(0,\eta)|\int_{\tilde{C}}\left|\Phi_\eta\bar{K}_\eta\zeta_{s_0}^2\right|\mathrm{d}s &\lesssim s_0^{-\frac{5}{2}}|b(0,\eta)|\left\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\right\|_{L^2(\tilde{C})}\|\zeta_{s_0}K_\eta\|_{L^2(\tilde{C})} \\ &\lesssim s_0^{-\frac{5}{2}}|b(0,\eta)|\left(\left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|^2_{L^2(\tilde{C})} + \|\zeta_{s_0}K_\eta\|^2_{L^2(\tilde{C})}\right).\end{aligned}$$

For the last term, since $\zeta'_{s_0} \equiv 0$ except on $\tilde{C}\setminus B^c := \{s_0 \leqslant |s_1| \leqslant 2s_0 \leqslant 2|s'|\}$ where $|\zeta'_{s_0}(s_1)| \lesssim \frac{1}{s_0}$, and since $\tilde{C}\setminus B^c \subset A^c$, then

$$\int_{\tilde{C}} |\zeta'_{s_0}K_\eta|^2\mathrm{d}s = \int_{\tilde{C}\setminus B^c} |\zeta'_{s_0}K_\eta|^2\mathrm{d}s \lesssim \frac{1}{s_0^2}\|K_\eta\|^2_{L^2(\tilde{C}\setminus B^c)} \lesssim \frac{1}{s_0^4}\left\|s_1H_\eta\right\|^2_{L^2(A^c)} + \frac{1}{s_0^2}\||s|_\eta^{-\gamma}\|^2_{L^2(A^c)}.$$

Therefore,

$$\begin{aligned}\|\zeta_{s_0}K_\eta\|^2_{L^2(\tilde{C})} \lesssim s_0^2\left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|_{L^2(\tilde{C})}\left\||s_1|^{\frac{1}{2}}\zeta_{s_0}K_\eta\right\|_{L^2(\tilde{C})} &+ s_0^{-\frac{1}{2}}|b(0,\eta)|\|\zeta_{s_0}K_\eta\|^2_{L^2(\tilde{C})} \\ + s_0^{-\frac{1}{2}}|b(0,\eta)|\left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|^2_{L^2(\tilde{C})} &+ \frac{1}{s_0^2}\left\|s_1H_\eta\right\|^2_{L^2(A^c)} + \||s|^{-\gamma}\|^2_{L^2(A^c)}.\end{aligned}$$

Since $|b(0,\eta)| \lesssim \eta^{\frac{2}{3}}$, thanks to (3.3.33) and (3.3.35), then for η small enough we get

$$\|\zeta_{s_0} K_\eta\|^2_{L^2(\tilde{C})} \lesssim s_0^2 \big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|_{L^2(\tilde{C})} \big\||s_1|^{\frac{1}{2}}\zeta_{s_0} K_\eta\big\|_{L^2(\tilde{C})} + \frac{1}{s_0^2}\big\|s_1 H_\eta\big\|^2_{L^2(A^c)} + s_0^{-\frac{1}{2}}|b(0,\eta)|\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{C})} + \||s|^{-\gamma}\|^2_{L^2(A^c)}.$$

Now, by (3.3.50), we get

$$\big\||s_1|^{\frac{1}{2}}\zeta_{s_0} K_\eta\big\|^2_{L^2(\tilde{C})} \leqslant 2s_0\big\|\zeta_{s_0} K_\eta\big\|^2_{L^2(\tilde{C})} \lesssim s_0^3 \left\|\partial_{s_1}\left(\frac{\zeta_{s_0} K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_{L^2(\tilde{C})}.$$

Then,

$$\big\||s_1|^{\frac{1}{2}}\zeta_{s_0} K_\eta\big\|^2_{L^2(\tilde{C})} \lesssim s_0^3 \big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|_{L^2(\tilde{C})} \big\||s_1|^{\frac{1}{2}}\zeta_{s_0} K_\eta\big\|_{L^2(\tilde{C})} + \frac{1}{s_0}\big\|s_1 H_\eta\big\|^2_{L^2(A^c)} + s_0^{\frac{1}{2}}|b(0,\eta)|\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{C})} + s_0\||s|^{-\gamma}\|^2_{L^2(A^c)}$$

Finally, since $\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(C)} \leqslant \big\||s_1|^{\frac{1}{2}}\zeta_{s_0} K_\eta\big\|^2_{L^2(\tilde{C})}$, we get:

$$\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(C)} \lesssim \frac{1}{s_0}\big\|s_1 H_\eta\big\|^2_{L^2(A^c)} + s_0^6\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{C})} + s_0\||s|^{-\gamma}\|^2_{L^2(A^c)} \tag{3.3.51}$$

Conclusion: Since $A = B \cup C$ then, by summing the two inequalities (3.3.49) and (3.3.51) we find:

$$\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)} \leqslant \frac{1}{4}\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)} + \frac{C}{s_0}\big\|s_1 H_\eta\big\|^2_{L^2(A^c)} + C(s_0^6 + |c_\eta - 1|)\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{A})} + Cs_0^3\||s|^{-\gamma}\|^2_{L^2(A^c)}$$

Hence,

$$\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)} \lesssim \frac{1}{s_0}\big\|s_1 H_\eta\big\|^2_{L^2(A^c)} + (s_0^6 + |c_\eta - 1|)\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{A})} + s_0^3\||s|^{-\gamma}\|^2_{L^2(A^c)}, \tag{3.3.52}$$

where $\tilde{A} := \{|s_1| \leqslant 2s_0\}$. So it remains to estimate $\big\|s_1 H_\eta\big\|_{L^2(A^c)}$, where $A^c := \{|s_1| \geqslant s_0\}$.

Estimation of $\big\|s_1 H_\eta\big\|_{L^2(A^c)}$. We have $\big\|s_1 H_\eta\big\|_{L^2(A^c)} \leqslant \big\|s_1 \chi_{s_0} H_\eta\big\|_{L^2(D)}$, where $\chi_{s_0}(s_1) := \chi\big(\frac{s_1}{s_0}\big)$, with $\chi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 0$ on $B(0, \frac{1}{2})$ and $\chi \equiv 1$ outside $B(0,1)$ and where $D := \{|s_1| \geqslant \frac{s_0}{2}\}$. Now, integrating the equation of H_η against $s_1 \bar{H}_\eta \chi^2_{s_0}$ and take the imaginary part, we obtain:

$$\int_D \big|s_1 \chi_{s_0} H_\eta\big|^2 \mathrm{d}s = -\mathrm{Im}\Big(\int_D Q_\eta(H_\eta) s_1 \bar{H}_\eta \chi^2_{s_0} \mathrm{d}s\Big) - \eta^{-\frac{\gamma+2}{3}}\mathrm{Im}\Big(b(0,\eta)\int_D \Phi_\eta s_1 \bar{H}_\eta \chi^2_{s_0}\mathrm{d}s\Big) \tag{3.3.53}$$

Let's start with the second term which is simpler, by (3.3.38) we have,

$$\begin{aligned}\eta^{-\frac{\gamma+2}{3}}\left|\mathrm{Im}\left(b(0,\eta)\int_D \Phi_\eta s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right)\right| &\lesssim \frac{1}{s_0^2}|b(0,\eta)|\||s|_\eta^{-\gamma}\|_{L^2(D)}\big\|s_1\chi_{s_0}H_\eta\big\|_{L^2(D)}\\ &\lesssim \frac{1}{s_0^2}|b(0,\eta)|\left(\||s|^{-\gamma}\|_{L^2(D)}^2+\big\|s_1\chi_{s_0}H_\eta\big\|_{L^2(D)}^2\right).\end{aligned}$$

For the first term, we will proceed exactly as for E_1^η (first step in the proof of the Proposition 3.3.1). By integration by parts, we write:

$$\begin{aligned}\left|\mathrm{Im}\int_D Q_\eta(H_\eta)s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| &= \left|\mathrm{Im}\int_D \partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\left[\chi_{s_0}\bar{H}_\eta+2s_1\bar{H}_\eta\chi_{s_0}'\right]\mathrm{d}s\right|\\ &\leqslant \left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}\left(\|\chi_{s_0}H_\eta\|_{L^2(D)}+2\big\|s_1\chi_{s_0}'H_\eta\big\|_{L^2(D)}\right)\\ &\leqslant \frac{1}{2}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2+\frac{2}{s_0^2}\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2\\ &+\frac{s_0}{2}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2+\frac{1}{s_0}\big\|s_1\chi_{s_0}'H_\eta\big\|_{L^2(D)}^2.\end{aligned}$$

Now, since $\chi_{s_0}'\equiv 0$ except on: $D\setminus A^c := \{\frac{s_0}{2}\leqslant |s_1|\leqslant s_0\}\subset A$ where $|\chi_{s_0}'(s_1)|\lesssim \frac{1}{s_0}$, and since $|H_\eta|\leqslant |K_\eta|+|s|_\eta^{-\gamma}$ then,

$$\begin{aligned}\big\|s_1\chi_{s_0}'H_\eta\big\|_{L^2(D)}^2 \lesssim \|H_\eta\|_{L^2(D\setminus A^c)}^2 \lesssim \|K_\eta\|_{L^2(D\setminus A^c)}^2+\||s|_\eta^{-\gamma}\|_{L^2(D\setminus A^c)}^2\\ \lesssim \frac{1}{s_0}\left(\big\||s_1|^{\frac{1}{2}}K_\eta\big\|_{L^2(A)}^2+\big\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\big\|_{L^2(A)}^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\mathrm{Im}\int_D Q_\eta(H_\eta)s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| \leqslant s_0\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2+\frac{2}{s_0^2}\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2\\ +\frac{C}{s_0^2}\left(\big\||s_1|^{\frac{1}{2}}K_\eta\big\|_{L^2(A)}^2+\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|_{L^2(A)}^2\right).\end{aligned} \tag{3.3.54}$$

Let us now deal with the term $\big\|\partial_{s_1}(\frac{H_\eta}{|s|_\eta^{-\gamma}})|s|_\eta^{-\gamma}\chi_{s_0}\big\|_{L^2(D)}$. By an integration by parts, we can see that:

$$\left\|\nabla_s\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2 = \mathrm{Re}\int_D\left[Q_\eta(H_\eta)\bar{H}_\eta\chi_{s_0}^2-2\chi_{s_0}\chi_{s_0}'\frac{\bar{H}_\eta}{|s|_\eta^{-\gamma}}\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-2\gamma}\right]\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2 &\leqslant \left|\mathrm{Re}\int_D Q_\eta(H_\eta)\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| \\ &\quad + 2\|\chi_{s_0}'H_\eta\|_{L^2(D)}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)} \\ &\leqslant \left|\mathrm{Re}\int_D Q_\eta(H_\eta)\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| \\ &\quad + 2\|\chi_{s_0}'H_\eta\|_{L^2(D)}^2 + \frac{1}{2}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2.\end{aligned}$$

Which implies that,

$$\begin{aligned}\left\|\partial_{s_1}\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\chi_{s_0}\right\|_{L^2(D)}^2 &\leqslant 2\left|\mathrm{Re}\int_D Q_\eta(H_\eta)\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| + 4\|\chi_{s_0}'H_\eta\|_{L^2(D\setminus A^c)}^2 \\ &\lesssim \frac{1}{s_0^2}\left(|b(0,\eta)|\||s|_\eta^{-\gamma}\|_{L^2(D)}\|\chi_{s_0}H_\eta\|_{L^2(D)} + \|H_\eta\|_{L^2(D\setminus A^c)}^2\right) \\ &\lesssim \frac{1}{s_0^3}|b(0,\eta)|\left(\left\|s_1\chi_{s_0}H_\eta\right\|_{L^2(D)}^2 + \||s|^{-\gamma}\|_{L^2(D)}^2\right) \\ &\quad + \frac{1}{s_0^3}\left(\left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 + \left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|_{L^2(A)}^2\right).\end{aligned}$$

Thus, injecting this last inequality into (3.3.54) we obtain:

$$\begin{aligned}\left|\mathrm{Im}\int_D Q_\eta(H_\eta)s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\right| &\lesssim \frac{1}{s_0^2}\Big[(1+|b(0,\eta)|)\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2 + \left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 \\ &\quad + \left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|_{L^2(A)}^2 + |b(0,\eta)|\||s|^{-\gamma}\|_{L^2(D)}^2\Big],\end{aligned}$$

and going back to (3.3.53), using the fact that $|b(0,\eta)| \lesssim 1$ by Remark 3.2.19, we get:

$$\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2 \lesssim \frac{1}{s_0^2}\Big[\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2 + \left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 + \left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|_{L^2(A)}^2 + \||s|^{-\gamma}\|_{L^2(D)}^2\Big].$$

Finally, for s_0 large enough, the term $\frac{1}{s_0^2}\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2$ is absorbed and we obtain thanks to the inequality $\|s_1H_\eta\|_{L^2(A^c)}^2 \leqslant \|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2$:

$$\|s_1H_\eta\|_{L^2(A^c)}^2 \lesssim \frac{1}{s_0^2}\left(\left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 + \left\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\right\|_{L^2(A)}^2 + \||s|^{-\gamma}\|_{L^2(D)}^2\right). \tag{3.3.55}$$

Now, by injecting the inequality (3.3.55) into (3.3.52), we get:

$$\left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 \leqslant C\left(\frac{1}{s_0^3}\left\||s_1|^{\frac{1}{2}}K_\eta\right\|_{L^2(A)}^2 + s_0^6\left\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\right\|_{L^2(\tilde{A})}^2 + s_0^3\left\||s|_\eta^{-\gamma}\right\|_{L^2(D)}^2\right).$$

Where we used the fact that $A^c \subset D$ and $|c_\eta - 1| \lesssim 1$ by Remark 3.3.9. Finally, for s_0 large enough, the norm $\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)}$ which appears in the right hand side of the previous inequality is absorbed, from where:

$$\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)} \lesssim s_0^6\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(\tilde{A})} + s_0^3\big\||s|^{-\gamma}\big\|^2_{L^2(A^c)} \lesssim 1, \tag{3.3.56}$$

since for $\gamma \in]\frac{d}{2}, \frac{d+1}{2}[$: $|s_1|^{\frac{1}{2}}|s|^{-\gamma} \in L^2(\tilde{A})$ and $|s|^{-\gamma} \in L^2(A^c)$. From the inequality (3.3.55) we deduce that $\big\||s_1|^{\frac{1}{2}}K_\eta\big\|^2_{L^2(A)} \lesssim 1$ implies that $\|s_1H_\eta\|^2_{L^2(A^c)} \lesssim 1$. Thus we obtain (3.3.45).

2. Recall that $\tilde{K}_\eta := H_\eta - c_\eta|s|_\eta^{-\gamma}$ satisfies the orthogonality condition (3.2.5) of Hardy-Poincaré inequality (3.2.4) and that $\mathrm{Im}c_\eta = 0$. It follows that, $\mathrm{Im}\tilde{K}_\eta = \mathrm{Im}K_\eta = \mathrm{Im}H_\eta$, so by (3.2.4), we get on the one hand

$$\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} = \left\|\frac{\mathrm{Im}\tilde{K}_\eta}{|s|_\eta}\right\|^2_{L^2(A)} \leqslant \left\|\frac{\tilde{K}_\eta}{|s|_\eta}\right\|^2_2 \lesssim_{\gamma,d} \left\|\nabla_s\left(\frac{\tilde{K}_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_2 = \left\|\nabla_s\left(\frac{K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_2.$$

On the other hand,

$$\left\|\nabla_s\left(\frac{K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_2 = \int_{\mathbb{R}^d} Q_\eta(K_\eta)\bar{K}_\eta \mathrm{d}s = \mathrm{Re}\left(-\mathrm{i}\int_{\mathbb{R}^d} s_1|s|_\eta^{-\gamma}\bar{K}_\eta\mathrm{d}s\right) - \eta^{-\frac{\gamma+2}{3}}b(0,\eta)\int_{\mathbb{R}^d}\Phi_\eta\bar{K}_\eta\mathrm{d}s.$$

We have: $b(0,\eta)\int_{\mathbb{R}^d}\Phi_\eta\bar{K}_\eta\mathrm{d}s \geqslant 0$. Indeed, by performing the change of variable $s = \eta^{\frac{1}{3}}v$, we obtain:

$$\int_{\mathbb{R}^d}\Phi_\eta\bar{K}_\eta\mathrm{d}s = \eta^{\frac{\gamma+d}{3}}\int_{\mathbb{R}^d}\Phi\bar{N}_{0,\eta}\mathrm{d}v = \eta^{\frac{\gamma+d}{3}}\overline{b(0,\eta)}.$$

Now, since $\mathrm{Re}\left(-i\int_{\mathbb{R}^d} s_1|s|_\eta^{-\gamma}\bar{K}_\eta\mathrm{d}s\right) = \int_{\mathbb{R}^d} s_1|s|_\eta^{-\gamma}\mathrm{Im}K_\eta\mathrm{d}s$, then we write:

$$\begin{aligned}\left\|\nabla_s\left(\frac{K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_2 &\leqslant \int_{\mathbb{R}^d}|s_1|s|_\eta^{-\gamma}\mathrm{Im}K_\eta|\mathrm{d}s \\ &= \int_A |s_1|s|_\eta^{-\gamma}\mathrm{Im}K_\eta|\mathrm{d}s + \int_{A^c}|s_1|s|_\eta^{-\gamma}\mathrm{Im}H_\eta|\mathrm{d}s\end{aligned} \tag{3.3.57}$$

Therefore,

$$\begin{aligned}\left\|\nabla_s\left(\frac{K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_2 \leqslant{}& \big\|s_1|s|^{1-\gamma}\big\|_{L^2(A)}\left\|\frac{\mathrm{Im}\tilde{K}_\eta}{|s|_\eta}\right\|_{L^2(A)} \\ &+ s_0^{-\frac{1}{2}}\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|_{L^2(A^c)}\|s_1H_\eta\|_{L^2(A^c)}.\end{aligned} \tag{3.3.58}$$

It remains to estimate the norm $\|s_1H_\eta\|_{L^2(A^c)}$. Recall that $D := \{|s_1| \geqslant \frac{s_0}{2}\}$. We start by estimating $\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}$.

We have; as before; the two equalities:

$$\int_D \big|s_1\chi_{s_0}H_\eta\big|^2 \mathrm{d}s = -\mathrm{Im}\bigg(\int_D Q_\eta(H_\eta)s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\bigg) - \eta^{-\frac{\gamma+2}{3}}\mathrm{Im}\bigg(b(0,\eta)\int_D \Phi_\eta s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\bigg)$$

and

$$\bigg|\mathrm{Im}\int_D Q_\eta(H_\eta)s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\bigg| = \bigg|\mathrm{Im}\int_D \partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)\chi_{s_0}|s|_\eta^{-\gamma}\big[\chi_{s_0}\bar{H}_\eta + 2s_1\bar{H}_\eta\chi_{s_0}'\big]\mathrm{d}s\bigg|.$$

The term on the right in the first equality is treated in the same way as before and we have:

$$\begin{aligned}\eta^{-\frac{\gamma+2}{3}}\bigg|\mathrm{Im}\bigg(b(0,\eta)\int_D \Phi_\eta s_1\bar{H}_\eta\chi_{s_0}^2\mathrm{d}s\bigg)\bigg| &\lesssim s_0^{-\frac{5}{2}}|b(0,\eta)|\big\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\big\|_{L^2(D)}\big\|s_1\chi_{s_0}H_\eta\big\|_{L^2(D)} \\ &\lesssim s_0^{-\frac{5}{2}}|b(0,\eta)|\bigg(\big\||s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma}\big\|_{L^2(D)}^2 + \big\|s_1\chi_{s_0}H_\eta\big\|_{L^2(D)}^2\bigg)\end{aligned} \tag{3.3.59}$$

For the left term in the first equality we write:

$$\bigg|\mathrm{Im}\int_D \partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)\chi_{s_0}|s|_\eta^{-\gamma}\big[\chi_{s_0}\bar{H}_\eta + 2s_1\bar{H}_\eta\chi_{s_0}'\big]\mathrm{d}s\bigg| \leqslant I_1^\eta + I_2^\eta.$$

where

$$I_1^\eta := \bigg|\mathrm{Im}\int_D \chi_{s_0}\bar{H}_\eta\partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)\chi_{s_0}|s|_\eta^{-\gamma}\mathrm{d}s\bigg|$$

and

$$I_2^\eta := \bigg|\mathrm{Im}\int_D s_1\chi_{s_0}\bar{H}_\eta\partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)|s|_\eta^{-\gamma}\chi_{s_0}'\mathrm{d}s\bigg|.$$

Then we write

$$\begin{aligned}I_1^\eta &\leqslant \|\chi_{s_0}H_\eta\|_{L^2(D)}\bigg\|\partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)\chi_{s_0}|s|_\eta^{-\gamma}\bigg\|_{L^2(D)} \\ &\leqslant \frac{1}{s_0}\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}\bigg\|\nabla_s\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)|s|_\eta^{-\gamma}\bigg\|_2 \\ &\leqslant \frac{1}{2s_0}\bigg(\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2 + \bigg\|\nabla_s\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)|s|_\eta^{-\gamma}\bigg\|_2^2\bigg),\end{aligned} \tag{3.3.60}$$

and

$$\begin{aligned}I_2^\eta &\leqslant \|s_1\chi_{s_0}H_\eta\|_{L^2(D)}\|\chi_{s_0}'\|_{L^\infty(D\setminus A^c)}\bigg\|\partial_{s_1}\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)|s|_\eta^{-\gamma}\bigg\|_{L^2(D)} \\ &\lesssim \frac{1}{s_0}\bigg(\|s_1\chi_{s_0}H_\eta\|_{L^2(D)}^2 + \bigg\|\nabla_s\bigg(\frac{H_\eta}{|s|_\eta^{-\gamma}}\bigg)|s|_\eta^{-\gamma}\bigg\|_2^2\bigg).\end{aligned} \tag{3.3.61}$$

Hence, by the inequalities (3.3.59), (3.3.60) and (3.3.61) to estimate $\|s_1\chi_{s_0}H_\eta\|^2_{L^2(D)}$, and by inequality (3.3.58) to estimate $\big\|\nabla_s\big(\frac{H_\eta}{|s|_\eta^{-\gamma}}\big)|s|_\eta^{-\gamma}\big\|_2^2$, we get:

$$\begin{aligned}\|s_1\chi_{s_0}H_\eta\|^2_{L^2(D)} &\lesssim \frac{1}{s_0}\big\|s_1|s|^{1-\gamma}\big\|_{L^2(A)}\left\|\frac{\mathrm{Im}\tilde{K}_\eta}{|s|_\eta}\right\|_{L^2(A)} + s_0^{-\frac{3}{2}}\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|_{L^2(A^c)}\|s_1H_\eta\|_{L^2(A^c)} \\ &+ \frac{1}{s_0}\|s_1\chi_{s_0}H_\eta\|^2_{L^2(D)} + s_0^{-\frac{5}{2}}|b(0,\eta)|\left(\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)} + \big\|s_1\chi_{s_0}H_\eta\big\|^2_{L^2(D)}\right) \\ &\lesssim \frac{1}{s_0}\big\|s_1|s|^{1-\gamma}\big\|_{L^2(A)}\left\|\frac{\mathrm{Im}\tilde{K}_\eta}{|s|_\eta}\right\|_{L^2(A)} + \frac{1}{s_0}\|s_1\chi_{s_0}H_\eta\|^2_{L^2(D)} + \frac{1}{s_0^{3/2}}\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)}.\end{aligned}$$

Hence, for s_0 large enough and since $\|s_1H_\eta\|^2_{L^2(A^c)} \leqslant \|s_1\chi_{s_0}H_\eta\|^2_{L^2(D)}$:

$$\|s_1H_\eta\|^2_{L^2(A^c)} \lesssim \frac{1}{s_0}\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} + \frac{1}{s_0}\big\|s_1|s|^{1-\gamma}\big\|^2_{L^2(A)} + s_0^{-\frac{3}{2}}\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)}. \tag{3.3.62}$$

So, going back to (3.3.58) and using inequality $ab \leqslant Ca^2 + \frac{b^2}{4C}$, we get:

$$\begin{aligned}\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} \leqslant C\left\|\nabla_s\left(\frac{K_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_2^2 &\leqslant \frac{1}{4}\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} + C^2\big\|s_1|s|^{1-\gamma}\big\|^2_{L^2(A)} \\ &+ \frac{C}{2}\big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)} + \frac{C}{2s_0}\|s_1H_\eta\|^2_{L^2(A^c)}\end{aligned}$$

Thus, by (3.3.62) we obtain

$$\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} \leqslant \left(\frac{1}{4} + \frac{C'}{s_0^2}\right)\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} + C'\left(\big\|s_1|s|^{1-\gamma}\big\|^2_{L^2(A)} + \big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)}\right).$$

Finally, for $s_0$large enough

$$\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|^2_{L^2(A)} \lesssim \big\|s_1|s|^{1-\gamma}\big\|^2_{L^2(A)} + \big\||s_1|^{\frac{1}{2}}|s|^{-\gamma}\big\|^2_{L^2(D)} \lesssim 1. \tag{3.3.63}$$

By (3.3.62), it follows that $\|s_1H_\eta\|^2_{L^2(A^c)} \lesssim 1$. Note that for $\gamma \in]\frac{d+1}{2}, \frac{d+4}{2}[$ we have:

$$s_1|s|_\eta|s|_\eta^{-\gamma} \leqslant s_1|s|^{1-\gamma} \in L^2(A) \;\text{ and }\; |s_1|^{\frac{1}{2}}|s|_\eta^{-\gamma} \leqslant |s_1|^{\frac{1}{2}}|s|^{-\gamma} \in L^2(A^c).$$

Hence the inequality (3.3.46) holds. □

The third Lemma contains some complementary estimates on the rescaled solution.

Lemma 3.3.10 (Complementary estimates)**.** *For all $\eta \in [0, \eta_0]$ and for all $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$, the following estimate holds*

$$\left\|\frac{H_\eta - c_\eta|s|_\eta^{-\gamma}}{|s|_\eta}\right\|_{L^2(\mathbb{R}^d)} \lesssim_{\gamma,d} \left\|\nabla_s\left(\frac{H_\eta}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|_{L^2(\mathbb{R}^d)} \lesssim 1. \tag{3.3.64}$$

The last one gives the formula of the diffusion coefficient.

Lemma 3.3.11. *We have the following limit:*

$$\lim_{\eta\to 0} \mathrm{i}\eta^{1-\alpha}\int_{\{|v_1|\geqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v = -2\int_0^\infty\int_{\mathbb{R}^{d-1}} s_1|s|^{-\gamma}\mathrm{Im}H_0(s)\mathrm{d}s, \tag{3.3.65}$$

where H_0 is the unique solution to (3.3.41) *satisfying the conditions* (3.3.42)*.*

Proof of Lemma 3.3.10. We have by the Hardy-Poincaré inequality and the inequality (3.3.57):

$$\begin{aligned}\Lambda_{\gamma,d}\left\|\frac{H_\eta - c_\eta|s|_\eta^{-\gamma}}{|s|_\eta}\right\|^2_{L^2(\mathbb{R}^d)} &\leqslant \left\|\nabla_s\left(\frac{H_\eta - |s|_\eta^{-\gamma}}{|s|_\eta^{-\gamma}}\right)|s|_\eta^{-\gamma}\right\|^2_{L^2(\mathbb{R}^d)}\\ &\leqslant \int_{\mathbb{R}^d}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s\\ &= \int_{\{|s_1|\leqslant s_0\}}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s + \int_{\{|s_1|\geqslant s_0\}}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s.\end{aligned}$$

Case 1: $\gamma\in(\frac{d}{2},\frac{d+1}{2})$. By Cauchy-Schwarz and inequality (3.3.45) of Lemma 3.3.8 we get:

$$\int_{\{|s_1|\leqslant s_0\}}|s_1|s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s \leqslant \left\||s_1|^{\frac12}|s|_\eta^{-\gamma}\right\|_{L^2(\{|s_1|\leqslant s_0\})}\left\||s_1|^{\frac12}\mathrm{Im}H_\eta\right\|_{L^2(\{|s_1|\leqslant s_0\})}\lesssim 1$$

and

$$\int_{\{|s_1|\geqslant s_0\}}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s \leqslant \left\||s|_\eta^{-\gamma}\right\|_{L^2(\{|s_1|\geqslant s_0\})}\left\|s_1H_\eta\right\|_{L^2(\{|s_1|\geqslant s_0\})}\lesssim 1.$$

Case 2: $\gamma\in(\frac{d+1}{2},\frac{d+4}{2})$. Similarly, by Cauchy-Schwarz and inequality (3.3.46) we get:

$$\int_{\{|s_1|\leqslant s_0\}}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s \leqslant \left\|s_1|s|_\eta^{1-\gamma}\right\|_{L^2(\{|s_1|\leqslant s_0\})}\left\|\frac{\mathrm{Im}H_\eta}{|s|_\eta}\right\|_{L^2(\{|s_1|\leqslant s_0\})}\lesssim 1$$

and

$$\int_{\{|s_1|\geqslant s_0\}}|s_1||s|_\eta^{-\gamma}|\mathrm{Im}H_\eta|\mathrm{d}s \leqslant \left\||s|_\eta^{-\gamma}\right\|_{L^2(\{|s_1|\geqslant s_0\})}\left\|s_1H_\eta\right\|_{L^2(\{|s_1|\geqslant s_0\})}\lesssim 1.$$

This completes the proof of the Lemma. □

Proof of Lemma 3.3.11.
First of all, since $\overline{M}_{0,\eta}(-v_1,v') = M_{0,\eta}(v_1,v')$ and $M(-v_1,v') = M(v_1,v')$ for all $v_1\in\mathbb{R}$ and for all $v'\in\mathbb{R}^{d-1}$, thus

$$\mathrm{i}\int_{\{|v_1|\geqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v = -2\int_{\{v_1\geqslant R\}} v_1\mathrm{Im}M_{0,\eta}(v)M(v)\mathrm{d}v.$$

Then, in order to compute the limit

$$\lim_{\eta\to 0} \mathrm{i}\eta^{\frac{d+1-2\gamma}{3}} \int_{\{|v_1|\geqslant R\}} v_1 M_{0,\eta}(v) M(v)\mathrm{d}v = -2\lim_{\eta\to 0} \eta^{\frac{d+1-2\gamma}{3}} \int_{\{v_1\geqslant R\}} v_1 \mathrm{Im} M_{0,\eta}(v) M(v)\mathrm{d}v,$$

we proceed to a change of variable $v = \eta^{-\frac{1}{3}} s$, which means that we compute

$$\lim_{\eta\to 0} \int_{\{|s_1|\geqslant \eta^{\frac{1}{3}} R\}} s_1 |s|_\eta^{-\gamma} \mathrm{Im} H_\eta(s)\mathrm{d}s.$$

For that purpose, we use the *weak-strong* convergence in the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}^{d-1})$. The estimates of Lemma 3.3.8 imply that the sequence H_η defined by

$$\mathsf{H}_\eta(s) := \begin{cases} s_1^{\frac{1}{2}} \mathrm{Im} H_\eta(s), & \gamma \in (\frac{d}{2}, \frac{d+1}{2}], 0 < s_1 \leqslant s_0, \\ |s|_\eta^{-1} \mathrm{Im} H_\eta(s), & \gamma \in (\frac{d+1}{2}, \frac{d+4}{2}), 0 < s_1 \leqslant s_0, \\ s_1 \mathrm{Im} H_\eta(s) \quad \text{for all} & \gamma \in (\frac{d}{2}, \frac{d+4}{2}) \text{ and } s_1 \geqslant s_0, \end{cases}$$

is bounded in $L^2(\mathbb{R}^d)$, uniformly with respect to η, which implies that H_η converges weakly in $L^2(\mathbb{R}^d)$, up to a subsequence. Let's identify this limit that we denote by $\mathsf{H}_0 \in L^2(\mathbb{R}^d)$. We have on the one hand, H_η converges to H_0 in $\mathcal{D}'(\mathbb{R}^d \setminus \{0\})$. Indeed, recall that H_η satisfies the equation

$$\Big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|_\eta^2} + \mathrm{i}s_1 \Big] H_\eta(s) = \eta^{\frac{2}{3}} \frac{\gamma(\gamma+2)}{|s|_\eta^4} H_\eta(s) - \eta^{-\frac{2+\gamma}{3}} b(0,\eta)\Phi_\eta(s).$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$. Then, by integrating the previous equation against φ, we obtain:

$$\int_{\mathbb{R}^d\setminus\{0\}} \Big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|_\eta^2} + \mathrm{i}s_1 \Big] \varphi(s) H_\eta(s)\mathrm{d}s = \eta^{\frac{2}{3}} \int_{\mathbb{R}^d\setminus\{0\}} \frac{\gamma(\gamma+2)}{|s|_\eta^4} \varphi(s) H_\eta(s)\mathrm{d}s$$
$$- \eta^{-\frac{2+\gamma}{3}} b(0,\eta) \int_{\mathbb{R}^d\setminus\{0\}} \Phi_\eta(s)\varphi(s)\mathrm{d}s.$$

Thanks to the uniform bound (3.3.64) and since $\Phi_\eta(s) \lesssim \eta^{\frac{2+\gamma}{3}} |s|^{-2-\gamma}$ and $b(0,\eta) \to 0$ then, passing to the limit when η goes to 0 in the last equality, we obtain that H_η converges to H_0 in $\mathcal{D}'(\mathbb{R}^d \setminus \{0\})$, solution to the equation

$$\Big[-\Delta_s + \frac{\gamma(\gamma-d+2)}{|s|^2} + is_1 \Big] H_0(s) = 0. \tag{3.3.66}$$

Moreover, for all $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$, the function H_η satisfies the estimate

$$\Big\| \frac{H_\eta - c_\eta |s|_\eta^{-\gamma}}{|s|_\eta} \Big\|_{L^2(\{|s_1|\leqslant s_0\})} + \|s_1 H_\eta\|_{L^2(\{|s_1|\geqslant s_0\})} \lesssim 1,$$

thanks to the inequality (3.3.64) and the first point of Lemma 3.3.8 for $\gamma \in (\frac{d}{2}, \frac{d+1}{2}]$, and thanks to the second point of Lemma 3.3.8 for $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$. Therefore H_0 satisfies the estimate

$$\left\| \frac{H_0 - |s|^{-\gamma}}{|s|} \right\|_{L^2(\{|s_1| \leqslant s_0\})} + \|s_1 H_0\|_{L^2(\{|s_1| \geqslant s_0\})} \lesssim 1.$$

Now, $\|s_1 H_0\|_{L^2(\{|s_1| \geqslant s_0\})} \lesssim 1$ implies that $H_0 \in L^2(\{|s_1| \geqslant 1\})$ and $\left\| \frac{H_0 - |s|^{-\gamma}}{|s|} \right\|_{L^2(\{|s_1| \leqslant s_0)} \lesssim 1$ implies that $H_0(s) \underset{0}{\sim} |s|^{-\gamma}$, a different behaviour near zero would make the latter norm infinite. These two conditions imply that H_0 is the unique solution of the equation (3.3.66). Thanks to the uniqueness of this limit, the whole sequence H_η converges weakly to

$$\mathsf{H}_0(s) := \begin{cases} s_1^{\frac{1}{2}} \mathrm{Im} H_0(s), & \gamma \in (\frac{d}{2}, \frac{d+1}{2}], 0 < s_1 \leqslant s_0, \\ |s|^{-1} \mathrm{Im} H_0(s), & \gamma \in (\frac{d+1}{2}, \frac{d+4}{2}), 0 < s_1 \leqslant s_0, \\ s_1 \mathrm{Im} H_0(s) & \text{for all } \ \gamma \in (\frac{d}{2}, \frac{d+4}{2}) \text{ and } s_1 \geqslant s_0. \end{cases}$$

Finally, we conclude by passing to the limit in the scalar product $\langle \mathsf{H}_\eta, \mathsf{I}_\eta \rangle$, where I_η definded by

$$\mathsf{I}_\eta := \begin{cases} s_1^{\frac{1}{2}} |s|_\eta^{-\gamma}, & \gamma \in (\frac{d}{2}, \frac{d+1}{2}], \ 0 < |s_1| \leqslant s_0, \\ s_1 |s|_\eta^{1-\gamma}, & \gamma \in (\frac{d+1}{2}, \frac{d+4}{2}), \ 0 < s_1 \leqslant s_0, \\ |s|_\eta^{-\gamma}, & \gamma \in (\frac{d}{2}, \frac{d+4}{2}), \ s_1 \geqslant s_0, \end{cases}$$

converges strongly in $L^2(\mathbb{R}_+ \times \mathbb{R}^{d-1})$ to

$$\mathsf{I}_0 := \begin{cases} s_1^{\frac{1}{2}} |s|^{-\gamma}, & \gamma \in (\frac{d}{2}, \frac{d+1}{2}], \ 0 < s_1 \leqslant s_0, \\ s_1 |s|^{1-\gamma}, & \gamma \in (\frac{d+1}{2}, \frac{d+4}{2}), \ 0 < s_1 \leqslant s_0, \\ |s|^{-\gamma}, & \gamma \in (\frac{d}{2}, \frac{d+4}{2}), \ s_1 \geqslant s_0. \end{cases}$$

Hence the limit (3.3.65) holds true. □

Proof of Proposition 3.3.5. By doing an expansion in λ for B and by Proposition 3.3.4, we get

$$B(\lambda, \eta) = \eta^{-\frac{2}{3}} b(\lambda, \eta) = \eta^{-\frac{2}{3}} b(0, \eta) + \lambda \int_{\mathbb{R}^d} M_{0,\eta} M \mathrm{d}v + O(\lambda^2).$$

Then, for $\lambda = \tilde{\lambda}(\eta)$ and since $B(\tilde{\lambda}(\eta), \eta) = 0$, we obtain:

$$\tilde{\lambda}(\eta) = -\eta^{-\frac{2}{3}} b(0,\eta) \Big(\int_{\mathbb{R}^d} M_{0,\eta} M \mathrm{d}v \Big)^{-1} + o\big(\eta^{-\alpha} b(0,\eta)\big),$$

which implies that

$$\eta^{-\alpha}\mu(\eta) = \eta^{\frac{2}{3}-\alpha}\tilde{\lambda}(\eta) = -\eta^{-\alpha} b(0,\eta) \Big(\int_{\mathbb{R}^d} M_{0,\eta} M \mathrm{d}v \Big)^{-1}.$$

By (3.3.34) and (3.3.65),

$$\lim_{\eta\to 0} \int_{\mathbb{R}^d} M_{0,\eta}(v) M(v) \mathrm{d}v = \|M\|_2^2 = C_\beta^{-2}$$

and

$$\lim_{\eta\to 0} \eta^{-\alpha} b(0,\eta) = 2C_\beta^2 \int_0^\infty \int_{\mathbb{R}^{d-1}} s_1 |s|^{-\gamma} \mathrm{Im} H_0(s) \mathrm{d}s' \mathrm{d}s_1$$

respectively. Hence, $\lim\limits_{\eta\to 0} \eta^{-\alpha}\mu(\eta) = \kappa$. For $\eta \in [-\eta_0, 0]$, the symmetry $\mu(\eta) = \overline{\mu}(-\eta)$ holds by complex conjugation on the equation. So it remains to prove the positivity of κ. By integrating the equation of $M_\eta := M_{\tilde{\lambda}(\eta),\eta}$ against $\bar{M}_\eta$ we obtain:

$$\int_{\mathbb{R}^d} \Big| \nabla_v \Big(\frac{M_\eta}{M} \Big) \Big|^2 M^2 \mathrm{d}v + \mathrm{i}\eta \int_{\mathbb{R}^d} v_1 |M_\eta|^2 \mathrm{d}v = \mu(\eta) \int_{\mathbb{R}^d} |M_\eta|^2 \mathrm{d}v.$$

Now, taking the real part and using the equality $\mu(\eta)\|M_\eta\|_2^2 = \kappa\eta^\alpha\big(1 + o(\eta^\alpha)\big)$ we get:

$$\int_{\mathbb{R}^d} \Big| \nabla_v \Big(\frac{M_\eta}{M} \Big) \Big|^2 M^2 \mathrm{d}v = \kappa\eta^\alpha\big(1 + o(\eta^\alpha)\big). \tag{3.3.67}$$

Therefore, multiplying this last equality by $\eta^{-\alpha}$ and performing the change of variable $v = \eta^{-\frac{1}{3}} s$ we obtain:

$$\int_{\mathbb{R}^d} \Big| \nabla_s \Big(\frac{H_\eta}{|s|_\eta^{-\gamma}} \Big) \Big|^2 |s|_\eta^{-2\gamma} \mathrm{d}s = \kappa\big(1 + o_\eta(1)\big).$$

Thus, $\kappa \geqslant 0$. If $\kappa = 0$ then,

$$\int_{\mathbb{R}^d} \Big| \nabla_s \Big(\frac{H_0}{|s|^{-\gamma}} \Big) \Big|^2 |s|^{-2\gamma} \mathrm{d}s \leqslant \liminf \int_{\mathbb{R}^d} \Big| \nabla_s \Big(\frac{H_\eta}{|s|_\eta^{-\gamma}} \Big) \Big|^2 |s|_\eta^{-2\gamma} \mathrm{d}s = 0.$$

Therefore, $H_0 = |s|^{-\gamma}$. Which leads to a contradiction since H_0 is solution to equation (3.3.41). Hence, the proof of Proposition 3.3.5 is complete. □

Proof of Theorem 3.1.1. The existence of the eigen-solution $(\mu(\eta), M_\eta)$ is given by Proposition 3.3.4. The limit (4.1.6) follows from inequality (3.3.1) for $|\lambda| = |\tilde{\lambda}(\eta)| \lesssim \eta^{\frac{2\gamma-d}{3}} \underset{\eta\to 0}{\longrightarrow} 0$, thanks to (3.3.39), with the limit (3.2.21) obtained by Theorem 3.2.18. Finally, the second point of Theorem 3.1.1 is given by Proposition 3.3.5.

3.4 Derivation of the fractional diffusion equation

The goal of this section is to prove Theorem 3.1.2. The proof was taken from Section 3 in [LP19] and adapted for the dimension d.

Let's start by defining the two weighted L^p spaces, $L^p_{F^{1-p}}(\mathbb{R}^d)$ and $Y^p_F(\mathbb{R}^{2d})$:

$$L^p_{F^{1-p}}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{R}, \int_{\mathbb{R}^d} |f|^p \; F^{1-p} \; \mathrm{d}v < \infty \right\} \quad \text{and} \quad Y^p_F(\mathbb{R}^{2d}) := L^p\big(\mathbb{R}^d, L^p_{F^{1-p}}(\mathbb{R}^d)\big).$$

Recall that our goal is to show that the solution f^ε of the Fokker-Planck equation (3.1.3) converges; weakly star in $L^\infty\left([0,T], L^2_{F^{-1}}(\mathbb{R}^2)\right)$; towards $\rho(t,x)F(v)$ when ε goes to 0, where ρ is the solution of the following *fractional diffusion* equation:

$$\partial_t \rho + \kappa(-\Delta)^{\frac{\beta-d+2}{6}} \rho = 0, \quad \rho(0,x) = \int_{\mathbb{R}^d} f_0 \mathrm{d}v. \tag{3.4.1}$$

Remark 3.4.1. Note that we will work with the Fourier transform of ρ and we will prove that $\hat{\rho}(t,\xi) = \int e^{-\mathrm{i}x\cdot\xi} \rho(t,x)\mathrm{d}x$ satisfies

$$\partial_t \hat{\rho} + \kappa|\xi|^{\frac{\beta-d+2}{3}} \hat{\rho} = 0. \tag{3.4.2}$$

3.4.1 A priori estimates

We start by recalling the following compactness lemma:

Lemma 3.4.2 ([LP19, NP15]). *For initial datum $f_0 \in Y^p_F(\mathbb{R}^{2d})$ where $p \geqslant 2$ and a positive time T.*

1. *The solution f^ε of (3.1.3) is bounded in $L^\infty\big([0,T]; Y^p_F(\mathbb{R}^{2d})\big)$ uniformly with respect to ε since it satisfies*

$$\|f^\varepsilon(T)\|^p_{Y^p_F} + \frac{p(p-1)}{\theta(\varepsilon)} \int_0^T \int_{\mathbb{R}^{2d}} \left| \nabla_v \left(\frac{f^\varepsilon}{F} \right) \right|^2 \left| \frac{f^\varepsilon}{F} \right|^{p-2} F \; \mathrm{d}v\mathrm{d}x\mathrm{d}t \leqslant \|f_0\|^p_{Y^p_F}. \tag{3.4.3}$$

2. *The density $\rho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon \; \mathrm{d}v$ is such that*

$$\|\rho^\varepsilon(t)\|^p_p \leqslant C_\beta^{-2(p-1)} \|f_0\|^p_{Y^p_F(\mathbb{R}^{2d})} \quad \text{for all} \quad t \in [0,T]. \tag{3.4.4}$$

3. *Up to a subsequence, the density ρ^ε converges weakly star in $L^\infty\big([0,T]; L^p(\mathbb{R}^d)\big)$ to ρ.*
4. *Up to a subsequence, the sequence f^ε converges weakly star in $L^\infty\big([0,T]; Y^p_F(\mathbb{R}^{2d})\big)$ to the function $f = \rho(t,x)F(v)$.*

As a consequence, we have the following estimate:

Corollary 3.4.3 ([LP19]). *Let $F = C_\beta^2 M^2$ with $M = (1+|v|^2)^{-\frac{\gamma}{2}}$ and $\beta = 2\gamma \in (d, d+4)$. Let f^ε solution to (3.1.3) with $\theta(\varepsilon) = \varepsilon^{\frac{2\gamma-d+2}{3}}$. Assume that $\|f_0/F\|_\infty \leqslant C$. Then $g^\varepsilon = f^\varepsilon F^{-\frac{1}{2}}$ satisfies the following estimate*

$$\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left| g^\varepsilon - \rho^\varepsilon F^{\frac{1}{2}} \right|^2 \mathrm{d}v \right)^{\frac{2\gamma-d+2}{2\gamma-d}} \mathrm{d}s\mathrm{d}x \leqslant C\varepsilon^{\frac{2\gamma-d+2}{3}}. \tag{3.4.5}$$

Proof. Recall the Nash type inequality [CGGR10, RW01, BBCG08]: for any h such that $\int hF\mathrm{d}v = 0$, we have

$$\int_{\mathbb{R}^d} h^2 F\mathrm{d}v \leqslant C \left(\int_{\mathbb{R}^d} \left|\nabla_v h\right|^2 F\mathrm{d}v \right)^{\frac{2\gamma-d}{2\gamma-d+2}} (\|h\|_\infty^2)^{\frac{2}{2\gamma-d+2}} . \tag{3.4.6}$$

Define $h = g^\varepsilon F^{-\frac{1}{2}} - \rho^\varepsilon = \frac{f^\varepsilon}{F} - \rho^\varepsilon$ and define $\alpha = \frac{2\gamma-d+2}{3}$. Observe that from $\|f\|_{L^p_{F^{1-p}}(\mathbb{R}^{2d})} = \|\frac{f}{F}\|_{L^p_F}$ and Lemma 3.4.2, formula (3.4.3), we have

$$\|h_0\|_{L^\infty} = \lim_{p\to\infty} \|h_0\|_{L^p_{F^{1-p}}(\mathbb{R}^{2d})} \geqslant \lim_{p\to\infty} \|h\|_{L^p_{F^{1-p}}(\mathbb{R}^{2d})} \geqslant \|h\|_{L^\infty} .$$

Thus by Lemma 3.4.2, formula (3.4.3) taking $p = 2$, we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g^\varepsilon - \rho^\varepsilon F^{\frac{1}{2}}|^2 \mathrm{d}v \right)^{\frac{2\gamma-d+2}{2\gamma-d}} \mathrm{d}x\mathrm{d}s &= \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h^2 F\mathrm{d}v \right)^{\frac{2\gamma-d+2}{2\gamma-d}} \mathrm{d}x\mathrm{d}s \\
&\leqslant C \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\nabla_v h\right|^2 F\mathrm{d}v \right) (\|h\|_\infty^2)^{\frac{2}{2\gamma-d}} \mathrm{d}x\mathrm{d}s \\
&\leqslant C \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\nabla_v \left(\frac{f^\varepsilon}{F}\right)\right|^2 F\mathrm{d}v \right) \mathrm{d}x\mathrm{d}s \leqslant C\varepsilon^\alpha.
\end{aligned}$$

□

3.4.2 Weak limit and proof of Theorem 3.1.2

By solving equation (3.1.8), we write

$$\hat{g}^\varepsilon(t,\xi,v) = e^{-t\theta(\varepsilon)\mathcal{L}_\eta} \hat{g}(0,\xi,v),$$

which gives going back to the rescaled space variable x

$$g^\varepsilon(t,x,v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi}\, \hat{g}^\varepsilon(t,\xi,v)\, \mathrm{d}\xi.$$

Our purpose is to pass to the limit when $\varepsilon \to 0$.
Recall that $f^\varepsilon(t,x,v) \geqslant 0$ and $\int f^\varepsilon(t,x,v)\mathrm{d}x\mathrm{d}v = \int f_0(x,v)\mathrm{d}x\mathrm{d}v$ for all $t \geqslant 0$.

Let $\hat\rho^\varepsilon(t,\xi)=\int e^{-\mathrm{i}x\cdot\xi}\rho^\varepsilon(t,x)\mathrm{d}x$ be the Fourier transform in x of $\rho^\varepsilon=\int f^\varepsilon \mathrm{d}v=\int g^\varepsilon F^{\frac12}\mathrm{d}v$.

Proposition 3.4.4. *For all $\xi\in\mathbb{R}^d$, $\hat\rho^\varepsilon(\cdot,\xi)$ converges to $\hat\rho(\cdot,\xi)$, unique solution to the ode*

$$\partial_t\hat\rho+\kappa|\xi|^\alpha\hat\rho=0,\quad \hat\rho_0=\int_{\mathbb{R}^d}\hat f_0\mathrm{d}v\ . \tag{3.4.7}$$

Proof. Let $\xi\in\mathbb{R}^d$ and let M_η be the unique solution in $L^2(\mathbb{R}^d,\mathbb{C})$ of $\mathcal{L}_\eta(M_\eta)=\mu(\eta)M_\eta$ given in Theorem 3.1.1. One has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\hat g^\varepsilon(t,\xi,v)M_\eta\mathrm{d}v &= \int\partial_t\hat g^\varepsilon M_\eta\mathrm{d}v=-\varepsilon^{-\alpha}\int\mathcal{L}_\varepsilon(\hat g^\varepsilon)M_\eta\mathrm{d}v\\ &= -\varepsilon^{-\alpha}\int\hat g^\varepsilon\mathcal{L}_\varepsilon(M_\eta)\mathrm{d}v=-\varepsilon^{-\alpha}\mu(\eta)\int\hat g^\varepsilon(t,\xi,v)M_\eta\mathrm{d}v.\end{aligned}$$

Therefore one has, with $F^\varepsilon(t,x)=C_\beta\int g^\varepsilon(t,x,v)M_\eta\mathrm{d}v$,

$$\hat F^\varepsilon(t,\xi)=e^{-t\varepsilon^{-\alpha}\mu(\varepsilon|\xi|)}\hat F^\varepsilon(0,\xi),\quad \forall t\geqslant 0. \tag{3.4.8}$$

By Theorem 3.1.1, we have $\varepsilon^{-\alpha}\mu(\varepsilon|\xi|)\to\kappa|\xi|^\alpha$. Moreover, the following limit holds true:

$$\forall\xi\in\mathbb{R}^d,\quad \hat F^\varepsilon(0,\xi)=C_\beta\int\hat g^\varepsilon(0,\xi,v)M_\eta\mathrm{d}v\to\hat\rho_0(\xi)\ . \tag{3.4.9}$$

The verification of (3.4.9) is easy. One has $\hat g^\varepsilon(0,v,\xi)=\hat f_0(v,\xi)F^{-\frac12}(v)=\frac{\hat f_0(v,\xi)}{C_\beta M(v)}$ and $M_\eta\to M$ in $L^2(\mathbb{R}^d)$ thanks to (4.1.6). Thus, (3.4.9) holds true by Cauchy-Schwarz inequality by writing:

$$\left|C_\beta\int\hat g^\varepsilon(0,\xi,v)M_\eta\mathrm{d}v-\hat\rho_0(\xi)\right|\leqslant C_\beta\Big(\int\frac{f_0^2}{F}\mathrm{d}v\Big)^{\frac12}\Big(\int|M_\eta-M|^2\mathrm{d}v\Big)^{\frac12}.$$

It remains to verify

$$\forall\xi\in\mathbb{R}^d,\quad C_\beta\int\hat g^\varepsilon(t,\xi,v)M_\eta\mathrm{d}v\longrightarrow\hat\rho(t,\xi)\quad\text{in }\mathcal{D}'\big(]0,\infty[\times\mathbb{R}^d\big). \tag{3.4.10}$$

By (3.4.8) and (3.4.9), for all $\xi\in\mathbb{R}^d$ and $t\geqslant 0$, one has $\lim_{\varepsilon\to0}\hat F^\varepsilon(t,\xi)=e^{-t\kappa|\xi|^\alpha}\hat\rho_0(\xi)$, thus (3.4.10) will be consequence of the weaker

$$C_\beta\int g^\varepsilon(t,x,v)M_\eta\mathrm{d}v\to\rho(t,x)\quad\text{in }\mathcal{D}'\big(]0,\infty[\times\mathbb{R}^d\big)\ . \tag{3.4.11}$$

Let us now verify (3.4.11). For that purpose, we write

$$C_\beta\int g^\varepsilon M_\eta\mathrm{d}v-\rho=C_\beta\int(g^\varepsilon-\rho^\varepsilon F^{\frac12})M_\eta\mathrm{d}v+\rho^\varepsilon\int(C_\beta M_\eta-F^{\frac12})F^{\frac12}\mathrm{d}v+\rho^\varepsilon-\rho\ .$$

By using (3.4.5) and the first point of Theorem 3.1.1, the limit (4.1.6), we pass to the limit. The proof of Proposition 3.4.4 is complete. □

Proof of Theorem **3.1.2.** From the two last items in Lemma 3.4.2, we have just to prove that for any given ξ, the Fourier transform $\hat{\rho}(t,\xi)$ of the weak limit $\rho(t,y)$, is solution of the equation (3.4.2), which is precisely Proposition 3.4.4. □

CHAPTER 4

Fractional diffusion for Fokker-Planck equation with drift

Résumé

Dans ce petit chapitre, nous allons étendre les résultats des deux derniers chapitres aux équilibres plus généraux, n'ayant pas forcément une formule explicite telle que $F(v) = C_\beta^2 \langle v \rangle^{-\beta}$, et ne satisfaisants pas l'hypothèse de symétrie. On montre que sous certaines hypothèses sur le comportement de F à l'infini ainsi que le potentiel W, nous obtenons que la densité tend vers une solution de l'équation de la diffusion fractionnaire avec ou sans drift, selon la décroissance de F. Ceci nous permet de montrer rigoureusement les résultats mentionnés dans [BM22, Section 9] "Remarks and extensions".

Abstract

In this short chapter, we extend the results of the last two chapters to more general equilibria, not necessarily having an explicit formula such as $F(v) = C_\beta^2 \langle v \rangle^{-\beta}$, and not satisfying the symmetry assumption. We show that under some assumptions on the behavior of F at infinity and the potential W, we obtain that the density tends to a solution of the fractional diffusion equation with or without drift, depending on the decay of F. This allows us to rigorously show the results mentioned in [BM22, Section 9] "Remarks and extensions".

Contents

4.1 Introduction

The proof of Theorems 3.1.1 and 3.1.2 presented in the previous chapter, allows us to tackle the case of a more general equilibrium, since we didn't use the fact that F (or M) was given explicitly, but rather used the decomposition of the potential W into two parts (its tail for high speeds and the remainder which is of lower order) to show the existence of solutions for the penalized equation, and the symmetry of the equilibrium, in order to use the Hardy-Poincaré inequality to establish some estimates.

The proofs in this chapter are identical to those in the previous one. We only give the results obtained and the changes needed to be made.

4.1.1 Notations and assumptions on equilibrium

Let $\beta > d+1$. We denote by j_F the vector

$$j_F := \int_{\mathbb{R}^d} vF(v)\, \mathrm{d}v,$$

and by j_1 the scalar

$$j_1 := \int_{\mathbb{R}^d} v_1 F(v)\, \mathrm{d}v.$$

Let f^ε be the solution of the Fokker-Planck equation

$$\begin{cases} \theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot \left(F\nabla_v \left(\frac{f^\varepsilon}{F}\right)\right), & x \in \mathbb{R}^d, v \in \mathbb{R}^d, t > 0, \\ f^\varepsilon(0,x,v) = f_0(x,v), & x \in \mathbb{R}^d, v \in \mathbb{R}^d. \end{cases} \tag{4.1.1}$$

We denote by $\tilde{f}_\varepsilon$ the function defined by

$$\tilde{f}_\varepsilon(t,x,v) := f^\varepsilon(t, x - \varepsilon^{1-\alpha} j_F t, v),$$

solution to the following equation:

$$\theta(\varepsilon)\partial_t \tilde{f}_\varepsilon + \varepsilon(v - j_F)\cdot\nabla_x \tilde{f}_\varepsilon = \nabla_v \cdot \left(F\nabla_v\left(\frac{\tilde{f}_\varepsilon}{F}\right)\right). \tag{4.1.2}$$

Finally, we denote by $\tilde{\rho}_\varepsilon$ the function $\tilde{\rho}_\varepsilon(t,x) := \rho^\varepsilon(t, x - \varepsilon^{1-\alpha} j_F t)$. Note that we also keep the same notations introduced in the two previous chapters ($M := F^{\frac{1}{2}}, \ldots$). Thus, for the change of unknown $g = \frac{f}{M}$, the function $\tilde{g}_\varepsilon$ satisfies the following equation for F smooth:

$$\theta(\varepsilon)\partial_t \tilde{g}_\varepsilon + \varepsilon(v - j_F)\cdot\nabla_x \tilde{g}_\varepsilon = \Delta_v \tilde{g}_\varepsilon - W(v)\tilde{g}_\varepsilon,$$

with

$$W(v) := \frac{\Delta M}{M}.$$

Assumptions 4.1.1.

(A1) Assume that there exists an equilibrium F such that $F(v) > 0$ for all $v \in \mathbb{R}^d$ and there exists $\beta > d$ and a constant C_β such that:

$$F(v) \underset{|v|\to\infty}{\sim} \frac{C_\beta}{|v|^\beta} \quad \text{and} \quad \int_{\mathbb{R}^d} F(v)\mathrm{d}v = \int_{\mathbb{R}^d} M^2(v)\mathrm{d}v = 1. \tag{4.1.3}$$

Here, C_β is a constant of normalization.

(A2) Concerning the potential W, we assume that there exists a constant $\sigma \in (0, 2\gamma - d + 2)$ such that

$$0 < W(v) - \frac{\gamma(\gamma - d + 2)}{|v|^2} \underset{|v|\to\infty}{=} O\left(\frac{1}{|v|^{2+\sigma}}\right), \tag{4.1.4}$$

where $\gamma := \frac{\beta}{2}$.

The assumptions on the behavior of W is recovered by the assumption (A2) above. Concerning the non-symmetry of the equilibrium, it is treated as follows: in order to use Hardy's inequality, we add and subtract a term corresponding to the projection of the function $M_{\mu,\eta}$ on M in the space L^2 endowed with measure $\frac{\mathrm{d}v}{\langle v\rangle^2}$. The term that is added is used in the inequality and appears in the approximation of the eigenvalue, while the other term is put in the equation. So that amounts to making a kind of correction on the equation, which gives exactly the drift term j_F for $\beta > d+1$. This last term is dominated when $\beta \in (d, d+1)$ and does not appear in the equation. Observe that $\beta = d+1$ is the absolute convergence threshold of the integral $j_1 = \displaystyle\int_{\mathbb{R}^d} v_1 M^2(v)\mathrm{d}v$ defining the macroscopic drift.

4.1.2 Main results

Assume (A1) and (A2). Let's start with the spectral problem:

Theorem 4.1.2 (Eigen-solution for the Fokker-Planck operator/with drift)**.** *Assume that $d < \beta < d+4$ with $\beta \neq d+1$. Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Then, for all $\eta \in [0, \eta_0]$, there exists a unique eigen-couple $\big(\mu(\eta), M_\eta\big)$ in $\{\mu \in \mathbb{C}, |\mu| \leqslant \eta^{\frac{2}{3}}\lambda_0\} \times L^2(\mathbb{R}^d, \mathbb{C})$, solution to the spectral problem*

$$\mathcal{L}_\eta(M_{\mu,\eta})(v) = \big[- \Delta_v + W(v) + \mathrm{i}\eta v_1\big] M_{\mu,\eta}(v) = \mu M_{\mu,\eta}(v), \;\; v \in \mathbb{R}^d. \tag{4.1.5}$$

Moreover,

1. The following convergence in the Sobolev space $H^1(\mathbb{R}^d)$ holds:

$$\big\| M_\eta - M \big\|_{H^1(\mathbb{R}^d)} \underset{\eta \to 0}{\longrightarrow} 0. \tag{4.1.6}$$

2. The eigenvalue $\mu(\eta)$ satisfies

- *For $\beta \in (d, d+1)$:*
$$\mu(\eta) = \kappa \eta^{\frac{\beta-d+2}{3}} \big(1 + O(\eta^{\frac{\beta-d+2}{3}})\big), \tag{4.1.7}$$
- *For $\beta \in (d+1, d+4)$:*
$$\mu(\eta) - \mathrm{i}\eta\; j_1 = \kappa \eta^{\frac{\beta-d+2}{3}} \big(1 + O(\eta^{\frac{\beta-d+2}{3}})\big), \tag{4.1.8}$$

where κ is a positive constant given by

$$\kappa := \begin{cases} \mathrm{i} \displaystyle\int_{\{|s_1|>0\}} s_1 |s|^{-\gamma} H_0(s)\; \mathrm{d}s, & \text{for } \;\gamma \in (\tfrac{d}{2}, \tfrac{d+1}{2}), \\ \mathrm{i} \displaystyle\int_{\{|s_1|>0\}} s_1 |s|^{-\gamma} [H_0(s) - |s|^{-\gamma}]\; \mathrm{d}s, & \text{for } \;\gamma \in (\tfrac{d+1}{2}, \tfrac{d+4}{2}). \end{cases} \tag{4.1.9}$$

and where H_0 is the unique solution of equation (3.1.13) *satisfying* (3.1.14)*.*

As a consequence of this result, we have the following Diffusion Limit Theorem:

Theorem 4.1.3 (Fractional diffusion limit for the Fokker-Planck equation/with drift)**.** *Assume that $d < \beta < d+4$ with $\beta \neq d+1$. Assume that $f_0 \in L^2_{F^{-1}}(\mathbb{R}^{2d}) \cap L^\infty_{F^{-1}}(\mathbb{R}^{2d})$ is a positive function. Let f^ε be the solution of* (4.1.1) *in Y, with initial data f_0 with $\theta(\varepsilon) = \varepsilon^\alpha$ and $\alpha := \frac{\beta-d+2}{3}$. Let κ be the constant defined in* (4.1.9)*. Then,*

1. *For $\beta \in (d, d+1)$, the sequence f^ε converges weakly star in $L^\infty\big([0,T]; L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ towards $\rho(t,x)F(v)$ where ρ is solution to the equation*
$$\partial_t \rho + \kappa(-\Delta)^{\frac{\alpha}{2}} \rho = 0, \quad \rho(0,x) = \int_{\mathbb{R}^d} f_0 \; \mathrm{d}v. \tag{4.1.10}$$
2. *For $\beta \in (d+1, d+4)$, the sequence $\tilde{f}_\varepsilon$ converges weakly star in $L^\infty\big([0,T]; L^2_{F^{-1}}(\mathbb{R}^{2d})\big)$ to $\rho(t,x)F(v)$ where ρ is solution of equation* (4.1.10)*.*

Remark 4.1.4. In the case of a symmetric equilibrium, we indeed recover the results of the previous chapter, since $j_1 = 0,\ \ j_F = 0_{\mathbb{R}^d}$ and H_0 realizes the symmetry

$$\overline{H_0}(-s_1, s') = H_0(s_1, s'), \quad \forall (s_1, s') \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Remark 4.1.5. The hypothesis $\beta \neq d+1$ is technical. It avoids to introduce logarithmic terms in the expression of $\mu(\eta)$.

4.2 Existence of the eigen-solution

Let us recall the procedure of the proof of Theorem 4.1.2. It is done in three steps: The first consists in showing the existence of a unique solution for the penalized equation. In the second step, we improve the functional space in which the solution M_η belongs by establishing some L^2 estimates. Finally, these estimates are used to study the additive term in the penalized equation and show the existence of a function $\mu = \mu(\eta)$ such that the additive term disappears.

4.2.1 Existence of solution for the penalized equation

Under Assumption (A2), all the proofs in Section 3.2 remain unchanged as well as the existence of solutions for the penalized equation and we have:

Theorem 4.2.1. *There is a unique function $M_{\lambda,\eta}$ in $\mathcal{H}_0$ (defined in (3.2.2)) solution to the penalized equation*

$$[-\Delta_v + W(v) + \mathrm{i}\eta v_1 - \lambda \eta^{\frac{2}{3}}] M_{\lambda,\eta}(v) = b(\lambda,\eta)\Phi(v), \quad v \in \mathbb{R}^d. \tag{4.2.1}$$

where $b(\lambda,\eta) := \langle N_{\lambda,\eta}, \Phi\rangle$ with $N_{\lambda,\eta} := M_{\lambda,\eta} - M$. Moreover,

$$\|N_{\lambda,\eta}\|_{\mathcal{H}_0} = \|M_{\lambda,\eta} - M\|_{\mathcal{H}_0} \underset{\eta \to 0}{\longrightarrow} 0. \tag{4.2.2}$$

Recall that Φ is given by $\Phi(v) := c_{\gamma,d}\frac{M(v)}{\langle v\rangle^2}$, where $c_{\gamma,d}$ is a constant such that $\langle M, \Phi\rangle_{L^2} = 1$.

4.2.2 L^2 estimates and eigenvalue

There are no changes in this subsection. We have the following two Propositions:

Proposition 4.2.2. *Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. Let $M_{\lambda,\eta}$ be the solution of the penalised equation (4.2.1). Then, for all $\eta \in [0, \eta_0]$ and for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant \lambda_0$, one has*

1. *For all $\gamma > \frac{d}{2}$, the function $M_{\lambda,\eta}$ is uniformly bounded, with respect to λ and η, in $L^2(\mathbb{R}^d, \mathbb{C})$. Moreover, the following estimate holds*

$$\|N_{\lambda,\eta}\|^2_{L^2(\mathbb{R}^d)} = \|M_{\lambda,\eta} - M\|^2_{L^2(\mathbb{R}^d)} \lesssim |\lambda| + \nu_\eta, \tag{4.2.3}$$

where $\nu_\eta \underset{\eta\to 0}{\longrightarrow} 0.$

2. *For all* $\gamma > \frac{d+1}{2}$, *the function* $|v_1|^{\frac{1}{2}} M_{\lambda,\eta}$ *is uniformly bounded, with respect to* λ *and* η, *in* $L^2(\mathbb{R}^d, \mathbb{C})$.

The second is on the existence of the eigenvalue:

Proposition 4.2.3. *Define*

$$B(\lambda,\eta) := \eta^{-\frac{2}{3}} b(\lambda,\eta).$$

There exists $\tilde{\eta}_0, \tilde{\lambda}_0 > 0$ *small enough, a function* $\tilde{\lambda} : \{|\eta| \leqslant \tilde{\eta}_0\} \longrightarrow \{|\lambda| \leqslant \tilde{\lambda}_0\}$ *such that, for all* $(\lambda,\eta) \in [0,\tilde{\eta}_0[\times\{|\lambda| < \tilde{\lambda}_0\}$, $\lambda = \tilde{\lambda}(\eta)$ *and the constraint is satisfied:*

$$B(\lambda,\eta) = B(\tilde{\lambda}(\eta),\eta) = 0.$$

Consequently, $\mu(\eta) := \eta^{\frac{2}{3}} \tilde{\lambda}(\eta)$ *is the eigenvalue associated to the eigenfunction* $M_\eta := M_{\tilde{\lambda}(\eta),\eta}$ *for the operator* $\mathcal{L}_\eta$, *and the couple* $\big(\mu(\eta), M_\eta\big)$ *is solution to the spectral problem* (4.1.5).

This last Proposition gives the first part of Theorem 4.1.2. The second part consists in studying the behavior of $\mu(\eta)$ for η small enough.

4.2.3 Approximation of the eignenvalue

It is in this phase that the symmetry of equilibrium plays an important role.

Proposition 4.2.4. *Let* $\beta = 2\gamma \in (d, d+4)$ *and let* $\alpha := \frac{2\gamma-d+2}{3}$. *Then, the eigenvalue* $\mu(\eta)$ *associated with the solution of the problem* (4.1.5) *satisfies* (4.1.7) *for* $\beta \in (d, d+1)$ *and* (4.1.8) *for* $\beta \in (d+1, d+4)$.

The proof of this Proposition relies on a series of Lemmas. Let us first recall some notations used in the previous chapter and then give the necessary changes. The function H_η, is the rescaled of the function $M_{0,\eta}$, defined by

$$H_\eta(s) := \eta^{-\frac{\gamma}{3}} M_{0.\eta}(\eta^{-\frac{1}{3}} s).$$

We denote by m_η the rescaled equilibrium M

$$m_\eta(s) := \eta^{-\frac{\gamma}{3}} M(\eta^{-\frac{1}{3}} s).$$

Also, the constant c_η is defined by

$$c_\eta := \left(\int_{\mathbb{R}^d} \frac{M^2}{\langle v\rangle^2} \mathrm{d}v \right)^{-1} \int_{\mathbb{R}^d} \frac{M_{0,\eta}(v) M(v)}{\langle v\rangle^2}\, \mathrm{d}v = \left(\int_{\mathbb{R}^d} \frac{m_\eta^2}{|s|_\eta^2} \mathrm{d}s \right)^{-1} \int_{\mathbb{R}^d} \frac{H_\eta(s) m_\eta(s)}{|s|_\eta^2}\, \mathrm{d}s,$$

so that it ensures the orthogonality condition in the Hardy-Poincaré inequality

$$\int_{\mathbb{R}^d} \frac{[M_{0,\eta}(v) - c_\eta M(v)]M(v)}{\langle v\rangle^2} \,\mathrm{d}v = \int_{\mathbb{R}^d} \frac{[H_\eta(s) - c_\eta m_\eta(s)]m_\eta(s)}{|s|_\eta^2} \,\mathrm{d}s = 0.$$

Finally, the expression of $b(0,\eta)$ is given by

$$b(0,\eta) := \langle M_{0,\eta} - M, \Phi\rangle = -\mathrm{i}\eta \int_{\mathbb{R}^d} v_1 M_{0,\eta}(v)M(v) \,\mathrm{d}v.$$

Recall that $c_\eta \to 1$ when $\eta \to 0$, for all $\gamma > \frac{d}{2}$. Moreover, thanks to the second point of Proposition 4.2.2, for $\gamma > \frac{d+1}{2}$ we get

$$|c_\eta - 1| = c_{\gamma,d}^{-1}|b(0,\eta)| \lesssim \eta. \tag{4.2.4}$$

The proof of the following two Lemmas does not change.

Lemma 4.2.5 (Large velocities)**.** *Let $s_0 > 0$ be fixed, large enough. We have the following estimates, uniform with respect to η, for the rescaled solution:*

1. *For all $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$, one has*

$$\big\||s_1|^{\frac{1}{2}}(H_\eta - m_\eta)\big\|_{L^2(\{|s_1|\leqslant s_0\})} + \|s_1(H_\eta - m_\eta)\|_{L^2(\{|s_1|\geqslant s_0\})} \lesssim 1. \tag{4.2.5}$$

2. *For all $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$, one has*

$$\left\|\frac{H_\eta - m_\eta}{|s|_\eta}\right\|_{L^2(\{|s_1|\leqslant s_0\})} + \|s_1(H_\eta - m_\eta)\|_{L^2(\{|s_1|\geqslant s_0\})} \lesssim 1. \tag{4.2.6}$$

Lemma 4.2.6. *For all $\eta \in [0,\eta_0]$ and for all $\gamma \in (\frac{d}{2}, \frac{d+4}{2})$, the following estimate holds*

$$\left\|\frac{H_\eta - c_\eta m_\eta}{|s|_\eta}\right\|_{L^2(\mathbb{R}^d)} \lesssim_{\gamma,d} \left\|\nabla_s\left(\frac{H_\eta}{m_\eta}\right)m_\eta\right\|_{L^2(\mathbb{R}^d)} \lesssim 1. \tag{4.2.7}$$

Concerning the passage to the limit, the impact of the non-symmetry comes into play, and we have the first result on the small velocities:

Lemma 4.2.7 (Small velocities)**.** *1. For all $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$, one has*

$$\lim_{\eta\to 0} \eta^{1-\alpha} \int_{\{|v_1|\leqslant R\}} v_1 M_{0,\eta}(v)M(v) \,\mathrm{d}v = 0. \tag{4.2.8}$$

2. For all $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$, one has the estimate

$$\int_{\{|v_1|\leqslant R\}} \left|\frac{M_{0,\eta}(v) - c_\eta M(v)}{\langle v\rangle}\right|^2 \mathrm{d}v \lesssim \eta, \tag{4.2.9}$$

and the limit

$$\lim_{\eta\to 0} \eta^{1-\alpha} \int_{\{|v_1|\leqslant R\}} v_1[M_{0,\eta}(v) - M(v)]M(v)\,\mathrm{d}v = 0. \tag{4.2.10}$$

Proof. 1. For this point, the proof does not change and we write for $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$:

$$\eta^{1-\alpha}\left|\int_{\{|v_1|\leqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v\right| \leqslant R\eta^{1-\alpha}\|M_{0,\eta}\|_2\|M\|_2 \underset{\eta\to 0}{\longrightarrow} 0,$$

since $1-\alpha = \frac{1+d-2\gamma}{3} > 0$ for all $\gamma < \frac{d+1}{2}$.
2. The inequality of the second point follows from the Hardy-Poincaré inequality, see proof of Lemma 3.3.7. For the limit (4.2.10), an additional term comes into play to use inequality (4.2.9), but it is a factor of $|c_\eta - 1|$ which is smaller than η for $\gamma > \frac{d+1}{2}$:

$$\begin{aligned}\left|\int_{\{|v_1|\leqslant R\}} v_1[M_{0,\eta}(v) - M(v)]M(v)\mathrm{d}v\right| &\leqslant \left|\int_{\{|v_1|\leqslant R\}} v_1[M_{0,\eta}(v) - c_\eta M(v)]M(v)\mathrm{d}v\right| \\ &+ |c_\eta - 1|\left|\int_{\{|v_1|\leqslant R\}} v_1 M^2(v)\mathrm{d}v\right| \\ &\leqslant \left\|v_1\langle v\rangle M\right\|_{L^2(\{|v_1|\leqslant R\})}\left\|\frac{M_{0,\eta} - c_\eta M}{\langle v\rangle}\right\|_{L^2(\{|v_1|\leqslant R\})} \\ &+ |c_\eta - 1|\left\||v_1|^{\frac{1}{2}}M\right\|_{L^2(\{|v_1|\leqslant R\})} \\ &\lesssim \eta,\end{aligned}$$

thanks to the first point of this Lemma, inequality (4.2.4) and Proposition 4.2.2. Hence,

$$\eta^{1-\alpha}\left|\int_{\{|v_1|\leqslant R\}} v_1[M_{0,\eta}(v) - M(v)]M(v)\mathrm{d}v\right| \lesssim \eta^{2-\alpha} \underset{\eta\to 0}{\longrightarrow} 0,$$

since $2-\alpha > 0$ for $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$. □

For the Lemma which gives the coefficient κ, the proof requires some modifications.

Lemma 4.2.8. *We have the following limits:*
1. For $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$:

$$\lim_{\eta\to 0} \eta^{1-\alpha} \int_{\{|v_1|\geqslant R\}} v_1 M_{0,\eta}(v)M(v)\mathrm{d}v = \int_{\{|s_1|>0\}} s_1|s|^{-\gamma}H_0(s)\mathrm{d}s, \tag{4.2.11}$$

2. For $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$:

$$\lim_{\eta\to 0} \eta^{1-\alpha} \int_{\{|v_1|\geqslant R\}} v_1[M_{0,\eta}(v) - M(v)]M(v)\mathrm{d}v = \int_{\{|s_1|>0\}} s_1|s|^{-\gamma}\left[H_0(s) - |s|^{-\gamma}\right]\mathrm{d}s, \tag{4.2.12}$$

where H_0 is the unique solution to (3.3.41) satisfying the conditions (3.3.42).

Proof. First, by performing the change of variable $v = \eta^{-\frac{1}{3}}s$, we write:

$$\eta^{1-\alpha}\int_{\{|v_1|\geqslant R\}} v_1 M_{0,\eta}(v)M(v)\ \mathrm{d}v = \int_{\{|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta(s)H_\eta(s)\ \mathrm{d}s. \tag{4.2.13}$$

As in the previous chapter, the passage to the limit in the preceding integral is based on the argument of strong-weak convergence in the Hilbert space L^2. Thanks to the equivalence (4.1.3), we have the convergence of $m_\eta(s)$ to $|s|^{-\gamma}$ when $\eta \to 0$, and thanks to the inequalities of Lemmas 4.2.5 and 4.2.6, we have the weak convergence of $|s_1|^{\frac{1}{2}}(H_\eta - m_\eta)$ to $|s_1|^{\frac{1}{2}}(H_0 - |s|^{-\gamma})$ in $L^2(\{|s_1|\leqslant 1\})$ and of H_η to H_0 in $L^2(\{|s_1|\geqslant 1\})$. Thus, to ensure the convergence of the integral (4.2.13), we need to subtract the first moment in certain cases.

1. Let $\gamma \in (\frac{d}{2}, \frac{d+1}{2})$. We have:

$$\int_{\{|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta(s)H_\eta(s)\ \mathrm{d}s = \int_{\{1\geqslant|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta(s)[H_\eta(s) - m_\eta(s)]\ \mathrm{d}s + \int_{\{1\geqslant|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta^2(s)\ \mathrm{d}s + \int_{\{|s_1|\geqslant 1\}} s_1 m_\eta(s)H_\eta(s)\mathrm{d}s.$$

Now, since we have the strong convergence of $|s_1|^{\frac{1}{2}}m_\eta$ to $|s_1|^{\frac{1}{2}}|s|^{-\gamma}$ in $L^2(\{|s_1|\leqslant 1\})$ and of m_η to $|s|^{-\gamma}$ in $L^2(\{|s_1|\geqslant 1\})$, and this is true for $\gamma \in \left(\frac{d}{2}, \frac{d+1}{2}\right)$. Then, the previous integrals converge to

$$\int_{\{1\geqslant|s_1|>0\}} s_1|s|^{-\gamma}[H_0(s) - |s|^{-\gamma}]\mathrm{d}s + \int_{\{1\geqslant|s_1|>0\}} s_1|s|^{-2\gamma}\mathrm{d}s + \int_{\{|s_1|\geqslant 1\}} s_1|s|^{-\gamma}H_0(s)\mathrm{d}s$$
$$= \int_{\{|s_1|>0\}} s_1|s|^{-\gamma}H_0(s)\mathrm{d}s.$$

2. Similarly for $\gamma \in (\frac{d+1}{2}, \frac{d+4}{2})$, we have

$$\int_{\{|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta(s)[H_\eta(s) - m_\eta(s)]\ \mathrm{d}s = \int_{\{1\geqslant|s_1|\geqslant R\eta^{\frac{1}{3}}\}} s_1 m_\eta(s)[H_\eta(s) - m_\eta(s)]\mathrm{d}s + \int_{\{|s_1|\geqslant 1\}} s_1 m_\eta(s)H_\eta(s)\mathrm{d}s - \int_{\{|s_1|\geqslant 1\}} s_1 m_\eta^2(s)\mathrm{d}s.$$

The same arguments as for the previous point allow us to pass to the limit in order to obtain:

$$\int_{\{1\geqslant|s_1|>0\}} s_1|s|^{-\gamma}[H_0(s) - |s|^{-\gamma}]\mathrm{d}s + \int_{\{|s_1|\geqslant 1\}} s_1|s|^{-\gamma}H_0(s)\mathrm{d}s - \int_{\{|s_1|\geqslant 1\}} s_1|s|^{-2\gamma}\mathrm{d}s$$
$$= \int_{\{|s_1|>0\}} s_1|s|^{-\gamma}[H_0(s) - |s|^{-\gamma}]\mathrm{d}s.$$

□

Remark 4.2.9. Note that for $\gamma > \frac{d+1}{2}$, we have $\int_{|s_1|\geqslant 1} s_1|s|^{-2\gamma}\mathrm{d}s = 0$, but the integral of $s_1|s|^{-2\gamma}$ is not well-defined on $\{|s_1| \leqslant 1\}$.

With all these Lemmas, the rest of the proof of Proposition 4.2.4 is identical to that of Chapter 3.

4.3 Derivation of the fractional diffusion equation with drift

For the diffusion limit, the convergence of $\tilde{f}_\varepsilon$ and $\tilde{\rho}_\varepsilon$, up to subsequences, is ensured thanks to the following complementary estimates:

Lemma 4.3.1. *Let $\beta > d+1$. For an initial datum $f_0 \in Y^p_F(\mathbb{R}^{2d})$ where $p \geqslant 2$, and for a time $T > 0$, we have the following estimates:*

1. *The solution $\tilde{f}_\varepsilon$ of (4.1.2) is bounded in $L^\infty\big([0,T];Y^p_F(\mathbb{R}^{2d})\big)$ uniformly with respect to ε. Moreover,*
$$\|\tilde{f}_\varepsilon(T)\|^p_{Y^p_F} + \frac{p(p-1)}{\theta(\varepsilon)}\int_0^T\int_{\mathbb{R}^{2d}}\left|\nabla_v\left(\frac{\tilde{f}_\varepsilon}{F}\right)\right|^2\left|\frac{\tilde{f}_\varepsilon}{F}\right|^{p-2} F\ \mathrm{d}v\mathrm{d}x\mathrm{d}t \leqslant \|f_0\|^p_{Y^p_F}.$$
2. *The density $\tilde{\rho}^\varepsilon$ satisfies:*
$$\|\tilde{\rho}_\varepsilon(t)\|^p_{L^p(\mathbb{R}^d)} = \|\rho^\varepsilon(t)\|^p_{L^p(\mathbb{R}^d)} \leqslant C\|f_0\|^p_{Y^p_F(\mathbb{R}^{2d})} \quad \textit{for all} \quad t\in[0,T].$$
3. *Suppose that $\|f_0/F\|_\infty \leqslant C$. Then,*
$$\int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\big|\tilde{f}_\varepsilon - \tilde{\rho}_\varepsilon F\big|^2\frac{\mathrm{d}v}{F}\right)^{\frac{\beta-d+2}{\beta-d}}\mathrm{d}x\mathrm{d}t \leqslant C\varepsilon^{\frac{\beta-d+2}{3}}.$$

The proof of this Lemma is identical to that of Lemma 3.4.2.

Proof of Theorem 4.1.3. For $\beta\in(d,d+1)$, the proof given in Section 3.4 does not change. For $\beta \in (d+1, d+4)$, we define the function $h_\varepsilon(t,x,v) := g^\varepsilon(t, x-\varepsilon^{1-\alpha}j_F, v)$ which satisfies the equation
$$\varepsilon^\alpha\partial_t h_\varepsilon + \varepsilon(v - j_F)\cdot\nabla_x h_\varepsilon = Q(h_\varepsilon).$$
In Fourier (in x), one has
$$\hat{h}_\varepsilon(t,\xi,v) = \hat{g}^\varepsilon(t,\xi,v)e^{\mathrm{i}\varepsilon^{1-\alpha}j_F\cdot\xi}.$$

Thus, by integrating the equation of $\hat{h}_\varepsilon$ against M_η, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \hat{h}_\varepsilon(t,\xi,v)M_\eta \,\mathrm{d}v &= \int \partial_t\hat{h}_\varepsilon M_\eta \,\mathrm{d}v = -\varepsilon^{-\alpha}\int [\mathcal{L}_\eta(\hat{h}_\varepsilon) - \mathrm{i}\eta j_1\hat{h}_\varepsilon]M_\eta \,\mathrm{d}v \\ &= -\varepsilon^{-\alpha}\int \hat{h}_\varepsilon[\mathcal{L}_\eta(M_\eta) - \mathrm{i}\eta j_1 M_\eta] \,\mathrm{d}v \\ &= -\varepsilon^{-\alpha}[\mu(\eta) - \mathrm{i}\eta j_1]\int \hat{h}_\varepsilon(t,\xi,v)M_\eta \,\mathrm{d}v.\end{aligned}$$

By the second item of Theorem 4.1.2,

$$\varepsilon^{-\alpha}[\mu(\eta) - \mathrm{i}\eta j_1] \underset{\varepsilon\to 0}{\longrightarrow} \kappa|\xi|^\alpha.$$

The rest of the proof is analogous to the case $\beta \in (d, d+1)$.

Gevrey regularity for the Vlasov-Navier-Stokes system

CHAPTER 5

Gevrey regularity and analyticity for the solutions of the Vlasov-Navier-Stokes system

Ce chapitre fait l'objet du papier [Dec23].

Résumé

Dans ce chapitre, nous prouvons la propagation de la régularité $\frac{1}{s}$-Gevrey ($s \in (0,1)$) et de l'analyticité ($s = 1$) pour le système de Vlasov-Navier-Stokes sur $\mathbb{T}^d \times \mathbb{R}^d$ (ou $\mathbb{R}^d \times \mathbb{R}^d$) en utilisant une méthode d'espace de Fourier en analogie avec les résultats prouvés pour le système d'Euler dans [KV09] et [LO97] et pour le système de Vlasov-Poisson dans [VR21]. Plus précisément, nous donnons des estimations quantitatives pour la croissance de la norme de $\frac{1}{s}$-Gevrey et la décroissance du rayon de régularité pour la solution du système en termes de champ de force $\nabla_x u$, la densité locale ρ_f et le volume du support de la variable de vitesse de la distribution des particules f. Comme application, nous obtenons l'existence de solutions $\frac{1}{s}$-Gevrey ($s \in (0,1)$) et analytiques ($s = 1$) pour le système de Vlasov-Navier-Stokes dans $\mathbb{T}^d \times \mathbb{R}^d$ (ou $\mathbb{R}^d \times \mathbb{R}^d$). En particulier, cela implique l'existence globale de solutions Gevrey ($s \in (0,1)$) dans $\mathbb{T}^3 \times \mathbb{R}^3$ (ou $\mathbb{R}^3 \times \mathbb{R}^3$).

This chapter is the subject of the paper [Dec23].

Abstract

In this chapter, we prove propagation of $\frac{1}{s}$-Gevrey regularity ($s \in (0,1)$) and Analyticity ($s = 1$) for the Vlasov-Navier-Stokes system on $\mathbb{T}^d \times \mathbb{R}^d$ (or $\mathbb{R}^d \times \mathbb{R}^d$) using a Fourier space method in analogy to the results proved for the Euler system in [KV09] and [LO97] and for Vlasov-Poisson system in [VR21]. More precisely, we give quantitative estimates for the growth of the $\frac{1}{s}$-Gevrey norm and decay of the regularity radius for the solution of the system in terms of the force field $\nabla_x u$, the local density ρ_f and the volume of the support in the velocity variable of the distribution of particles f. As an application, we obtain existence of $\frac{1}{s}$-Gevrey ($s \in (0,1)$) and Analytic ($s = 1$) solutions for the Vlasov-Navier-Stokes system in $\mathbb{T}^d \times \mathbb{R}^d$ (or $\mathbb{R}^d \times \mathbb{R}^d$). In particular, this implies existence of a unique global Gevrey solutions in $\mathbb{T}^3 \times \mathbb{R}^3$.

Contents

5.1 Introduction

5.1.1 Setting of the problem and historical context

In this chapter, we deal with the Vlasov-Navier-Stokes system (VNS):

$$\text{(VNS)}\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[(u-v)f] &= 0, & \text{in } (0,T)\times\mathbb{T}^d\times\mathbb{R}^d,\\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p &= j_f - \rho_f u, & \text{in } (0,T)\times\mathbb{T}^d,\\ \nabla_x\cdot u &= 0, & \text{in } (0,T)\times\mathbb{T}^d,\\ u(0,\cdot)=u_0, \quad f(0,\cdot,\cdot) &= f_0, \end{cases} \tag{5.1.1}$$

where ρ_f and j_f are the local density and local current respectively:

$$\rho := \rho_f(t,x) := \int_{\mathbb{R}^d} f \, \mathrm{d}v \quad \text{and} \quad j := j_f(t,x) := \int_{\mathbb{R}^d} vf \, \mathrm{d}v.$$

This system of nonlinear PDEs describes the transport of particles (described by their density function f) within a homogeneous incompressible fluid (described by its velocity u and its pressure p). This description corresponds to a regime where the particles volume fraction is small compared to that of the surrounding fluid. It belongs to the broad family of *fluid-kinetic systems* or *couplings*, which were introduced in the pioneering works of O'Rourke [O'R81] and Williams [Wil85] for the description of sprays involving a large number of particles. We also refer to [Des10] for a general overview on the description of multiphase flows, as well as to [Rei96]. Among all possible couplings (we refer to the introduction of [GHKM18] or [EHK23] for other examples), the Vlasov-Navier-Stokes system has been intensively studied because of both its physical relevance and the mathematical challenges that it offers. It has been for instance shown to provide a good description of medical aerosols in the upper part of the lung (see e.g. [BGLM15, BM21]). The VNS system (5.1.1) is fully coupled: both unknowns f and u depend on each other. This is due to the Brinkman force (the source term in the fluid equation) and the drag acceleration (the inertial term in the kinetic equation). We refer to [BGLM15] for the physical justification of these, and to [DGR08, BDGR17, BDGR18, Hil18, HMS17] for the (partial) mathematical derivation of the former. The physical constants are all normalized in (5.1.1). The VNS system can also be considered with inhomogeneous or compressible Navier-Stokes equations [CK15, Cho17, CJ22b] and additional terms in kinetic equations [CKKK22, CJ22a]. Note that the case of compressible Euler equations for the fluid, coupled to a kinetic equation, has also been investigated [BD06].

The study of the VNS system, from a mathematical point of view, has been the topic of several research papers in the last twenty years and in many directions of research. The Cauchy theory, addressing the existence of weak global solutions for the VNS system, has been tackled in dimension 2 and 3 in various domains of space (see for instance [ABdMB97, Ham98, BDGM09]), and also allows for more complex physics in the model (see [BGM17, BMM20]). It mainly consists in obtaining a *Leray weak* solution for u and a *renormalized weak* solution (in the sense of Di-Perna and Lions [DL89]) for f, using a remarkable energy-dissipation identity that is satisfied by solutions to the system. In dimension 2, the uniqueness of such solutions has been shown in [HKMMM19]. In [CK15], Choi and Kwon showed the existence of a unique strong solutions to the inhomogeneous VNS system in a time interval that depends on the initial data (provided that the initial data is sufficiently small and regular), and also established an a priori estimate for the large-time behavior of the solutions to the last system in a spatial periodic domain, i.e. in $\mathbb{T}^3$. This last question, concerning the long-time behavior of (VNS) solutions and large-scale dynamics, has attracted a lot of attention and has been the subject of several advances over the last few years. Indeed, it is expected that the cloud of particles aligns its velocity on that of the fluid,

$$u(t) \underset{t\to+\infty}{\longrightarrow} v^{\infty}, \quad f(t) \underset{t\to+\infty}{\longrightarrow} \rho^{\infty} \otimes \delta_{v=v^{\infty}},$$

for some asymptotic velocity $v^{\infty} \in \mathbb{R}^3$ and profile $\rho^{\infty}(t,x)$. The answer to this question was obtained first in [HKMM20] for Fujita-Kato type solutions, in the $3d$ torus case and in [HK22] in the whole space $\mathbb{R}^3$, while the case of a $3d$ bounded domain (with absorption boundary conditions for f) and the half-space case are investigated in [EHKM21] and [Ert21] respectively. Another type of asymptotics has been studied on VNS, we refer to [HKM21, HÏ8, Ert22] for more details. It should be noted that the “rigorous” derivation of the VNS system from an ODE system written at the microscopic scale remains an open problem. We refer to [DGR08, Hil18, HMS19, CH20, Hil21, HJ20] for a partial answer.

In this work, we investigate the propagation of higher regularity for smooth solutions to the Vlasov-Navier-Stokes system. Our method is based on the notion of *Gevrey class* regularity, which is a stronger concept than the C^∞ regularity. It not only asserts that all derivatives of the solution are bounded, but also that these bounds depend on the order of the derivatives in some prescribed way. Gevrey [Gev18] used this notion as a setting in which to extend Cauchy-Kowalevski existence arguments to classes of functions that are not necessarily analytic (for a review of the analytic case see, e.g., [Joh82]). In fact, they are special cases of the quasianalytic classes [Had12]. La Vallée Poussin [LVP⁺24a] showed that, among the quasianalytic functions, the Gevrey classes are characterized by an exponential decay of their Fourier coefficients, see [LO97, Section 2] for a definition of Gevrey classes and the proof of their characterization by Fourier transformation and Sobolev spaces (see also [KM60]). An equivalent definition is given in Subsection 5.1.2. In turn, this characterization has proven useful in investigating different questions for the solutions of various nonlinear partial differential equations. For example, this notion played a very important role in the proof of *Landau damping* in the paper by Bedrossian, Masmoudi and Mouhot [BMM16], where the authors improved the Mouhot-Villani result [MV11].

Concerning the question of propagation of higher regularity, more precisely the propagation of Gevrey regularity and Analyticity, has been the subject of several papers since the seventies. In [FT89], Foias and Temam proved that the solution of the Navier-Stokes equations (in dimension 2 and 3) is in the Gevrey class for a Sobolev initial data and a Gevrey source term which does not depend on the solution, with an affine (in time) radius of regularity. The fact that the source term here is not complicated (since it is independent of the solution) plays an important role in this gain of regularity and it allows to take advantage of the dissipative term. Contrary to this last result, for Euler's equations, the radius of analyticity (of regularity in the Gevrey case) decays exponentially in time (and as long as the quantity $\int_0^t \|\nabla u(s)\|_\infty \mathrm{d}s$, which appears in the exponent, remains finite). The first result on the Euler system have been derived by Bardos, Benachour and Zerner (see references [BBZ76, Bar76, Ben76]) who use estimates on the Green function of the Poisson kernel in the complex plane to describe the region of analyticity. For results on the local propagation of analyticity see the works by Baouendi and Goulaouic [BG75], Alinhac and Metivier [AM85, AM86], and references therein. These results were continued by the work of Levermore and Oliver [LO97] where they proved the propagation of the analytic regularity in dimension 2 on $\mathbb{T}^2$ using a method of Fourier space based on the notion of Gevrey regularity. However, the analyticity radius decay rate obtained by these last two authors was $\exp(-\exp(\exp t))$, which is faster than the $\exp(-\exp t)$ obtained by Bardos, Benachour and Zerner in [BBZ76]. Subsequently, Levermore and Oliver's result was improved by Kukavica and Vicol in [KV09] who showed the same rate of decay for the analyticity radius obtained in [BBZ76], but using a Fourier space method instead.

Recently, using a Fourier space method in analogy to the results proved for the $2d$-Euler system in [KV09] and [LO97] and applying techniques used in the proof of Landau damping [BMM16], Velozo Ruiz [VR21] proved the Gevrey regularity propagation for solutions of the Vlasov-Poisson system, giving a quantitative estimate of the decay in the radius of regularity (it is an $\exp(-\exp t)$ decrease which was obtained, as for Euler) for the solution of the system in terms of the force field and the volume of the support in the velocity variable of the distribution of matter.

In this chapter we address the problem of propagation of Gevrey regularity for the VNS system on $\mathbb{T}^d \times \mathbb{R}^d$ as long as there exists a Sobolev solution (f, u) for this system. More precisely, we give quantitative estimates for the growth of the Gevrey norm and decay of regularity radius for the solution of (5.1.1) in terms of the Sobolev norm which is itself estimated in terms of the force field $\|u\|_{W^{1,\infty}}$, the local density $\|\rho_f\|_\infty$ and the volume of the support in the velocity variable of the distribution f. As an application, we show global existence of Gevrey solutions for the VNS system in $\mathbb{T}^3 \times \mathbb{R}^3$ for initial *modulated energy* small enough, due to the result proved by Han-Kwan, Moussa and Moyano in [HKMM20]. Furthermore, the propagation of Gevrey regularity remains true on $\mathbb{R}^d \times \mathbb{R}^d$ even if it means replacing the Fourier series by integrals and minor modifications.

5.1.2 Notations, definitions and preliminaries

In order to write the main theorems of the chapter, let introduce the usual Gevrey norms. In the following, we use the multi-index notations

$$v^\alpha := v_1^{\alpha_1} \dots (v_d)^{\alpha_d} \quad \text{and} \quad \mathrm{D}_\eta^\alpha := (\mathrm{i}\partial_1)^{\alpha_1} \dots (\mathrm{i}\partial_d)^{\alpha_d},$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d, v \in \mathbb{R}^d$, $\eta \in \mathbb{R}^d$ and $\mathrm{i}^2 = -1$.
We define the usual Fourier coefficient (transformation) of $f \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ by

$$\hat{f}_k(\eta) := \frac{1}{(2\pi)^d} \iint_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\mathrm{i}x\cdot k - \mathrm{i}v\cdot\eta} f(x, v) \mathrm{d}x\mathrm{d}v$$

and of $u \in L^2(\mathbb{T}^d)$ by

$$\hat{u}_k := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-\mathrm{i}x\cdot k} u(x) \mathrm{d}x.$$

We denote by $\langle \cdot, \cdot \rangle_{L^2}$ the scalar product in the Hilbert space L^2 and we define the Japanese brackets: $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$ and $\langle k, \eta \rangle := (1 + |k|^2 + |\eta|^2)^{\frac{1}{2}}$ for all $k, \eta \in \mathbb{R}^d$.
Finally, we denote the standard Sobolev norm of f in $H^\sigma_{x,v}(\mathbb{T}^d \times \mathbb{R}^d)$ by $\|f\|_\sigma$ and we

denote by $H^\sigma_{x,v;M}(\mathbb{T}^d \times \mathbb{R}^d)$ (as in [VR21]) the weighted Sobolev space with the norm

$$\|f\|^2_{\sigma,M} := \sum_{|\alpha|\leqslant M} \|v^\alpha f\|^2_\sigma,$$

which can be written, in Fourier variables, as

$$\|f\|^2_{\sigma,M} := \sum_{|\alpha|\leqslant M} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \left|\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)\right|^2 \langle k,\eta\rangle^{2\sigma} \mathrm{d}\eta.$$

Definition 5.1.1. **($\frac{1}{s}$-Gevrey Classes in $\mathbb{T}^d \times \mathbb{R}^d$).** A real-valued function $f \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is said to be of Gevrey class $\frac{1}{s}$ with radius of regularity $\lambda > 0$, Sobolev correction $\sigma > 0$ and weight $M \in \mathbb{N}$, if for some $s \in (0,1]$, we have $f \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ and

$$\|f\|^2_{\lambda,\sigma,M,s} := \sum_{|\alpha|\leqslant M} \|v^\alpha f\|^2_{\lambda,\sigma,s} < +\infty,$$

with

$$\|v^\alpha f\|^2_{\lambda,\sigma,s} := \|A\mathrm{D}^\alpha_\eta \hat{f}\|^2_{L^2_{k,\eta}} := \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{2\sigma} e^{2\lambda\langle k,\eta\rangle^s} |\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)|^2 \mathrm{d}\eta,$$

and where

$$A := A^\sigma_k(\eta) = \langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}$$

is the Fourier multiplier. We denote by $\mathcal{G}^{\lambda,\sigma,M,\frac{1}{s}}(\mathbb{T}^d \times \mathbb{R}^d)$ the space of functions of this class.

Definition 5.1.2. **($\frac{1}{s}$-Gevrey Classes in $\mathbb{T}^d$).** A real vector function $u \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ is said to be of Gevrey class $\frac{1}{s}$ with radius of regularity $\lambda > 0$ and Sobolev correction $\sigma > 0$ if, for some $s \in (0,1]$, we have $u \in L^2(\mathbb{T}^d)$ and

$$\|u\|^2_{\lambda,\sigma,s} := \|e^{\lambda\Lambda^s} u\|^2_\sigma := \|\Lambda^\sigma e^{\lambda\Lambda^s} u\|^2_2 := \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma} e^{2\lambda\langle k\rangle^s} |\hat{u}_k|^2 < +\infty,$$

where

$$\Lambda := (\mathrm{Id} - \Delta_x)^{\frac{1}{2}}.$$

In Fourier variables, $\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s} =: A^\sigma_k(0)$ is the Fourier multiplier and $\hat{u}_k$ the Fourier coefficients of u on $\mathbb{T}^d$. We denote by $\mathcal{G}^{\lambda,\sigma,\frac{1}{s}}(\mathbb{T}^d)$ the space of functions of this class.

The transport equation. The Vlasov equation

$$\partial_t f + v\cdot\nabla_x f + \nabla_v \cdot \left[(u-v)f\right] = 0$$

can be rewritten as

$$\partial_t f + v\cdot\nabla_x f + (u-v)\cdot\nabla_v f - df = 0.$$

For $s,t \geqslant 0$ and $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, we define (see [HKMM20, Definition 4.1]) the characteristic curves $(X(s,t,x,v), V(s,t,x,v))$ as the solutions to the system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}X(s,t,x,v) = V(s,t,x,v), & X(t,t,x,v)=x, \\ \\ \dfrac{\mathrm{d}}{\mathrm{d}s}V(s,t,x,v) = u(s,X(s,t,x,v)) - V(s,t,x,v), & V(t,t,x,v)=v. \end{cases} \tag{5.1.2}$$

By the method of characteristics, for a smooth vector field u, we can write the solution f to the Vlasov equation as

$$f(t,x,v) = e^{dt} f_0(X(0,t,x,v), V(0,t,x,v)). \tag{5.1.3}$$

As a consequence, for almost all $t \geqslant 0$,

$$\|f(t)\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \leqslant \|f_0\|_{L^\infty(\mathbb{T}^d\times\mathbb{R}^d)} e^{dt}.$$

It is clear thanks to (5.1.3) that if f_0 has a compact support, then $f(t,\cdot,\cdot)$ will also have a compact support for all t. We denote

$$X_\infty(t) := \sup\left\{|x| : \exists\ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d \text{ such that } f(t,x,v) > 0\right\}.$$

In other words,

$$\mathrm{supp} f(t,\cdot,\cdot) \subset \mathbb{T}^d \times B\big(0, V_\infty(t)\big).$$

Note that, under this notations, we can write

$$\|f(t)\|_{M,\infty} \leqslant V_\infty^M(t)\|f(t)\|_\infty \leqslant V_\infty^M(t)\|f_0\|_\infty e^{dt}. \tag{5.1.4}$$

Notation: We denote by $V^M(t)$ the quantity

$$V^M(t) := V_\infty^{2M}(t)\|f_0\|_\infty^2 e^{2dt}. \tag{5.1.5}$$

On the existence of strong solutions for VNS. In [CK15], the existence of a unique strong solution was proved for the inhomogeneous Vlasov-Navier-Stokes system in $\Omega\times\mathbb{R}^3$ with $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$, under some assumptions on the density ϱ, taking f_0 with compact support in position and velocity and under the smallness of $\|f_0\|_{H^2(\Omega\times\mathbb{R}^3)} + \|u_0\|_{H^2(\Omega)}$. For our system (5.1.1) which corresponds to $\varrho \equiv 1$ in [CK15, Theorem 1.1], the solution is given in the following sense:
For any $T > 0$, there exists $\varepsilon := \varepsilon(T) > 0$ depending only on T such that if

$$\|f_0\|_{H^2(\Omega\times\mathbb{R}^3)} + \|u_0\|_{H^2(\Omega)} \leqslant \varepsilon,$$

then, the VNS system (5.1.1) admits a unique strong solution (f,u) satisfying

- $f \in C(0,T; H^2(\Omega\times\mathbb{R}^3))$;
- $u \in C(0,T;H^2(\Omega) \cap L^2(0,T;H^3(\Omega)$ and $\partial_t u \in C(0,T;L^2(\Omega) \cap L^2(0,T;H^1(\Omega)$;

- $\nabla_x p \in C(0,T;L^2(\Omega) \cap L^2(0,T;H^1(\Omega)$.

A *modulated* version, denoted by $\mathcal{E}(t)$, of the energy $E(t)$ (see Definition 1.7 and Definition 1.1 in [HKMM20] respectively) was introduced in the paper [CK15] and played an important role in the work of [HKMM20]. In particular, for $\mathcal{E}(0)$ defined by

$$\mathcal{E}(0) := \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_0(x,v)|v-\langle j_{f_0}\rangle|^2 \,\mathrm{d}v\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |u_0(x)-\langle u_0\rangle|^2 \,\mathrm{d}x + \frac{1}{4}\big|\langle j_{f_0}\rangle - \langle u_0\rangle\big|^2, \tag{5.1.6}$$

small enough, we get

$$\int_1^\infty \|\nabla_x u(t)\|_\infty \mathrm{d}t + \|\rho_f\|_{L^\infty((0,\infty)\times\mathbb{T}^3)} \lesssim 1, \tag{5.1.7}$$

where $\langle j_{f_0}\rangle := \int_{\mathbb{R}^3} j_{f_0}(x)\mathrm{d}x$ and $\langle u_0\rangle := \int_{\mathbb{R}^3} u_0(x)\mathrm{d}x$. This last estimate will allow us to obtain the global existence of Gevrey solutions to the VNS system (5.1.1) in $\mathbb{T}^3\times\mathbb{R}^3$ later on.

5.1.3 Main results

From now on, the parameter $s \in (0,1]$ is fixed, while $\lambda(t)$ can vary over time.

Theorem 5.1.3 (Propagation of Gevrey regularity). *Let (f_0,u_0) be initial data for the VNS system (5.1.1) on $\mathbb{T}^d\times\mathbb{R}^d$ such that, f_0 has a compact support in velocity and $\|f_0\|_{\lambda_0,\sigma,M,s} + \|u_0\|_{\lambda_0,\sigma,s}$ is finite for some $s\in(0,1]$, $\lambda_0>0$, $\sigma > \frac{d}{2}+\frac{s}{2}+2$ and $M > \frac{d}{2}+1$. Then, the unique classical solution $(f,u) \in C(0,T_{max};H^\sigma_{x,v;M}) \times C(0,T_{max};H^\sigma_x)$ satisfies for all $t\in[0,T_{max})$ the upper bounds*

$$\|f\|_{\lambda,\sigma,M,s} \leqslant C_1(1+t)g(t) \tag{5.1.8}$$

and

$$\|u\|_{\lambda,\sigma,s} \leqslant \left(\|u_0\|_{\lambda_0,\sigma,s} + C_2\int_0^t (1+s)g(s)e^{-C_2\int_0^s g(\tau)\mathrm{d}\tau}\mathrm{d}s\right) e^{C_2\int_0^t g(\tau)\mathrm{d}\tau}, \tag{5.1.9}$$

and for all $t\in[0,T_{max})$ the lower bound

$$\lambda(t) \geqslant \left(2C_3 t + \lambda_0^{-1}\right)^{-1} \exp\left[-C_3\int_0^t \left(1+\|u\|_\sigma + \|f\|_{\sigma,M}\right)\mathrm{d}\tau\right] > 0, \tag{5.1.10}$$

where

$$g(t) := \exp\left[C_0\int_0^t \left(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(\tau) + 1\right)\mathrm{d}\tau\right],$$

and where T_{max} is the maximal time of existence. The constants C_0, C_1, C_2 and C_3 depend on the initial data (f_0,u_0), the radius of regularity λ_0, the Sobolev correction σ, the weight M and the dimension d.

Remark 5.1.4. We could remove the compact support assumption on the initial data, however the estimates for the growth of the Gevrey norm and the radius of regularity would be bounded in terms of $\|f\|_{\sigma,M}$ and $\|u\|_{\sigma}$, instead of $\|u\|_{W^{1,\infty}} + \|\rho_f\|_{\infty} + V^M(t) + 1$, and for a *short time*, due to the proof given in Section 5.2. If we want to have a propagation for global solutions (for small data), we must have more finite moments (in velocities) and show a propagation of the moments in this case. This is analogous to the results of Pfaffelmoser [Pfa92] and Lions-Perthame [LP91] for the Vlasov-Poisson system. This last constraint comes from the estimate of the commutator that one needs to control the force term which comes from the Vlasov equation. In particular, in both cases, we obtain propagation of analyticity for the Vlasov-Navier-Stokes system in $\mathbb{T}^d \times \mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem 5.1.5 (Propagation of Analyticity)**.** *Let (f_0, u_0) be initial data for the VNS system (5.1.1) on $\mathbb{T}^d \times \mathbb{R}^d$ such that $\|f_0\|_{\lambda_0,\sigma,M,1} + \|u_0\|_{\lambda_0,\sigma,1}$ is finite for some $\lambda_0 > 0$, $\sigma > \frac{d+1}{2} + 2$ and $M > \frac{d}{2} + 1$. Then, the classical solution $(f, u) \in C(0, T_{max}; H^{\sigma}_{x,v;M}) \times C(0, T_{max}; H^{\sigma}_{x})$ satisfies for all $t \in [0, T_{max})$ the upper bounds*

$$\|f(t)\|_{\lambda,\sigma,M,s} \leqslant \|f_0\|_{\lambda_0,\sigma,M,s} \exp\big[C_4 \int_0^t (1 + \|u(\tau)\|_{\sigma}) \mathrm{d}\tau\big] \tag{5.1.11}$$

and

$$\|u\|_{\lambda,\sigma,1} \leqslant \left(\|u_0\|_{\lambda_0,\sigma,s} + C_5 \int_0^t \|f(\zeta)\|_{\lambda,\sigma,M,s} e^{-C_5 \int_0^{\zeta} Y(\tau)\mathrm{d}\tau} \mathrm{d}\zeta \right) e^{C_5 \int_0^t Y(\tau)\mathrm{d}\tau}, \tag{5.1.12}$$

and the lower bound (5.1.10), where $Y(\tau) := \|u(t)\|_{\sigma}^2 + \|f(t)\|_{\sigma,M}^2$ and T_{max} is the maximal time of existence. The constants C_4 and C_5 depend on the initial data (f_0, u_0), the radius of regularity λ_0, the Sobolev correction σ, the weight M and the dimension d.

Remark 5.1.6. For all $t < \frac{1}{C} \ln\left(\frac{1+Y_0}{Y_0}\right)$ with $Y_0 := Y(0)$, we have the estimate

$$\|u(t)\|_{\sigma}^2 + \|f(t)\|_{\sigma,M}^2 \leqslant \left(1 - \frac{Y_0}{1+Y_0} e^{Ct}\right)^{-1}.$$

Theorem 5.1.7 (Blow up criterion)**.** *Let (f_0, u_0) be initial data for the VNS system (5.1.1) on $\mathbb{T}^d \times \mathbb{R}^d$ such that $\|f_0\|_{\lambda_0,\sigma,M,s} + \|u_0\|_{\lambda_0,\sigma,s}$ is finite for some $s \in (0, 1]$, $\lambda_0 > 0$, $\sigma > \frac{d}{2} + 1$ and $M > \frac{d}{2} + 1$. Let $T_{\max}$ be the maximal time of existence of the Gevrey solution (f, u) of the VNS system (5.1.1). Then, if for some $T \in [0, T_{\max}]$, we have*

$$\limsup_{t \to T} \left(\|f(t)\|_{\sigma,M} + \|u(t)\|_{\sigma} \right) < +\infty, \tag{5.1.13}$$

then $T < T_{\max}$.

In other words, the propagation of Gevrey regularity on $[0, T]$ follows as long as $\|f\|_{\sigma,M} + \|u\|_{\sigma}$ is uniformly bounded on $[0, T]$.

As an application of Theorem 5.1.3, we obtain global existence of Gevrey solutions for the VNS system (5.1.1) in $\mathbb{T}^3 \times \mathbb{R}^3$. This result follows directly by using the results of Han-Kwan, Moussa and Moyano in [HKMM20].

Corollary 5.1.8 (Global existence of Gevrey solutions for $s \in (0,1)$). *Let (f_0, u_0) be initial data for the VNS system (5.1.1) in $\mathbb{T}^3 \times \mathbb{R}^3$ such that, f_0 has a compact support in velocity and $\|f_0\|_{\lambda_0,\sigma,M,s} + \|u_0\|_{\lambda_0,\sigma,s}$ is finite for some $s \in (0,1)$, $\lambda_0 > 0$, $\sigma > \frac{7}{2} + \frac{s}{2}$ and $M > \frac{5}{2}$. Let $\mathcal{E}(0)$ (defined in (5.1.6)) small enough in the sense of Theorem 2.1 in [HKMM20]. Then, there exist a unique global classical solution $(f, u) \in C(0, \infty; H^\sigma(\mathbb{T}^3 \times \mathbb{R}^3)) \times C(0, \infty; H^\sigma(\mathbb{T}^3)) \cap L^2(0, \infty; H^{\sigma+1}(\mathbb{T}^3))$ of the VNS system (5.1.1) satisfies for all $t \in [0, \infty)$ the upper bounds (5.1.8) and (5.1.9), and the radius of regularity $\lambda(t)$ satisfies the lower bound (5.1.10).*

Comments.

1. Note that we have recovered the same Gevrey estimates for the Vlasov solution as those for Vlasov-Poisson, and with a lower bound like $\exp(-\exp t)$ for the radius of regularity.

2. In [FT89], the radius of analyticity for the Navier-Stokes equations is given by $\lambda(t) = \min(t, \lambda_1, T^*)$, where λ_1 is the radius of analyticity of the source term F (and which does not depend on u) and T^* is the maximal time of existence which depends on u_0 and the source term F. This makes that at $t = 0$, $\lambda(0) = 0$. Then, we recover the Sobolev norm of u_0 instead of a Gevrey norm at $t = 0$. Thus, a control of the Gevrey norm of u at time t by the Gevrey norm at the initial time $t = 0$, allowed them to get the Gevrey propagation for u_0 just Sobolev.

Idea of the proof and outline of the paper

The main result follow by energy estimates based on a Fourier space method motivated by the approach used in [KV09] to study the propagation of analytic regularity for the $2d$-Euler system and [VR21] for the Gevrey regularity for the Vlasov-Poisson system. The Gevrey norm will be estimated by the Sobolev norm, so we will start with the Sobolev estimates for the solution (f, u) in Section 5.2, then move on to Gevrey estimates in Section 5.3.

The parameters s, σ and M are fixed, while λ is a function in t. The weight in which M intervenes is made to control the moments ρ_f and j_f in term of the density distribution f, and the function λ will be chosen so that the norms whose Sobolev regularity overflows (due to time derivatives in the energy method) are absorbed.

5.2 Sobolev estimates

The purpose of this section is to show the following Proposition on the propagation of the Sobolev regularity. For this purpose, we prove a quantitative bound for the growth of weighted Sobolev norms of the solutions (u, f) of the VNS system in terms of $\|u\|_{W^{1,\infty}}$, $\|\rho\|_\infty$ and the support of f in velocity $V^M(t)$.

Proposition 5.2.1 (Sobolev estimates for VNS). *Let $\sigma > 0$ and $M > \frac{d}{2} + 1$. Let (f, u) satisfying (5.1.1) such that f has a compact support in velocity and $\|f_0\|_{\sigma,M} + \|u_0\|_\sigma$ is finite. Then, the following estimate holds*

$$\|f\|_{\sigma,M}^2 + \|u\|_\sigma^2 \leqslant (\|f_0\|_{\sigma,M}^2 + \|u_0\|_\sigma^2) g(t) \tag{5.2.1}$$

where

$$g(t) := \exp\big[C_0 \int_0^t \big(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(\tau) + 1\big)\, \mathrm{d}\tau\big],$$

and C_0 is a positive constant which depends only on σ, M and d.

In order to prove Proposition 5.2.1, we will establish estimates on the time derivative of the Sobolev norm of each of the Vlasov and Navier-Stokes solutions, then we conclude with Gronwall's Lemma applied to a combination of the two estimates.

5.2.1 Sobolev estimates for solutions of Vlasov's equation

The goal of this subsection is to prove the following lemma:

Lemma 5.2.2 (Sobolev estimates for Vlasov). *Let $\sigma > 0$ and $M > 0$. Let (u, f) be the solution of (5.1.1) such that f has a compact support in velocity. Then, one has the following estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\sigma,M}^2 \lesssim (\|u\|_{W^{1,\infty}} + 1)\|f\|_{\sigma,M}^2 + \sqrt{V^M(t)}\|u\|_{\sigma+1}\|f\|_{\sigma,M}. \tag{5.2.2}$$

We will start with two lemmas. The first one is on the Gevrey norm estimate for the density ρ_f and the moment j_f, which gives in particular the Sobolev estimates for $\lambda = 0$.

Lemma 5.2.3 (Density and first moment estimates). *Let $s \in (0, 1], \lambda \geqslant 0$ and $\sigma > 0$. Let $f \in \mathcal{G}^{\lambda,\sigma,M,\frac{1}{s}}(\mathbb{T}^d \times \mathbb{R}^d)$. Then, for $M > \frac{d}{2} + 1$, one has*

$$\|\rho_f\|_{\lambda,\sigma,s} \lesssim \|f\|_{\lambda,\sigma,M,s} \qquad \text{and} \qquad \|j_f\|_{\lambda,\sigma,s} \lesssim \|f\|_{\lambda,\sigma,M,s}. \tag{5.2.3}$$

Remark 5.2.4. The inequality

$$\|\rho_f\|_{\lambda,\sigma,s} \lesssim \|f\|_{\lambda,\sigma,M,s}$$

is valid as soon as $M > \frac{d}{2}$.

Proof. The proof of Lemma 5.2.3 is obtained by the Cauchy-Schwarz inequality. Another proof concerning the inequality on ρ_f is given in [VR21].
We have:

$$\begin{aligned}
\|\rho_f\|^2_{\lambda,\sigma,s} = \|\Lambda^\sigma e^{\lambda\Lambda^s}\rho_f\|^2_{L^2} &= \int_{\mathbb{T}^d}\left|\int_{\mathbb{R}^d}\Lambda^\sigma e^{\lambda\Lambda^s} f\mathrm{d}v\right|^2\mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{R}^d}\big(1+|v|\big)^{-2M}\mathrm{d}v\right)\iint_{\mathbb{T}^d\times\mathbb{R}^d}\left|\big(1+|v|\big)^M\Lambda^\sigma e^{\lambda\Lambda^s} f\right|^2\mathrm{d}v\mathrm{d}x \\
&\lesssim \|f\|^2_{\lambda,\sigma,M,s} \quad \text{for } M > \frac{d}{2}.
\end{aligned}$$

Similarly for j_f, we have:

$$\begin{aligned}
\|j_f\|^2_{\lambda,\sigma,s} &= \int_{\mathbb{T}^d}\left|\int_{\mathbb{R}^d} v\Lambda^\sigma e^{\lambda\Lambda^s} f\mathrm{d}v\right|^2\mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{R}^d}\big(1+|v|\big)^{2-2M}\mathrm{d}v\right)\iint_{\mathbb{T}^d\times\mathbb{R}^d}\left|\big(1+|v|\big)^M\Lambda^\sigma e^{\lambda\Lambda^s} f\right|^2\mathrm{d}v\mathrm{d}x \\
&\lesssim \|f\|^2_{\lambda,\sigma,M,s} \quad \text{for } M > \frac{d}{2}+1.
\end{aligned}$$

□

The second Lemma concerns the estimation of the commutator.

Lemma 5.2.5 (Inequality on the commutator term)**.**
Let $u, v \in W^{1,\infty}(\mathbb{T}^d)\bigcap H^\sigma(\mathbb{T}^d)$ and let $f \in L^\infty(\mathbb{T}^d\times\mathbb{R}^d)\bigcap H^\sigma(\mathbb{T}^d\times\mathbb{R}^d)$ be a function with compact support in velocity

$$\operatorname{supp} f \subset \mathbb{T}^d\times \mathrm{B}(0, C_f), \quad C_f \in (0,+\infty).$$

Then, the following inequalities hold

1. $\|uf\|_\sigma \lesssim_{C_f} \|u\|_\infty\|f\|_\sigma + \|u\|_\sigma\|f\|_\infty.$
2. $\displaystyle\sum_{|\beta|\leqslant\sigma}\|\mathrm{D}^\beta(uf) - u\mathrm{D}^\beta(f)\|_2 \lesssim_{C_f} \|\nabla u\|_\infty\|f\|_{\sigma-1} + \|u\|_\sigma\|f\|_\infty.$
3. $\|uv\|_\sigma \lesssim \|\nabla_x u\|_\infty\|v\|_{\sigma-1} + \|u\|_\sigma\|v\|_\infty.$
4. $\displaystyle\sum_{|\beta|\leqslant\sigma}\|\mathrm{D}^\beta(uv) - u\mathrm{D}^\beta(v)\|_2 \lesssim \|\nabla_x u\|_\infty\|v\|_{\sigma-1} + \|u\|_\sigma\|v\|_\infty.$

Remark 5.2.6. The condition of f with compact support eliminates the Analytic case, i.e. $s = 1$. We will deal with this last case separately (to control the Sobolev norm only).

We are going to prove the first two inequalities and the last two are obtained in the same way. Before starting the proof, let us recall the Gagliardo-Nirenberg inequality.

Lemma 5.2.7 (Gagliardo-Nirenberg inequality)**.** *Let $1 \leqslant q \leqslant +\infty$ be a positive extended real quantity. Let j and m be non-negative integers such that $j < m$. Furthermore, let $1 \leqslant r \leqslant +\infty$ be a positive extended real quantity, $p \geqslant 1$ be real and $\theta \in [0,1]$ such that the relations*

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad \frac{j}{m} \leq \theta \leq 1$$

hold. Then,

$$\|\mathrm{D}^j u\|_{L^p(\mathbb{R}^n)} \leqslant C\|\mathrm{D}^m u\|^{\theta}_{L^r(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}$$

for any $u \in L^q(\mathbb{R}^n)$ such that $\mathrm{D}^m u \in L^r(\mathbb{R}^n)$, with two exceptional cases:

1. *if $j = 0$ (with the understanding that $\mathrm{D}^0 u = u$), $q = +\infty$ and $rm < n$, then an additional assumption is needed: either u tends to 0 at infinity, or $u \in L^s(\mathbb{R}^n)$ for some finite value of s;*

2. *if $r > 1$ and $m - j - \dfrac{n}{r}$ is a non-negative integer, then the additional assumption $\dfrac{j}{m} \leqslant \theta < 1$ (notice the strict inequality) is needed.*

In any case, the constant $C > 0$ depends on the parameters j, m, n, q, r, θ, but not on u.

Proof of Lemma 5.2.5. The proof of this Lemma has been taken from [VR21] (inspired by Klainermann and Majda [KM81]), we recall it here to specify at what point the condition of support on f comes up.

1. Let f be a function with compact support such that $f \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d) \cap H^\sigma(\mathbb{T}^d \times \mathbb{R}^d)$ and let $u \in L^\infty(\mathbb{T}^d) \cap H^\sigma(\mathbb{T}^d)$. On the one hand, we have

$$\|uf\|_\sigma = \sum_{|\beta|\leqslant\sigma} \|\mathrm{D}^\beta(uf)\|_{L^2} \lesssim \sum_{\beta_1+\beta_2=\beta} \|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}f\|_{L^2}.$$

On the other hand, by using Hölder inequality for $p = \frac{|\beta|}{|\beta_1|}$ and $q = \frac{|\beta|}{|\beta_2|}$, we write

$$\begin{aligned}\|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}f\|^2_{L^2} &= \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |\mathrm{D}^{\beta_1}u|^2|\mathrm{D}^{\beta_2}f|^2 \mathrm{d}x\mathrm{d}v \\ &\leqslant \left(\int_{\mathbb{T}^d} |\mathrm{D}^{\beta_1}u|^{2\frac{|\beta|}{|\beta_1|}}\mathrm{d}x\right)^{\frac{|\beta_1|}{|\beta|}} \int_{\mathbb{R}^d}\left(\int_{\mathbb{T}^d} |\mathrm{D}^{\beta_2}f|^{2\frac{|\beta|}{|\beta_2|}}\mathrm{d}x\right)^{\frac{|\beta_2|}{|\beta|}} \mathrm{d}v.\end{aligned}$$

Thus, by Jensen's inequality applied to the integral in v with the convex function $t \mapsto t^{\frac{|\beta|}{|\beta_2|}}$, we obtain

$$\begin{aligned}\|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}f\|_{L^2_{x,v}(\mathbb{T}^d\times\mathbb{R}^d)} &\leqslant C_f\left(\int_{\mathbb{T}^d} |\mathrm{D}^{\beta_1}u|^{2\frac{|\beta|}{|\beta_1|}}\mathrm{d}x\right)^{\frac{|\beta_1|}{|\beta|}} \left(\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} |\mathrm{D}^{\beta_2}f|^{2\frac{|\beta|}{|\beta_2|}}\mathrm{d}x\mathrm{d}v\right)^{\frac{|\beta_2|}{|\beta|}} \\ &= C_f\|\mathrm{D}^{\beta_1}u\|_{L_x^{2|\beta|/|\beta_1|}}\|\mathrm{D}^{\beta_2}f\|_{L_{x,v}^{2|\beta|/|\beta_2|}}.\end{aligned}$$

Note that this is where the assumption of "f with compact support" comes into play, in order to be able to apply Jensen's inequality on $\mathbb{R}^d$ with the Lebesgue measure. Note also that the constant C_f in the last inequality depends on the support of f. Now, we have:

$$\|\mathrm{D}^\beta(uf)\|_{L^2} \lesssim \sum_{\beta_1+\beta_2=\beta} \|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}f\|_{L^2} \leqslant \sum_{\beta_1+\beta_2=\beta} \|\mathrm{D}^{\beta_1}u\|_{L_x^{2|\beta|/|\beta_1|}} \|D^{\beta_2}f\|_{L_{x,v}^{2|\beta|/|\beta_2|}},$$

and by applying Gagliardo-Nirenberg inequality for $j=|\beta_i|$, $m=|\beta|$, $\alpha=\frac{|\beta_i|}{|\beta|}=\frac{j}{m}\leqslant 1$, $q=\infty$ and $r=2$, $p=2\frac{|\beta|}{|\beta_i|}$ with $i\in\{1,2\}$, we write

$$\begin{aligned}
\|\mathrm{D}^\beta(uf)\|_{L^2} &\lesssim \sum_{\beta_1+\beta_2=\beta} \|u\|_\infty^{1-|\beta_1|/|\beta|}\|u\|_{|\beta|}^{|\beta_1|/|\beta|}\|f\|_\infty^{1-|\beta_2|/|\beta|}\|f\|_{|\beta_2|/|\beta|}^{|\beta|} \\
&= \sum_{\beta_1+\beta_2=\beta} \left(\|u\|_{|\beta|}\|f\|_\infty\right)^{|\beta_1|/|\beta|}\left(\|u\|_\infty\|f\|_{|\beta|}\right)^{|\beta_2|/|\beta|} \\
&\leqslant \sum_{\beta_1+\beta_2=\beta} \left(\frac{|\beta|}{|\beta_1|}\|u\|_{|\beta|}\|f\|_\infty + \frac{|\beta|}{|\beta_2|}\|u\|_\infty\|f\|_{|\beta|}\right).
\end{aligned}$$

Then, by Young's inequality: $ab\leqslant\frac{a^p}{p}+\frac{b^q}{q}$ with $p=\frac{|\beta|}{|\beta_1|}$ and $q=\frac{|\beta|}{|\beta_2|}$. Therefore,

$$\|uf\|_\sigma = \left(\sum_{|\beta|\leqslant\sigma}\|\mathrm{D}^\beta(uf)\|_{L^2}^2\right)^{\frac{1}{2}} \lesssim \|u\|_\sigma\|f\|_\infty + \|u\|_\infty\|f\|_\sigma. \tag{5.2.4}$$

2. For $u\in W^{1,\infty}(\mathbb{T}^d)\cap H^\sigma(\mathbb{T}^d)$, we write

$$\begin{aligned}
\sum_{|\beta|\leqslant\sigma}\|\mathrm{D}^\beta(uf)-u\mathrm{D}^\beta f\|_{L^2} &\lesssim \sum_{|\beta|\leqslant\sigma}\sum_{\beta_1+\beta_2=\beta}\|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}f\|_{L^2} \\
&\lesssim \sum_{|\beta|\leqslant\sigma}\sum_{\beta_1+\beta_2=\beta}\|\mathrm{D}^{\beta_1-1}(\mathrm{D}u)\mathrm{D}^{\beta_2}f\|_{L^2} \\
&\lesssim \|\nabla_x u\|_\infty\|f\|_{\sigma-1} + \|u\|_\sigma\|f\|_\infty \qquad \big(\text{by } (5.2.4)\big).
\end{aligned}$$

In the proof of the last two points, we do not need to apply Jensen's inequality because from the first step, by applying Hölder's inequality, we get:

$$\|\mathrm{D}^{\beta_1}u\mathrm{D}^{\beta_2}v\|_{L^2} \leqslant \|\mathrm{D}^{\beta_1}u\|_{L^{2|\beta|/|\beta_1|}}\|\mathrm{D}^{\beta_2}v\|_{L^{2|\beta|/|\beta_2|}}.$$

□

Remark 5.2.8. The condition "f with compact support" can be replaced by a weight for the velocity variable in order to be able to apply Jensen's inequality on $\mathbb{R}^d$ with the Lebesgue measure in the previous proof, but this requires more finite moments on f and which amounts to proving a propagation of moments (see e.g. [HKR16, Lemma 1]).

Proof of Lemma 5.2.2. Recall that

$$\|f\|^2_{\sigma,M} := \sum_{|\alpha|\leqslant M} \|v^\alpha f\|^2_\sigma = \sum_{|\beta|\leqslant\sigma}\sum_{|\alpha|\leqslant M} \|\mathrm{D}^\beta v^\alpha f\|^2_{L^2}.$$

We will show that for all $|\beta| \leqslant \sigma$ and $|\alpha| \leqslant M$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{D}^\beta(v^\alpha f)\|^2_{L^2} \lesssim (\|u\|_{W^{1,\infty}}+1)\Big[\|\mathrm{D}^\beta(v^\alpha f)\|^2_{L^2} + \|\mathrm{D}^\beta(v^{\alpha-e_i} f)\|^2_{L^2}\Big] + \|f\|_{\infty,M}\|u\|_{\sigma+1}\|\mathrm{D}^\beta(v^\alpha f)\|_{L^2},$$

where e_i denote the multi-index worth 1 in the i-th position and 0 elsewhere, i.e. $e_i := (0,\dots,1,\dots,0)$. Then, the estimate (5.2.2) is obtained after summation over α and β in the previous inequality. By using the Vlasov equation, we write

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\|\mathrm{D}^\beta(v^\alpha f)\|^2_{L^2} &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta(v^\alpha\partial_t f)\mathrm{d}x\mathrm{d}v\\ &= -\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta(v^\alpha v\cdot\nabla_x f)\mathrm{d}x\mathrm{d}v\\ &\quad -\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta\big(v^\alpha\nabla_v\cdot\big[(u-v)f\big]\big)\mathrm{d}x\mathrm{d}v\\ &=: -(E+F),\end{aligned}$$

with

$$E := \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta(v^\alpha v\cdot\nabla_x f)\mathrm{d}x\mathrm{d}v \;\text{ and }\; F := \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta\big(v^\alpha\nabla_v\cdot\big[(u-v)f\big]\big)\mathrm{d}x\mathrm{d}v.$$

Estimation of E: we have

$$\begin{aligned}E &= \sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\partial_{x_i}\big(\mathrm{D}^\beta[v_i(v^\alpha f)]\big)\mathrm{d}x\mathrm{d}v\\ &= \sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\sum_{\beta_1+\beta_2=\beta}\binom{\beta}{\beta_1}\mathrm{D}^{\beta_1}(v_i)\partial_{x_i}\big(\mathrm{D}^{\beta_2}(v^\alpha f)\big)\mathrm{d}x\mathrm{d}v,\end{aligned}$$

where we used Leibniz's formula in the last line, and where each term in this line is zero for $|\beta_1|>1$. Moreover, for the case $|\beta_1|\leqslant 1$, either $\mathrm{D}^{\beta_1}(v_i)=0$, $\mathrm{D}^{\beta_1}(v_i)=1$ or $\mathrm{D}^{\beta_1}(v_i)=v_i$. Then,

$$E = \sum_{i=1}^d\int\Big[v_i\mathrm{D}^\beta(v^\alpha f)\partial_{x_i}\big(\mathrm{D}^\beta(v^\alpha f)\big) + \mathrm{D}^\beta(v^\alpha f)\partial_{x_i}\big(\mathrm{D}^{\beta-e_i}(v^\alpha f)\big)\Big]\mathrm{d}x\mathrm{d}v := E_1+E_2,$$

where

$$E_1 := \sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d} v_i\mathrm{D}^\beta(v^\alpha f)\partial_{x_i}\big(\mathrm{D}^\beta(v^\alpha f)\big)\mathrm{d}x\mathrm{d}v = \frac{1}{2}\sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d} v_i\partial_{x_i}|\mathrm{D}^\beta(v^\alpha f)|^2\mathrm{d}x\mathrm{d}v = 0$$

and

$$
\begin{aligned}
|E_2| &:= \left| \sum_{i=1}^{d} \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\partial_{x_i}\big(\mathrm{D}^{\beta-e_i}(v^\alpha f)\big)\mathrm{d}x\mathrm{d}v \right| \\
&\leqslant \frac{1}{2}\sum_{i=1}^{d}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} |\mathrm{D}^\beta(v^\alpha f)|^2\mathrm{d}x\mathrm{d}v + \int_{\mathbb{T}^d\times\mathbb{R}^d} \left|\partial_{x_i}\big(\mathrm{D}^{\beta-e_i}(v^\alpha f)\big)\right|^2\mathrm{d}x\mathrm{d}v\right) \lesssim \|\mathrm{D}^\beta(v^\alpha f)\|_{L^2}^2.
\end{aligned}
$$

Estimation of F: we will expand the scalar product and make the commutator appear in order to use inequality 2 of Lemma 5.2.5 as follows:

$$
\begin{aligned}
F &:= \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta\big(v^\alpha\nabla_v\cdot\big[(u-v)f\big]\big)\mathrm{d}x\mathrm{d}v \\
&= \sum_{i=1}^{d}\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta\big(v^\alpha\partial_{v_i}\big[(u_i-v_i)f\big]\big)\mathrm{d}x\mathrm{d}v \\
&= \sum_{i=1}^{d}\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\Big[\mathrm{D}^\beta(u_iv^\alpha\partial_{v_i}f) - \mathrm{D}^\beta(v^\alpha f) - \mathrm{D}^\beta(v^\alpha v_i\partial_{v_i}f)\Big]\mathrm{d}x\mathrm{d}v \\
&= \sum_{i=1}^{d}\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathrm{D}^\beta(v^\alpha f)\Big[\partial_{v_i}\mathrm{D}^\beta(u_iv^\alpha f) - u_i\partial_{v_i}\mathrm{D}^\beta(v^\alpha f) + u_i\partial_{v_i}\mathrm{D}^\beta(v^\alpha f) - \mathrm{D}^\beta(u_iv^{\alpha-e_i}f) \\
&\qquad\qquad - \mathrm{D}^\beta(v^\alpha f) - \mathrm{D}^\beta(v^\alpha v_i\partial_{v_i}f)\Big]\mathrm{d}x\mathrm{d}v \\
&= F_1 + F_2 - F_3 - d\|\mathrm{D}^\beta(v^\alpha f)\|_{L^2}^2 - F_4,
\end{aligned}
$$

where

$$
F_1 := \sum_{i=1}^{d}\int \mathrm{D}^\beta(v^\alpha f)\big[\partial_{v_i}\mathrm{D}^\beta(u_iv^\alpha f) - u_i\partial_{v_i}\mathrm{D}^\beta(v^\alpha f)\big]\mathrm{d}x\mathrm{d}v,
$$

$$
F_2 := \sum_{i=1}^{d}\int u_i\mathrm{D}^\beta(v^\alpha f)\partial_{v_i}\mathrm{D}^\beta(v^\alpha f)\mathrm{d}x\mathrm{d}v,
$$

$$
F_3 := \sum_{i=1}^{d}\int \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta(u_iv^{\alpha-e_i}f)\mathrm{d}x\mathrm{d}v \quad\text{and}\quad F_4 := \sum_{i=1}^{d}\int \mathrm{D}^\beta(v^\alpha f)\mathrm{D}^\beta(v^\alpha v_i\partial_{v_i}f)\mathrm{d}x\mathrm{d}v.
$$

For F_1, by inequality 2 of Lemma 5.2.5, we obtain

$$
\begin{aligned}
|F_1| &\lesssim \sum_{i=1}^{d}\|\mathrm{D}^\beta(v^\alpha f)\|_{L^2}\big(\|\nabla u_i\|_\infty\|v^\alpha f\|_\sigma + \|u_i\|_{\sigma+1}\|v^\alpha f\|_\infty\big) \\
&\lesssim \|\mathrm{D}^\beta(v^\alpha f)\|_{L^2}\big(\|\nabla u\|_\infty\|f\|_{\sigma,M} + \|u\|_{\sigma+1}\|f\|_{\infty,M}\big).
\end{aligned}
$$

For F_2, we have

$$F_2 = \frac{1}{2} \sum_{i=1}^{d} \int u_i \partial_{v_i} |\mathrm{D}^\beta (v^\alpha f)|^2 \mathrm{d}x\mathrm{d}v = 0.$$

The term F_3, dealt with the same way as F_1, so we write

$$\begin{aligned}
F_3 &= \sum_{i=1}^{d} \int \mathrm{D}^\beta (v^\alpha f) \big[\mathrm{D}^\beta (u_i v^{\alpha - e_i} f) - u_i \mathrm{D}^\beta (v^{\alpha - e_i} f) + u_i \mathrm{D}^\beta (v^{\alpha - e_i} f)\big] \mathrm{d}x\mathrm{d}v \\
&\lesssim \|\mathrm{D}^\beta (v^\alpha f)\|_{L^2} \big(\|\nabla u_i\|_\infty \|v^\alpha f\|_{\sigma-1} + \|u_i\|_\sigma \|v^\alpha f\|_\infty + \|u_i\|_\infty \|\mathrm{D}^\beta (v^{\alpha - e_i} f)\|_{L^2}\big) \\
&\lesssim \|\mathrm{D}^\beta (v^\alpha f)\|_{L^2} \big(\|\nabla u\|_\infty \|f\|_{\sigma,M} + \|u\|_\sigma \|f\|_{\infty,M} + \|u\|_\infty \|\mathrm{D}^\beta (v^{\alpha - e_i} f)\|_{L^2}\big).
\end{aligned}$$

Finally, for F_4, we have

$$\begin{aligned}
F_4 &= \sum_{i=1}^{d} \sum_{\beta_1 + \beta_2 = \beta} \binom{\beta}{\beta_1} \int \mathrm{D}^\beta (v^\alpha f) \mathrm{D}^{\beta_1} (v_i) \partial_{v_i} \mathrm{D}^{\beta_2} (v^\alpha f) \mathrm{d}x\mathrm{d}v \\
&= \sum_{i=1}^{d} \int \mathrm{D}^\beta (v^\alpha f) \big[v_i \partial_{v_i} \mathrm{D}^\beta (v^\alpha f) + \partial_{v_i} \mathrm{D}^{\beta - e_i} (v^\alpha f)\big] \mathrm{d}x\mathrm{d}v \\
&= \sum_{i=1}^{d} \int \frac{v_i}{2} \partial_{v_i} |\mathrm{D}^\beta (v^\alpha f)|^2 + \sum_{i=1}^{d} \int \mathrm{D}^\beta (v^\alpha f) \partial_{v_i} \mathrm{D}^{\beta - e_i} (v^\alpha f)\big] \mathrm{d}x\mathrm{d}v,
\end{aligned}$$

which implies that

$$|F_4| \leqslant \frac{1}{2} \sum_{i=1}^{d} \int \left[2|\mathrm{D}^\beta (v^\alpha f)|^2 + |\partial_{v_i} \mathrm{D}^{\beta - e_i} (v^\alpha f)|^2\right] \mathrm{d}x\mathrm{d}v \lesssim \|\mathrm{D}^\beta (v^\alpha f)\|_{L^2}^2.$$

Thus, by combining the inequalities on E_i and F_i, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathrm{D}^\beta (v^\alpha f)\|_{L^2}^2 \lesssim &(\|u\|_{W^{1,\infty}} + 1) \left[\|\mathrm{D}^\beta (v^\alpha f)\|_{L^2}^2 + \|\mathrm{D}^\beta (v^{\alpha - e_i} f)\|_{L^2}^2\right] \\
&+ \|f\|_{\infty,M} \|u\|_{\sigma+1} \|\mathrm{D}^\beta (v^\alpha f)\|_{L^2},
\end{aligned}$$

and by summing over α and β, we obtain for $\sigma > 0$ and $M > 0$

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\sigma,M}^2 &\lesssim (\|u\|_{W^{1,\infty}} + 1) \|f\|_{\sigma,M}^2 + \|f\|_{\infty,M} \|u\|_{\sigma+1} \|f\|_{\sigma,M} \\
&\lesssim (\|u\|_{W^{1,\infty}} + 1) \|f\|_{\sigma,M}^2 + \sqrt{V^M(t)} \|u\|_{\sigma+1} \|f\|_{\sigma,M}.
\end{aligned}$$

where we used the inequality (5.1.4) in the second line. □

5.2.2 Sobolev estimates for Navier-Stokes

In this subsection, we estimate the Sobolev norm of the solution of the Navier-Stokes equations, which we summarize in the following Lemma:

Lemma 5.2.9 (Sobolev estimates for NS). *Let $\sigma > 0$ and $M > \frac{d}{2}+1$. Let (f,u) satisfying equations (5.1.1). Then, one has the following estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\sigma^2 + \|u\|_{\sigma+1}^2 \lesssim (\|\nabla_x u\|_\infty + \|\rho\|_\infty)\|u\|_\sigma^2 + (\|u\|_\infty + 1)\|u\|_\sigma \|f\|_{\sigma,M}. \tag{5.2.5}$$

Proof. Recall that u satisfies the equations

$$\partial_t u + u\cdot\nabla_x u - \Delta_x u + \nabla_x p = j_f - \rho_f u \quad \text{and} \quad \nabla_x\cdot u = 0,$$

and one has:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\sigma^2 = \frac{1}{2}\sum_{|\alpha|\leqslant\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{D}^\alpha u\|_{L^2}^2.$$

Then, applying D^α to the first equation of u and integrating it against $\mathrm{D}^\alpha u$, we get:

$$\frac{\mathrm{d}}{2\mathrm{d}t}\|\mathrm{D}^\alpha u\|_{L^2}^2 - \langle\Delta_x(\mathrm{D}^\alpha u), \mathrm{D}^\alpha u\rangle_{L^2} = -\langle \mathrm{D}^\alpha(u\cdot\nabla_x u), \mathrm{D}^\alpha u\rangle_{L^2} + \langle \mathrm{D}^\alpha j, \mathrm{D}^\alpha u\rangle_{L^2} - \langle \mathrm{D}^\alpha(\rho u), \mathrm{D}^\alpha u\rangle_{L^2}.$$

Since $\nabla_x\cdot u = 0$ then, $\langle u\cdot\nabla_x(\mathrm{D}^\alpha u), \mathrm{D}^\alpha u\rangle_{L^2} = 0$. Therefore,

$$\begin{aligned}\left|\langle \mathrm{D}^\alpha(u\cdot\nabla_x u), \mathrm{D}^\alpha u\rangle_{L^2}\right| &= \left|\langle \mathrm{D}^\alpha(u\cdot\nabla_x u) - u\cdot\nabla_x(\mathrm{D}^\alpha u), \mathrm{D}^\alpha u\rangle_{L^2}\right| \\ &\leqslant \|\mathrm{D}^\alpha u\|_{L^2}\left\|\mathrm{D}^\alpha(u\cdot\nabla_x u) - u\cdot\nabla_x(\mathrm{D}^\alpha u)\right\|_{L^2}.\end{aligned}$$

Thus, by Lemma 5.2.5 we obtain:

$$\left|\langle \mathrm{D}^\alpha(u\cdot\nabla_x u), \mathrm{D}^\alpha u\rangle_{L^2}\right| \lesssim \|\nabla_x u\|_\infty \|\mathrm{D}^\alpha u\|_{L^2}^2$$

and

$$\|\mathrm{D}^\alpha(\rho u)\|_{L^2} \lesssim \|\rho\|_\infty\|\mathrm{D}^\alpha u\|_{L^2} + \|\mathrm{D}^\alpha\rho\|_{L^2}\|u\|_\infty.$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\sigma^2 + \|u\|_{\sigma+1}^2 \lesssim \big(\|\nabla_x u\|_\infty + \|\rho\|_\infty\big)\|u\|_\sigma^2 + \big(\|u\|_\infty\|\rho\|_\sigma + \|j\|_\sigma\big)\|u\|_\sigma.$$

Finally, thanks to inequalities (5.2.3) of Lemma 5.2.3 (for $\lambda = 0$), we get for $M > \frac{d}{2}+1$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\sigma^2 + \|u\|_{\sigma+1}^2 \lesssim \big(\|\nabla_x u\|_\infty + \|\rho\|_\infty\big)\|u\|_\sigma^2 + \big(\|u\|_\infty + 1\big)\|f\|_{\sigma,M}\|u\|_\sigma.$$

Hence the inequality (5.2.5) holds true. ∎

5.2.3 Sobolev estimates for VNS and proof of Proposition 5.2.1

In this subsection, we will combine the estimates obtained on the solutions of the Vlasov and Navier-Stokes equations, as we said above, in order to be able to apply Gronwal's Lemma and control the Sobolev norm of (f, u) at time t by that at initial time $t = 0$.

Lemma 5.2.10 (Sobolev estimates for VNS). *Let (f, u) satisfying equations (5.1.1) such that f has a compact support in velocity. Let $\sigma > 0$ and $M > \frac{d}{2} + 1$. Then, one has the following estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right) \lesssim \left(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(t) + 1\right)\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right). \tag{5.2.6}$$

Proof of Lemma 5.2.10. We obtain (5.2.6) by combining the two inequalities (5.2.2) and (5.2.5) and using Young's inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right) + \|u\|_{\sigma+1}^2 \leqslant C\left(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(t) + 1\right)\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right) + \frac{1}{2}\|u\|_{\sigma+1}^2.$$

□

Proof of Proposition 5.2.1. Inequality (5.2.1) follows from (5.2.6) and Gronwall's Lemma. □

5.3 Gevrey estimates for VNS

The goal of this section is to prove Theorem 5.1.3. For this purpose, we will establish quantitative estimates on the Gevrey norms of the Vlasov and Navier-stokes solutions respectively, using a Fourier space method in analogy to the results proved for the 2D-Euler system in [KV09] and [LO97] and the 3D-Vlasov-Poisson system in [VR21].

5.3.1 Preliminary inequalities

This subsection is summarized in two Lemmas that we use throughout this section. The first contains two discrete Young inequalities (we find a variant of this Lemma in [BMM16]).

Lemma 5.3.1 (Young's inequality).

Let $f, \langle k, \eta\rangle^\sigma g \in L^2(\mathbb{Z}^d \times \mathbb{R}^d), \langle k\rangle^\sigma r \in L^2(\mathbb{Z}^d)$ and let $\nu, \beta, \gamma \in \mathbb{R}$. Then, *1. For $\sigma > \frac{d}{2} + \nu$, one has:*

$$\left|\sum_{k,l} \int_{\mathbb{R}^d} f_k(\eta)\langle l\rangle^\nu r_l \langle k-l,\eta\rangle^{\sigma+\beta} g_{k-l}(\eta)\mathrm{d}\eta\right| \lesssim \|f\|_{L^2_{k,\eta}} \|\langle k\rangle^\sigma r\|_{L^2_k} \|\langle k,\eta\rangle^{\sigma+\beta} g\|_{L^2_{k,\eta}}. \tag{5.3.1}$$

2. For $\sigma > \frac{d}{2} + \gamma - \beta$, one has:

$$\left|\sum_{k,l} \int_{\mathbb{R}^d} f_k(\eta)\langle l\rangle^\sigma r_l \langle k-l,\eta\rangle^{\gamma} g_{k-l}(\eta)\mathrm{d}\eta\right| \lesssim \|f\|_{L^2_{k,\eta}} \|\langle k\rangle^\sigma r\|_{L^2_k} \|\langle k,\eta\rangle^{\sigma+\beta} g\|_{L^2_{k,\eta}}. \tag{5.3.2}$$

The constant in the two inequalities depends only on $\nu, \beta, \gamma, \sigma$ and d.

Proof of Lemma 5.3.1. 1. Let $\nu, \beta \in \mathbb{R}$ and let $\sigma > \frac{d}{2} + \nu$. We write:

$$\begin{aligned}
&\left|\sum_{k,l\in\mathbb{Z}^d} \int_{\mathbb{R}^d} f_k(\eta)\langle l\rangle^\nu r_l \langle k-l,\eta\rangle^{\sigma+\beta} g_{k-l}(\eta)\mathrm{d}\eta\right| \\
&\leqslant \sum_k \left(\int_{\mathbb{R}^d} |f_k(\eta)|^2 \mathrm{d}\eta\right)^{\frac12} \sum_l \langle l\rangle^\nu |r_l| \left(\int_{\mathbb{R}^d} \langle k-l,\eta\rangle^{2(\sigma+\beta)} |g_{k-l}(\eta)|^2 \mathrm{d}\eta\right)^{\frac12} \\
&\leqslant \left(\sum_k \int_{\mathbb{R}^d} |f_k(\eta)|^2 \mathrm{d}\eta\right)^{\frac12} \left(\sum_k \left[\sum_l \langle l\rangle^\nu |r_l| \left(\int_{\mathbb{R}^d} \langle k-l,\eta\rangle^{2(\sigma+\beta)} |g_{k-l}(\eta)|^2 \mathrm{d}\eta\right)^{\frac12}\right]^2\right)^{\frac12}.
\end{aligned}$$

Now, by Young's inequality for convolution, we get

$$\begin{aligned}
&\left|\sum_{k,l\in\mathbb{Z}^d} \int_{\mathbb{R}^d} f_k(\eta)\langle l\rangle^\nu r_l \langle k-l,\eta\rangle^{\sigma+\beta} g_{k-l}(\eta)\mathrm{d}\eta\right| \\
&\lesssim \|f\|_{L^2_{k,\eta}} \sum_k \langle k\rangle^\nu |r_k| \left(\sum_k \int_{\mathbb{R}^d} \langle k,\eta\rangle^{2(\sigma+\beta)} |g_k(\eta)|^2 \mathrm{d}\eta\right)^{\frac12} \\
&\leqslant \left(\sum_k \langle k\rangle^{2\nu-2\sigma}\right)^{\frac12} \|\langle k\rangle^\sigma r\|_{L^2_k} \|f\|_{L^2_{k,\eta}} \|\langle k,\eta\rangle^{\sigma+\beta} g\|_{L^2_{k,\eta}} \\
&\lesssim_{\nu,\sigma,d} \|\langle k\rangle^\sigma r\|_{L^2_k} \|f\|_{L^2_{k,\eta}} \|\langle k,\eta\rangle^{\sigma+\beta} g\|_{L^2_{k,\eta}},
\end{aligned}$$

since $\sum_k \langle k\rangle^{2\nu-2\sigma} \lesssim 1$ for $\sigma > \frac{d}{2} + \nu$.

2. Similarly, we have:

$$\begin{aligned}
&\left|\sum_{k,l}\int_{\mathbb{R}^d} f_k(\eta)\langle l\rangle^\sigma r_l \langle k-l,\eta\rangle^\gamma g_{k-l}(\eta)\mathrm{d}\eta\right| \\
&\leqslant \int_{\mathbb{R}^d}\sum_k |f_k(\eta)|\Big(\sum_l \langle l\rangle^\sigma |r_l|\langle k-l,\eta\rangle^\gamma |g_{k-l}(\eta)|\Big)\mathrm{d}\eta \\
&\leqslant \int_{\mathbb{R}^d}\Big(\sum_k |f_k(\eta)|^2\Big)^{\frac12}\bigg(\sum_k\Big[\sum_l \langle l\rangle^\sigma |r_l|\langle k-l,\eta\rangle^\gamma |g_{k-l}(\eta)|\Big]^2\bigg)^{\frac12}\mathrm{d}\eta \\
&\lesssim \int_{\mathbb{R}^d}\|f(\eta)\|_{L^2_k}\|\langle k\rangle^\sigma r\|_{L^2_k}\Big(\sum_k \langle k,\eta\rangle^\gamma |g_k(\eta)|\Big)\mathrm{d}\eta \\
&\leqslant \int_{\mathbb{R}^d}\|f(\eta)\|_{L^2_k}\|\langle k\rangle^\sigma r\|_{L^2_k}\Big(\sum_k \langle k,\eta\rangle^{2(\gamma-\beta-\sigma)}\Big)^{\frac12}\Big(\sum_k \langle k,\eta\rangle^{2(\beta+\sigma)}|g_k(\eta)|^2\Big)^{\frac12}\mathrm{d}\eta \\
&\leqslant \Big(\sum_k \langle k\rangle^{2(\gamma-\sigma-\beta)}\Big)^{\frac12}\|\langle k\rangle^\sigma r\|_{L^2_k}\int_{\mathbb{R}^d}\|f(\eta)\|_{L^2_k}\|\langle k,\eta\rangle^{\sigma+\beta}g(\eta)\|_{L^2_k}\mathrm{d}\eta \\
&\lesssim_{\gamma,\sigma,\beta,d} \|\langle k\rangle^\sigma r\|_{L^2_k}\|f_k\|_{L^2_{k,\eta}}\|\langle k\rangle^{\sigma+\beta}g\|_{L^2_{k,\eta}},
\end{aligned}$$

since $\sum_k \langle k,\eta\rangle^{2(\gamma-\sigma-\beta)} \leqslant \sum_k \langle k\rangle^{2(\gamma-\sigma-\beta)} \lesssim_{\gamma,\sigma,\beta,d} 1$ for $\sigma > \frac{d}{2}+\gamma-\beta$. □

The second Lemma deals with triangular inequalities on Japanese brackets.

Lemma 5.3.2 (Some triangular inequalities). *Let $k, l, \eta \in \mathbb{R}^d$. The following inequalities hold:*

1. $\langle k+l\rangle^s \leqslant \langle k\rangle^s + \langle l\rangle^s, \quad \forall s \in (0,1].$
2. $\langle k+l\rangle^s \leqslant 2^{s-1}(\langle k\rangle^s + \langle l\rangle^s), \quad \forall s \geqslant 1.$
3. $|\langle k\rangle^s - \langle l\rangle^s| \leqslant \langle k-l\rangle^s, \quad \forall s \in (0,1].$
4. $|\langle k\rangle^s - \langle l\rangle^s| \lesssim_s \frac{\langle k-l\rangle}{\langle k\rangle^{1-s}+\langle l\rangle^{1-s}}, \quad \forall s \in (0,1].$
5. $|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s| \leqslant \langle l\rangle^s, \quad \forall s \in (0,1].$
6. $|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s| \leqslant 2^{s-1}(\langle l\rangle^s + \langle k-l,\eta\rangle^s), \quad \forall s \geqslant 1.$
7. $|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s| \lesssim_s \frac{\langle l\rangle}{\langle k,\eta\rangle^{1-s}+\langle k-l,\eta\rangle^{1-s}}, \quad \forall s \in (0,1].$

Some of these inequalities are found in [VR21] and some others in [BMM16], and in the two cited references the proof is not given.

Proof of Lemma 5.3.2. Let $k, l, \eta \in \mathbb{R}^d$.

1. First, let show that $\langle k+l\rangle \leqslant \langle k\rangle + \langle l\rangle$. For this, we consider the increasing convex function $f : [0, +\infty[\longrightarrow \mathbb{R}$ defined by

$$f(t) := (1 + t^2)^{\frac{1}{2}}.$$

Observe that $f(|k|) = \langle k \rangle$. We have

$$\frac{1}{4}\langle k + l \rangle = \frac{1}{4} f(|k + l|) \leqslant \frac{1}{4} f(|k| + |l|) \leqslant \frac{1}{2} f\left(\frac{|k| + |l|}{2}\right) \leqslant \frac{f(|k|) + f(|l|)}{4} = \frac{1}{4}(\langle k \rangle + \langle l \rangle).$$

Now, let $s \in [0, 1]$ and let consider the increasing function $g : [1, +\infty[\longrightarrow \mathbb{R}$ defined by

$$g(t) := 1 + t^s - (1 + t)^s.$$

We have for all $t \geqslant 1$, $g(t) \geqslant g(1) = 2 - 2^s \geqslant 0$. Thus, $1 + t^s \geqslant (1 + t)^s$ for all $t \geqslant 1$. Let $k, l \in \mathbb{R}^d$. Without loss of generality, we can assume that $|k| \geqslant |l|$. Therefore, for $t = \frac{\langle k \rangle}{\langle l \rangle} \geqslant 1$, we obtain

$$\left(1 + \frac{\langle k \rangle}{\langle l \rangle}\right)^s \leqslant 1 + \frac{\langle k \rangle^s}{\langle l \rangle^s}.$$

Which implies that

$$\langle k + l \rangle^s \leqslant (\langle k \rangle + \langle l \rangle)^s \leqslant \langle k \rangle^s + \langle l \rangle^s,$$

since the function $t \mapsto t^s$ is increasing for $t \geqslant 1$ and $s \geqslant 0$, and we have $\langle k + l \rangle \leqslant \langle k \rangle + \langle l \rangle$.

2. The function $t \mapsto t^s$ with $t \geqslant 1$, is convex for all $s \geqslant 1$. Therefore, for $a \geqslant 1$ and $b \geqslant 1$

$$\left(\frac{a + b}{2}\right)^s \leqslant \frac{a^s + b^s}{2}.$$

Thus, for $a = \langle k \rangle$ and $b = \langle l \rangle$, we get

$$\langle k + l \rangle^s \leqslant (\langle k \rangle + \langle l \rangle)^s \leqslant 2^{s-1}\big(\langle k \rangle^s + \langle l \rangle^s\big).$$

3. Let $s \in [0, 1]$. We have

$$\begin{cases} \langle k \rangle^s = \langle k - l + l \rangle^s \leqslant \langle k - l \rangle^s + \langle l \rangle^s, \\ \langle l \rangle^s = \langle k - l + l \rangle^s \leqslant \langle k - l \rangle^s + \langle k \rangle^s. \end{cases}$$

Hence,

$$\big|\langle k \rangle^s - \langle l \rangle^s\big| \leqslant \langle k - l \rangle^s.$$

4. Let $s \in [0, 1]$. By applying the mean value theorem to the function $t \mapsto t^s$ between $X = \langle k \rangle$ and $Y = \langle l \rangle$, we obtain

$$\langle k \rangle^s - \langle l \rangle^s = s\big(\theta\langle k \rangle + (1 - \theta)\langle l \rangle\big)^{s-1}\big(\langle k \rangle - \langle l \rangle\big),$$

where $\theta := \theta_{k,l} \in]0, 1[$. Thus, by concavity of the function $t \mapsto t^{s-1}$ for $s \in [0, 1]$ and inequality of the previous point for $s = 1$, we write

$$\big|\langle k\rangle^s - \langle l\rangle^s\big| \leqslant \frac{s\langle k-l\rangle}{\big(\theta\langle k\rangle + (1-\theta)\langle l\rangle\big)^{s-1}} \leqslant \frac{s\langle k-l\rangle}{\theta\langle k\rangle^{s-1} + (1-\theta)\langle l\rangle^{s-1}} \leqslant \frac{s}{\min(\theta, 1-\theta)} \frac{\langle k-l\rangle}{\langle k\rangle^{s-1} + \langle l\rangle^{s-1}}.$$

5. Let $s \in [0,1]$. By applying the mean value theorem to the function $t \mapsto (t^{2/s} + |\eta|^2)^{\frac{s}{2}}$ between $X = \langle k\rangle^s$ and $Y = \langle k-l\rangle^s$, there exists $\theta := \theta_{k,l,\eta} \in]0,1[$ such that

$$\big|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s\big| = \left|\frac{\big(\theta\langle k\rangle^s + (1-\theta)\langle k-l\rangle^s\big)^{\frac{s}{2}-1}}{\big[|\eta|^2 + (\theta\langle k\rangle^s + (1-\theta)\langle k-l\rangle^s)^{\frac{2}{s}}\big]^{1-\frac{s}{2}}}\big(\langle k\rangle^s - \langle k-l\rangle^s\big)\right| \leqslant \big|\langle k\rangle^s - \langle k-l\rangle^s\big|,$$

since for $s \in [0,1]$,

$$\left|\frac{\big(\theta\langle k\rangle^s + (1-\theta)\langle k-l\rangle^s\big)^{\frac{s}{2}-1}}{\big[|\eta|^2 + (\theta\langle k\rangle^s + (1-\theta)\langle k-l\rangle^s)^{\frac{2}{s}}\big]^{1-\frac{s}{2}}}\right| \leqslant 1.$$

Thus, inequality 5 follows from **3**.

6. Let $s \geqslant 1$. We have by **2**,

$$\langle k,\eta\rangle^s = \big(1 + |\eta|^2 + |k|^2\big)^{\frac{s}{2}} \leqslant \big(1 + |\eta|^2 + 2|k-l|^2 + 2|l|^2\big)^{\frac{s}{2}} \leqslant 2^{\frac{3s}{2}-1}\big(\langle k-l,\eta\rangle^s + \langle l\rangle^s\big).$$

7. The proof of this point is identical to that of **4**, with $X = \langle k,\eta\rangle$ and $Y = \langle k-l,\eta\rangle$.

□

Whether for Vlasov or for Navier-Stokes, the estimates that we are going to establish are based on the two previous Lemmas with different parameters.

5.3.2 Gevrey estimates for Vlasov

The aim of this subsection is to prove the following Proposition:

Proposition 5.3.3. *Assume that the radius of regularity $t \mapsto \lambda(t)$ depends smoothly on time. Let $s \in (0,1]$, $M > 0$ and $\sigma > \frac{d}{2} + \frac{s}{2} + 2$. Let $f \in \mathcal{G}^{\lambda,\sigma+\frac{s}{2},M,\frac{1}{s}}(\mathbb{T}^d \times \mathbb{R}^d)$ and let $u \in \mathcal{G}^{\lambda,\sigma+\frac{s}{2},\frac{1}{s}}(\mathbb{T}^d)$. Then, the following estimate holds*

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} &\lesssim (\|u\|_{W^{1,\infty}} + 1)\,\|f\|^2_{\lambda,\sigma,M,s} + \|u\|_\sigma \|f\|_{\sigma,M}\|f\|_{\lambda,\sigma,M,s} \\ &+ \Big(\dot\lambda + \lambda(1 + \|u\|_\sigma) + \lambda^2\|u\|_{\lambda,\sigma,s}\Big)\,\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} \\ &+ \big(\lambda\|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\big)\,\|u\|_{\lambda,\sigma+\frac{s}{2},s}\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \end{aligned} \tag{5.3.3}$$

where $\dot\lambda$ denotes the derivative of λ with respect to t.

Subsequently, we will choose λ so that the norm $\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}$ will be absorbed in order to control the Gevrey norm of f. Indeed, for λ such that

$$\dot{\lambda} + \lambda(1 + \|f\|_{\sigma,M} + \|u\|_{\sigma}) + \lambda^2(\|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}) \leqslant 0,$$

we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} \lesssim (\|u\|_{W^{1,\infty}} + 1)\,\|f\|^2_{\lambda,\sigma,M,s} + \|u\|_{\sigma}\|f\|_{\sigma,M}\|f\|_{\lambda,\sigma,M,s}.$$

Thus, if the norms $\|u\|_{W^{1,\infty}}$, $\|f\|_{\sigma,M}$ and $\|u\|_{\sigma}$ are finite then, Gronwall's Lemma allows us to conclude.

Remark 5.3.4. Note that the estimation of the previous Proposition does not require the condition that f has compact support in velocity.

As a consequence of Proposition 5.3.3, we have the following estimate which is useful in the case $\sigma = 1$.

Corollary 5.3.5. *Let $s \in (0,1]$, $M > 0$ and $\sigma > \frac{d}{2} + 2$. Let $f \in H^{\sigma}_M(\mathbb{T}^d \times \mathbb{R}^d)$ and let $u \in H^{\sigma}(\mathbb{T}^d)$. Then, the following estimate holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\sigma,M} \lesssim (\|u\|_{W^{1,\infty}} + 1)\,\|f\|^2_{\sigma,M} + \|u\|_{\sigma}\|f\|^2_{\sigma,M}. \tag{5.3.4}$$

Proof of Corollary 5.3.5. It suffices to take $\lambda = 0$ in Proposition 5.3.3 as we only look for Sobolev norms in this case. □

Proof of Proposition 5.3.3. We will work in Fourier variables in order to simplify the calculations and expressions and to use the inequalities of the Lemma 5.3.2. The Vlasov equation, in Fourier variables, is given by:

$$\partial_t \hat{f}_k(\eta) - k \cdot \nabla_\eta \hat{f}_k(\eta) + \eta \cdot \nabla_\eta \hat{f}_k(\eta) + \mathrm{i}\sum_{l\in\mathbb{Z}^d} \hat{u}_l \cdot \eta \hat{f}_{k-l}(\eta) = 0.$$

Recall that

$$\|f\|^2_{\lambda,\sigma,M,s} := \sum_{|\alpha|\leqslant M} \|A^{\sigma} v^{\alpha} f\|^2_{L^2} = \sum_{|\alpha|\leqslant M} \|A^{\sigma}_k(\eta)\mathrm{D}^{\alpha}_{\eta}\hat{f}\|^2_{L^2},$$

where

$$A^{\sigma} := A^{\sigma}_k(\eta) := \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s}$$

and $\lambda := \lambda(t)$ is a positive function, by assumption. First, we have for a complex-valued function g:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2_{L^2_\xi} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |g|^2\,\mathrm{d}\xi = \frac{1}{2}\left[\int \bar{g}\partial_t g\,\mathrm{d}\xi + \int g\partial_t\bar{g}\,\mathrm{d}\xi\right] = \mathrm{Re}\int \bar{g}\partial_t g\,\mathrm{d}\xi.$$

Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} = \frac{1}{2}\sum_{|\alpha|\leqslant M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\partial_t\big(e^{2\lambda\langle k,\eta\rangle^s}\big)\langle k,\eta\rangle^{2\sigma}|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)|^2\mathrm{d}\eta$$
$$+\mathrm{Re}\Big[\sum_{|\alpha|\leqslant M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{2\sigma}\overline{\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)}\;\mathrm{D}^\alpha_\eta\big(\partial_t\hat{f}(\eta)\big)\;\mathrm{d}\eta\Big].$$

Which implies, replacing $\partial_t\hat{f}_k(\eta)$ by $[k\cdot\nabla_\eta\hat{f}_k(\eta)-\eta\cdot\nabla_\eta\hat{f}_k(\eta)-\mathrm{i}\sum_l\hat{u}_k\cdot\eta\hat{f}_{k-l}(\eta)]$ in the last term, that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} = \dot{\lambda}\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}+\hat{E}-\hat{F}-\mathrm{i}\hat{G}, \tag{5.3.5}$$

where

$$\hat{E} = \mathrm{Re}\sum_{|\alpha|\leqslant M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{2\sigma}e^{2\lambda\langle k,\eta\rangle^s}\overline{\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)}\;\mathrm{D}^\alpha_\eta\big(k\cdot\nabla_\eta\hat{f}_k(\eta)\big)\;\mathrm{d}\eta,$$
$$\hat{F} = \mathrm{Re}\sum_{|\alpha|\leqslant M}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{2\sigma}e^{2\lambda\langle k,\eta\rangle^s}\overline{\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)}\;\mathrm{D}^\alpha_\eta\big(\eta\cdot\nabla_\eta\hat{f}_k(\eta)\big)\;\mathrm{d}\eta,$$
$$\hat{G} = \mathrm{Re}\sum_{|\alpha|\leqslant M}\sum_{k,l\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{2\sigma}e^{2\lambda\langle k,\eta\rangle^s}\overline{\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)}\;\hat{u}_l\cdot\mathrm{D}^\alpha_\eta\big(\eta\hat{f}_{k-l}(\eta)\big)\;\mathrm{d}\eta.$$

Estimations of $\hat{E}$. By expanding $k\cdot\nabla_\eta$ and integrating by parts with respect to η, we obtain:

$$\hat{E} = \mathrm{Re}\Big[\sum_{i=1}^{d}\sum_{\alpha,k}\int_{\mathbb{R}^d}A^\sigma_k(\eta)\overline{\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)}\;A^\sigma_k(\eta)k_i\partial_{\eta_i}\big(\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big)\;\mathrm{d}\eta\Big]$$
$$= -\frac{1}{2}\sum_{i=1}^{d}\sum_{\alpha,k}\int_{\mathbb{R}^d}k_i\partial_{\eta_i}\big(A^\sigma_k(\eta)^2\big)|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)|^2\mathrm{d}\eta.$$

Now, since

$$\partial_{\eta_i}\left(A^\sigma_k(\eta)\right) = \partial_{\eta_i}\left(\langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}\right) = \big[\sigma\eta_i\langle k,\eta\rangle^{\sigma-2}+\lambda s\eta_i\langle k,\eta\rangle^{\sigma+s-2}\big]e^{\lambda\langle k,\eta\rangle^s}.$$

Then,

$$\hat{E} = -\sum_{i,\alpha,k}\int_{\mathbb{R}^d}|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)|^2\big[\sigma k_i\eta_i\langle k,\eta\rangle^{2\sigma-2}+\lambda s k_i\eta_i\langle k,\eta\rangle^{2\sigma+s-2}\big]e^{2\lambda\langle k,\eta\rangle^s}\mathrm{d}\eta.$$

That leads to

$$\big|\hat{E}\big| \lesssim \|f\|^2_{\lambda,\sigma,M,s}+\lambda\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} \tag{5.3.6}$$

Estimations of $\hat{F}$. We will proceed as in $\hat{E}$.

$$\begin{aligned}\hat{F} &= \mathrm{Re}\sum_{i,\alpha,k}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \mathrm{D}_\eta^\alpha\big(\eta_i\partial_{\eta_i}\hat{f}_k(\eta)\big)\ \mathrm{d}\eta\\ &= \mathrm{Re}\sum_{i,\alpha,k}\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha_1}{\alpha_2}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \mathrm{D}_\eta^{\alpha_1}(\eta_i)\ \mathrm{D}_\eta^{\alpha_2}\big(\partial_{\eta_i}\hat{f}_k(\eta)\big)\ \mathrm{d}\eta\\ &= \sum_{i,\alpha,k}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\left[\frac{\eta_i}{2}\partial_{\eta_i}|\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)|^2+\mathrm{Re}\Big(\overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \partial_{\eta_i}\big(\mathrm{D}_\eta^{\alpha-e_i}\hat{f}_k(\eta)\big)\Big)\right]\mathrm{d}\eta\\ &=: \hat{F}_1+\hat{F}_2.\end{aligned}$$

For $\hat{F}_1$, by integrating by parts with respect to η we get:

$$\begin{aligned}\big|\hat{F}_1\big| &:= \left|\sum_{i,\alpha,k}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\frac{\eta_i}{2}\partial_{\eta_i}|\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)|^2\mathrm{d}\eta\right|\\ &= \left|\sum_{i,\alpha,k}\int_{\mathbb{R}^d}\left[-\frac{1}{2}[A_k^\sigma(\eta)]^2-\eta_i\partial_{\eta_i}\big(A_k^\sigma(\eta)\big)A_k^\sigma(\eta)\right]|\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)|^2\mathrm{d}\eta\right|\\ &\lesssim \|f\|^2_{\lambda,\sigma,M,s}+\lambda\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s},\end{aligned}\tag{5.3.7}$$

by the same token as for $\hat{E}$.

For $\hat{F}_2$, we write:

$$\begin{aligned}\big|\hat{F}_2\big| &:= \left|\mathrm{Re}\sum_{i,\alpha,k}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \partial_{\eta_i}\big(\mathrm{D}_\eta^{\alpha-e_i}\hat{f}_k(\eta)\big)\mathrm{d}\eta\right|\\ &\leqslant \frac{1}{2}\sum_{i,\alpha,k}\int_{\mathbb{R}^d}[A_k^\sigma(\eta)]^2\left[|\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)|^2+\big|\partial_{\eta_i}\big(\mathrm{D}_\eta^{\alpha-e_i}\hat{f}_k(\eta)\big|^2\right]\mathrm{d}\eta\\ &\lesssim \|f\|^2_{\lambda,\sigma,M,s}.\end{aligned}\tag{5.3.8}$$

Thus, from (5.3.7) and (5.3.8) we get:

$$|\hat{F}|\lesssim \|f\|^2_{\lambda,\sigma,M,s}+\lambda\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}.\tag{5.3.9}$$

Remark 5.3.6. Note that the only way to absorb the term $\|f\|_{\lambda,\sigma+\frac{s}{2},M,s}$, of overflowing Sobolev regularity, is to choose a suitable function λ. We already have $(\dot{\lambda}+\lambda)\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}$ which comes from the first three terms of $\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}$ and others will come from the term $\hat{G}$. This is where the exponential decay of the radius of regularity λ comes from.

Estimations of $\hat{G}$. As in $\hat{F}$, expanding $\hat{u}_l \cdot \eta$ and using Leibniz, we write:

$$\begin{aligned}\hat{G} &= \mathrm{Re} \sum_{j=1}^{d} \sum_{|\alpha|\leqslant M} \sum_{k,l\in\mathbb{Z}^d} \int_{\mathbb{R}^d} [A_k^\sigma(\eta)]^2 \ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)} \ \mathrm{D}_\eta^\alpha \big(\hat{u}_l^j \eta_j \hat{f}_{k-l}(\eta)\big) \ \mathrm{d}\eta \\ &= \mathrm{Re} \sum_{j,\alpha,k,l} \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha_1}{\alpha_2} \int_{\mathbb{R}^d} [A_k^\sigma(\eta)]^2 \ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)} \ \hat{u}_l^j \ \mathrm{D}_\eta^{\alpha_1}(\eta_j) \ \mathrm{D}_\eta^{\alpha_2}\big(\hat{f}_{k-l}(\eta)\big) \ \mathrm{d}\eta \\ &=: \hat{G}_1 + \hat{G}_2,\end{aligned}$$

with

$$\hat{G}_1 := \mathrm{Re} \sum_{\alpha,k,l} \int_{\mathbb{R}^d} [A_k^\sigma(\eta)]^2 \ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)} \ \hat{u}_l \cdot \eta \ \mathrm{D}_\eta^\alpha\big(\hat{f}_{k-l}(\eta)\big) \ \mathrm{d}\eta$$

and

$$\hat{G}_2 := \mathrm{Re} \sum_{j,\alpha,k,l} \int_{\mathbb{R}^d} [A_k^\sigma(\eta)]^2 \ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)} \ \hat{u}_l^j \ \mathrm{D}_\eta^{\alpha-e_j}\big(\hat{f}_{k-l}(\eta)\big) \ \mathrm{d}\eta.$$

Now, since we have in the physical variables

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} u(t,x)\cdot\nabla_v\big(A^\sigma v^\alpha f\big)^2 \mathrm{d}x\mathrm{d}v = 0,$$

by Plancherel:

$$\sum_{k,l\in\mathbb{Z}^d} \int_{\mathbb{R}^d} A_k^\sigma(\eta) \ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)} \ \hat{u}_l\cdot\eta \ A_{k-l}^\sigma(\eta) \ \mathrm{D}_\eta^\alpha \hat{f}_{k-l}(\eta)\mathrm{d}\eta = 0.$$

Thus, $\hat{G}_1$ can be written as follow:

$$\hat{G}_1 = \mathrm{Re} \sum_{k,l\in\mathbb{Z}^d} \int_{\mathbb{R}^d} A_k^\sigma(\eta)\overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\big[A_k^\sigma(\eta) - A_{k-l}^\sigma(\eta)\big]\hat{u}_l\cdot\eta\mathrm{D}_\eta^\alpha \hat{f}_{k-l}(\eta)\mathrm{d}\eta. \tag{5.3.10}$$

To estimate $\big[A_k^\sigma(\eta) - A_{k-l}^\sigma(\eta)\big]$, we will use inequalities of Lemma 5.3.2. First, we have:

$$A_k^\sigma(\eta) - A_{k-l}^\sigma(\eta) = \big[\langle k,\eta\rangle^\sigma - \langle k-l,\eta\rangle^\sigma\big]e^{\lambda\langle k-l,\eta\rangle^s} + \big[e^{\lambda\langle k,\eta\rangle^s} - e^{\lambda\langle k-l,\eta\rangle^s}\big]\langle k,\eta\rangle^\sigma =: I + J.$$

Let's start by estimating $\langle k,\eta\rangle^\sigma - \langle k-l,\eta\rangle^\sigma$. By applying the mean value theorem to the function $t\mapsto t^{\frac{\sigma}{2}}$ between $X = \langle k,\eta\rangle^2$ and $Y = \langle k-l,\eta\rangle^2$ then, there exists $\theta := \theta_{k,l,\eta} \in]0,1[$ such that:

$$\begin{aligned}\langle k,\eta\rangle^\sigma - \langle k-l,\eta\rangle^\sigma &= \frac{\sigma}{2}\big[\theta\langle k,\eta\rangle^2 + (1-\theta)\langle k-l,\eta\rangle^2\big]^{\frac{\sigma}{2}-1}\big(\langle k,\eta\rangle^2 - \langle k-l,\eta\rangle^2\big) \\ &= \frac{\sigma}{2}\left[\big(\theta\langle k,\eta\rangle^2 + (1-\theta)\langle k-l,\eta\rangle^2\big)^{\frac{\sigma}{2}-1} - \langle k-l,\eta\rangle^{\sigma-2}\right]\big(\langle k,\eta\rangle^2 - \langle k-l,\eta\rangle^2\big) \\ &\quad + \frac{\sigma}{2}\langle k-l,\eta\rangle^{\sigma-2}\big(\langle k,\eta\rangle^2 - \langle k-l,\eta\rangle^2\big).\end{aligned}$$

That we write as

$$\langle k,\eta\rangle^\sigma - \langle k-l,\eta\rangle^\sigma = B + \sigma\langle k-l,\eta\rangle^{\sigma-2}\sum_{j=1}^d l_j(k_j-l_j),$$

where

$$B := \frac{\sigma}{2}\Big[\big(\theta\langle k,\eta\rangle^2+(1-\theta)\langle k-l,\eta\rangle^2\big)^{\frac{\sigma}{2}-1}-\langle k-l,\eta\rangle^{\sigma-2}\Big]\big(\langle k,\eta\rangle^2-\langle k-l,\eta\rangle^2\big)+\frac{\sigma}{2}\langle k-l,\eta\rangle^{\sigma-2}|l|^2.$$

Applying the mean value theorem once again, this time to the function $t\mapsto t^{\frac{\sigma}{2}-1}$ between $X'=\theta\langle k,\eta\rangle^2+(1-\theta)\langle k-l,\eta\rangle^2$ and $Y'=\langle k-l,\eta\rangle^2$ then, there exists $\theta':=\theta'_{k,l,\eta,\theta}\in]0,1[$ such that:

$$\begin{aligned}&\big(\theta\langle k,\eta\rangle^2+(1-\theta)\langle k-l,\eta\rangle^2\big)^{\frac{\sigma}{2}-1}-\langle k-l,\eta\rangle^{\sigma-2}\\&=\big(\frac{\sigma}{2}-1\big)\theta\big(\langle k,\eta\rangle^2-\langle k-l,\eta\rangle^2\big)\Big[\theta\theta'\langle k,\eta\rangle^2+(1-\theta\theta')\langle k-l,\eta\rangle^2\Big]^{\frac{\sigma}{2}-2}.\end{aligned}$$

By inequality 3 of Lemma 5.3.2 we write:

$$\big|\langle k,\eta\rangle^2-\langle k-l,\eta\rangle^2\big|\leqslant\langle l\rangle\big(\langle k,\eta\rangle+\langle k-l,\eta\rangle\big).$$

Then, by inequality 6 of Lemma 5.3.2 we obtain:

$$\Big|\theta\theta'\langle k,\eta\rangle^2+(1-\theta\theta')\langle k-l,\eta\rangle^2\Big|^{\frac{\sigma}{2}-2}\lesssim\langle k,\eta\rangle^{\sigma-4}+\langle k-l,\eta\rangle^{\sigma-4}.$$

Therefore,

$$\begin{aligned}|B|&\lesssim\langle l\rangle^2\big(\langle k,\eta\rangle^2+\langle k-l,\eta\rangle^2\big)\big(\langle k,\eta\rangle^{\sigma-4}+\langle k-l,\eta\rangle^{\sigma-4}\big)+\langle l\rangle^2\langle k-l,\eta\rangle^{\sigma-2}\\&\lesssim\langle l\rangle^2\big(\langle k,\eta\rangle^{\sigma-2}+\langle k,\eta\rangle^{\sigma-4}\langle k-l,\eta\rangle^2+\langle k,\eta\rangle^{\sigma-4}\langle k-l,\eta\rangle^2+\langle k,\eta\rangle^{\sigma-2}\big).\end{aligned}$$

Finally, by using inequality 2 of Lemma 5.3.2, $\langle k,\eta\rangle^2\lesssim\langle l\rangle^2+\langle k-l,\eta\rangle^2$, we get:

$$|B|\lesssim\langle l\rangle^2\big(\langle l\rangle^{\sigma-2}+\langle k-l,\eta\rangle^{\sigma-2}\big). \tag{5.3.11}$$

In summary,

$$I=\Big[B+\sigma\langle k-l,\eta\rangle^{\sigma-2}\sum_{j=1}^d l_j(k_j-l_j)\Big]e^{\lambda\langle k-l,\eta\rangle^s}, \tag{5.3.12}$$

with B satisfying inequality (5.3.11).
Now, for J, we write:

$$J:=\big[e^{\lambda\langle k,\eta\rangle^s}-e^{\lambda\langle k-l,\eta\rangle^s}\big]\langle k,\eta\rangle^\sigma=e^{\lambda\langle k-l,\eta\rangle^s}\big[e^{\lambda[\langle k,\eta\rangle^s-\langle k-l,\eta\rangle^s]}-1\big]\langle k,\eta\rangle^\sigma.$$

Since $|e^x - 1| \leqslant |x|e^{|x|}$ for all $x \in \mathbb{R}$ then, by using inequality 5 of Lemma 5.3.2, we get

$$|J| \leqslant \lambda\big|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s\big|e^{\lambda\langle l\rangle^s}e^{\lambda\langle k-l,\eta\rangle^s}\langle k,\eta\rangle^\sigma.$$

Now, by using inequality 7 of Lemma 5.3.2 we obtain

$$\big|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s\big| \lesssim \langle l\rangle\langle k,\eta\rangle^{s-1},$$

and by inequality 6 of the same Lemma 5.3.2,

$$\langle k,\eta\rangle^{\sigma+\frac{s}{2}-1} \lesssim \langle l\rangle^{\sigma+\frac{s}{2}-1} + \langle k-l,\eta\rangle^{\sigma+\frac{s}{2}-1}.$$

Then, thanks to the last two inequalities, we write

$$\begin{aligned}\big|\langle k,\eta\rangle^s - \langle k-l,\eta\rangle^s\big|\langle k,\eta\rangle^\sigma &\lesssim \langle l\rangle\langle k,\eta\rangle^{s-1}\langle k,\eta\rangle^\sigma\\ &= \langle k,\eta\rangle^{\frac{s}{2}}\langle k,\eta\rangle^{\frac{s}{2}+\sigma-1}\langle l\rangle\\ &\lesssim \langle k,\eta\rangle^{\frac{s}{2}}\big[\langle l\rangle^{\sigma+\frac{s}{2}} + \langle l\rangle\langle k-l,\eta\rangle^{\frac{s}{2}+\sigma-1}\big].\end{aligned}$$

Hence,

$$|J| \lesssim \lambda\langle k,\eta\rangle^{\frac{s}{2}}\big[\langle l\rangle^{\sigma+\frac{s}{2}} + \langle l\rangle\langle k-l,\eta\rangle^{\frac{s}{2}+\sigma-1}\big]e^{\lambda\langle l\rangle^s}e^{\lambda\langle k-l,\eta\rangle^s}. \tag{5.3.13}$$

Estimation of $\hat{G}_1$. Since $A_k^\sigma(\eta) - A_{k-l}^\sigma(\eta) = I + J$ and thanks to equality (5.3.12), we can decompose $\hat{G}_1$ as follows:

$$\hat{G}_1 = \hat{G}_{I,1} + \hat{G}_{I,2} + \hat{G}_J,$$

where

$$\hat{G}_{I,1} := \mathrm{Re}\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}\overline{\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)}\ \hat{u}_l\cdot\eta\ B\ e^{\lambda\langle k-l,\eta\rangle^s}\mathrm{D}_\eta^\alpha\hat{f}_{k-l}(\eta)\ \mathrm{d}\eta,$$

$$\hat{G}_{I,2} := \sigma\mathrm{Re}\sum_{\alpha,k,l}\sum_{j=1}^d\int_{\mathbb{R}^d}\langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}\overline{\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)}\,l_j\,\hat{u}_l\cdot\eta(k_j-l_j)\langle k-l,\eta\rangle^{\sigma-2}e^{\lambda\langle k-l,\eta\rangle^s}\mathrm{D}_\eta^\alpha\hat{f}_{k-l}(\eta)\ \mathrm{d}\eta$$

and

$$\hat{G}_J := \mathrm{Re}\sum_{\alpha,k,l}\int_{\mathbb{R}^d}J\ A_k^\sigma(\eta)\ \overline{\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)}\ \hat{u}_l\cdot\eta\ \mathrm{D}_\eta^\alpha\hat{f}_{k-l}(\eta)\ \mathrm{d}\eta.$$

Estimation of $\hat{G}_{I,1}$. We have by inequality (5.3.11)

$$|\hat{G}_{I,1}| \lesssim \sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}_\eta^\alpha\hat{f}_k(\eta)\big|\langle l\rangle^2|\hat{u}_l|\big[\langle l\rangle^{\sigma-2}+\langle k-l,\eta\rangle^{\sigma-2}\big]|\eta|e^{\lambda\langle k-l,\eta\rangle^s}\big|\mathrm{D}_\eta^\alpha\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta.$$

Then, by using the inequality $e^x \leqslant e + x^2e^x$ for all $x \geqslant 0$, we get

$$|\hat{G}_{I,1}| \lesssim |\hat{G}_{I,11}| + |\hat{G}_{I,12}| + |\hat{G}_{I,13}| + |\hat{G}_{I,14}|,$$

where

$$\left|\hat{G}_{I,11}\right| \lesssim \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{\sigma} |\hat{u}_l||\eta| \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta,$$

$$\left|\hat{G}_{I,12}\right| \lesssim \lambda^2 \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{\sigma} |\hat{u}_l||\eta| \langle k-l,\eta\rangle^{2s} e^{\lambda\langle k-l,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta,$$

$$\left|\hat{G}_{I,13}\right| \lesssim \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{2} |\hat{u}_l||\eta| \langle k-l,\eta\rangle^{\sigma-2} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta,$$

$$\left|\hat{G}_{I,14}\right| \lesssim \lambda^2 \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{2} |\hat{u}_l||\eta| \langle k-l,\eta\rangle^{\sigma+2s-2} e^{\lambda\langle k-l,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta.$$

Note that $|\eta| \leqslant \langle k-l,\eta\rangle$. Now, by applying inequality (5.3.2) of Lemma 5.3.1 for the first two inequalities, with $\gamma = 1$ and $\beta = 0$ for $\hat{G}_{I,11}$, and with $\gamma = 2s+1$ and $\beta = \frac{s}{2}$ for $\hat{G}_{I,12}$, for $\sigma > \frac{d}{2}+1$, we obtain

$$\left|\hat{G}_{I,11}\right| \lesssim \|u\|_{\sigma} \|f\|_{\sigma,M} \|f\|_{\lambda,\sigma,M,s}, \tag{5.3.14}$$

and for $\sigma > \frac{d}{2}+\frac{3s}{2}+1$, we obtain

$$\left|\hat{G}_{I,12}\right| \lesssim \lambda^2 \|u\|_{\sigma} \|f\|_{\lambda,\sigma,M,s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \tag{5.3.15}$$

For $\hat{G}_{I,13}$, by applying inequality (5.3.1) for $\nu = 2$ and $\beta = -1$, for $\sigma > \frac{d}{2}+2$ we get

$$\left|\hat{G}_{I,13}\right| \lesssim \|u\|_{\sigma} \|f\|_{\sigma-1,M} \|f\|_{\lambda,\sigma,M,s}. \tag{5.3.16}$$

For $\hat{G}_{I,14}$, since $\langle k-l,\eta\rangle^{\frac{s}{2}} \leqslant \langle k,\eta\rangle^{\frac{s}{2}} + \langle l\rangle^{\frac{s}{2}}$ (by inequality 5 of Lemma 5.3.2) and $s \in (0,1]$ then,

$$\begin{aligned}\left|\hat{G}_{I,14}\right| &\lesssim \lambda^2 \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma+\frac{s}{2}} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{2} |\hat{u}_l| \langle k-l,\eta\rangle^{\sigma+\frac{s}{2}} e^{\lambda\langle k-l,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta \\ &\quad + \lambda^2 \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_k(\eta)\right| \langle l\rangle^{2+\frac{s}{2}} |\hat{u}_l||\eta| \langle k-l,\eta\rangle^{\sigma+\frac{s}{2}} e^{\lambda\langle k-l,\eta\rangle^s} \left|\mathrm{D}^{\alpha}_{\eta} \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta.\end{aligned}$$

Thus, by applying inequality (5.3.2) to the first line in the previous inequality, for $\nu = 2$ and $\beta = \frac{s}{2}$, and by applying inequality (5.3.1) to the second line, for $\nu = 2+\frac{s}{2}$ and $\beta = \frac{s}{2}$, we get for $\sigma > \frac{d}{2}+\frac{s}{2}+2$

$$\left|\hat{G}_{I,14}\right| \lesssim \lambda^2 \left(\|u\|_{\sigma} \|f\|_{\lambda,\sigma,M,s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s} + \|u\|_{2+\frac{s}{2}} \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} \right).$$

Hence, for $\sigma > \frac{d}{2}+\frac{s}{2}+2$

$$\left|\hat{G}_{I,14}\right| \lesssim \lambda^2 \|u\|_{\sigma} \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}. \tag{5.3.17}$$

Therefore, by summing inequalities (5.3.14), (5.3.15), (5.3.16) and (5.3.17) we obtain for

$\sigma > \frac{d}{2} + \frac{s}{2} + 2$:

$$\big|\hat{G}_{I,1}\big| \lesssim \|u\|_\sigma \left(\|f\|_{\sigma,M}\|f\|_{\lambda,\sigma,M,s} + \lambda^2\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}\right). \tag{5.3.18}$$

Estimation of $\hat{G}_{I,2}$. Recall that $\hat{G}_{I,2}$ is defined by

$$\hat{G}_{I,2} := \sigma \mathrm{Re} \sum_{\alpha,k,l}\sum_{j=1}^{d} \int_{\mathbb{R}^d} \langle k,\eta\rangle^\sigma e^{\lambda\langle k,\eta\rangle^s} \overline{\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)}\, l_j\, \hat{u}_l\cdot\eta(k_j-l_j)\langle k-l,\eta\rangle^{\sigma-2} e^{\lambda\langle k-l,\eta\rangle^s}\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\,\mathrm{d}\eta.$$

Thus, $\hat{G}_{I,2}$ can be seen, using the inverse Fourier transform, as

$$\hat{G}_{I,2} = \sigma\sum_{\alpha,k}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\overline{\mathcal{F}(A^\sigma v^\alpha f)}\mathcal{F}\left(\frac{\partial u}{\partial x_j}\cdot\nabla_v\frac{\partial}{\partial x_j}\big(A^{\sigma-2}v^\alpha f\big)\right)\mathrm{d}\eta.$$

Hence,

$$\big|\hat{G}_{I,2}\big| \lesssim \|\nabla_x u\|_\infty \|f\|^2_{\lambda,\sigma,M,s}. \tag{5.3.19}$$

Note that each time we want to have L^∞ estimates (for u), we go back through Fourier using Parseval.

Estimation of $\hat{G}_J$. By using inequality (5.3.13), we write

$$\begin{aligned}
\big|\hat{G}_J\big| &\leqslant \sum_{\alpha,k,l}\int_{\mathbb{R}^d}|J|A^\sigma_k(\eta)\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big||\hat{u}_l||\eta|\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta \\
&\lesssim \lambda\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big|\langle l\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle l\rangle^s}|\hat{u}_l||\eta|e^{\lambda\langle k-l,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta \\
&+ \lambda\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big|\langle l\rangle e^{\lambda\langle l\rangle^s}|\hat{u}_l|\langle k-l,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k-l,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta.
\end{aligned}$$

We proceed as in $\hat{G}_{I,1}$, using the inequality $e^x \leqslant 1 + xe^x$ for all $x \geqslant 0$, we get:

$$\big|\hat{G}_J\big| \lesssim \big|\hat{G}_{J1}\big| + \big|\hat{G}_{J2}\big| + \big|\hat{G}_{J3}\big| + \big|\hat{G}_{J4}\big|,$$

where

$$\begin{aligned}
\big|\hat{G}_{J1}\big| &\lesssim \lambda\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big|\langle l\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle l\rangle^s}|\hat{u}_l||\eta|\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta, \\
\big|\hat{G}_{J2}\big| &\lesssim \lambda^2\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big|\langle l\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle l\rangle^s}|\hat{u}_l|\langle k-l,\eta\rangle^{1+s}e^{\lambda\langle k-l,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta, \\
\big|\hat{G}_{J3}\big| &\lesssim \lambda\sum_{\alpha,k,l}\int_{\mathbb{R}^d}\langle k,\eta\rangle^{\sigma+\frac{s}{2}}e^{\lambda\langle k,\eta\rangle^s}\big|\mathrm{D}^\alpha_\eta\hat{f}_k(\eta)\big|\langle l\rangle|\hat{u}_l|\langle k-l,\eta\rangle^{\sigma+\frac{s}{2}}\big|\mathrm{D}^\alpha_\eta\hat{f}_{k-l}(\eta)\big|\mathrm{d}\eta
\end{aligned}$$

and

$$\left|\hat{G}_{J4}\right| \lesssim \lambda^2 \sum_{\alpha,k,l} \int_{\mathbb{R}^d} \langle k,\eta\rangle^{\sigma+\frac{s}{2}} e^{\lambda\langle k,\eta\rangle^s} \left|\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)\right| \langle l\rangle^{1+s} e^{\lambda\langle l\rangle^s} |\hat{u}_l| \langle k-l,\eta\rangle^{\sigma+\frac{s}{2}} e^{\lambda\langle k-l,\eta\rangle^s} \left|\mathrm{D}^\alpha_\eta \hat{f}_{k-l}(\eta)\right| \mathrm{d}\eta.$$

By applying inequality (5.3.2) to the first two inequalities, we obtain for $\sigma > \frac{d}{2}+1$,

$$\left|\hat{G}_{J1}\right| \lesssim \lambda \|f\|_{\sigma,M} \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}, \tag{5.3.20}$$

and for $\sigma > \frac{d}{2}+s+1$,

$$\left|\hat{G}_{J2}\right| \lesssim \lambda^2 \|f\|_{\lambda,\sigma,M,s} \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \tag{5.3.21}$$

For $\hat{G}_{J3}$ and $\hat{G}_{I,14}$, we have thanks to inequality (5.3.1), for $\sigma > \frac{d}{2}+1$

$$\left|\hat{G}_{J3}\right| \lesssim \lambda \|u\|_\sigma \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}, \tag{5.3.22}$$

and for $\sigma > \frac{d}{2}+s+1$,

$$\left|\hat{G}_{J4}\right| \lesssim \lambda^2 \|u\|_{\lambda,\sigma,s} \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s}. \tag{5.3.23}$$

Thus, by summing the last four inequalities, we get:

$$\begin{aligned}\left|\hat{G}_{J}\right| &\lesssim \left(\lambda \|u\|_\sigma + \lambda^2 \|u\|_{\lambda,\sigma,s}\right) \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} \\ &\quad + \left(\lambda \|f\|_{\sigma,M} + \lambda^2 \|f\|_{\lambda,\sigma,M,s}\right) \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \end{aligned} \tag{5.3.24}$$

Finally, by summing (5.3.18), (5.3.19) and (5.3.24), we obtain the following estimate for $\hat{G}_1$:

$$\begin{aligned}\left|\hat{G}_{1}\right| &\lesssim \|\nabla_x u\|_\infty \|f\|^2_{\lambda,\sigma,M,s} + \|u\|_\sigma \|f\|_{\sigma,M} \|f\|_{\lambda,\sigma,M,s} \\ &\quad + \left((\lambda+\lambda^2)\|u\|_\sigma + \lambda^2 \|u\|_{\lambda,\sigma,s}\right) \|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} \\ &\quad + \left(\lambda \|f\|_{\sigma,M} + \lambda^2 \|f\|_{\lambda,\sigma,M,s}\right) \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \end{aligned} \tag{5.3.25}$$

Thus, it only remains to estimate $\hat{G}_2$ to conclude the Gevrey estimates for the solution of the Vlasov equation.

Estimation of $\hat{G}_2$. Recall that $\hat{G}_2$ is defined by

$$\hat{G}_2 := \mathrm{Re} \sum_{\alpha,k,l} \sum_{j=1}^{d} \int_{\mathbb{R}^d} [A^\sigma_k(\eta)]^2 \, \overline{\mathrm{D}^\alpha_\eta \hat{f}_k(\eta)} \; \hat{u}^j_l \; \mathrm{D}^{\alpha-e_j}_\eta \left(\hat{f}_{k-l}(\eta)\right) \mathrm{d}\eta,$$

which we decompose as follows:

$$\begin{aligned}\hat{G}_2 &= \mathrm{Re} \sum_{j,\alpha,k,l} \int_{\mathbb{R}^d} A_k^\sigma(\eta)\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \hat{u}_l^j\ A_{k-l}^\sigma(\eta) \mathrm{D}_\eta^{\alpha-e_j}\big(\hat{f}_{k-l}(\eta)\big)\ \mathrm{d}\eta \\ &\quad + \mathrm{Re} \sum_{j,\alpha,k,l} \int_{\mathbb{R}^d} A_k^\sigma(\eta)\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \hat{u}_l^j \big[A_k^\sigma(\eta) - A_{k-l}^\sigma(\eta)\big] \mathrm{D}_\eta^{\alpha-e_j}\big(\hat{f}_{k-l}(\eta)\big)\ \mathrm{d}\eta \\ &\quad =: \hat{G}_{21} + \hat{G}_{22}.\end{aligned}$$

For $\hat{G}_{21}$, by the inverse Fourier transform theorem, we write:

$$\begin{aligned}\big|\hat{G}_{21}\big| &:= \left|\mathrm{Re} \sum_{j,\alpha,k,l} \int_{\mathbb{R}^d} A_k^\sigma(\eta)\ \overline{\mathrm{D}_\eta^\alpha \hat{f}_k(\eta)}\ \hat{u}_l^j\ A_{k-l}^\sigma(\eta) \mathrm{D}_\eta^{\alpha-e_j}\big(\hat{f}_{k-l}(\eta)\big)\ \mathrm{d}\eta\right| \\ &= \left|\mathrm{Re} \sum_{j,\alpha,k} \int_{\mathbb{R}^d} \overline{\mathcal{F}\left(A^\sigma v^\alpha f\right)}\ \mathcal{F}\left(u^j A^\sigma v^{\alpha-e_j} f\right)\ \mathrm{d}\eta\right| \\ &\lesssim \|u\|_\infty \|f\|_{\lambda,\sigma,M,s}^2. \end{aligned} \tag{5.3.26}$$

For the term $\hat{G}_{22}$, proceeding exactly in the same way as for $\hat{G}_1$, we obtain for $\sigma > \frac{d}{2} + 2$:

$$\begin{aligned}\big|\hat{G}_{22}\big| &\lesssim \|\nabla_x u\|_\infty \|f\|_{\lambda,\sigma,M,s}^2 + \|u\|_\sigma \|f\|_{\sigma,M} \|f\|_{\lambda,\sigma,M,s} \\ &\quad + \left((\lambda+\lambda^2)\|u\|_\sigma + \lambda^2 \|u\|_{\lambda,\sigma,s}\right) \|f\|_{\lambda,\sigma,M,s}^2 \\ &\quad + \left(\lambda \|f\|_{\sigma,M} + \lambda^2 \|f\|_{\lambda,\sigma,M,s}\right) \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \end{aligned} \tag{5.3.27}$$

Finally, returning to (5.3.5) and summing all the inequalities obtained on $\hat{E}$, $\hat{F}$ and $\hat{G}$, namely: (5.3.6), (5.3.9), (5.3.25), (5.3.26) and (5.3.27), we obtain the following estimate for $\sigma > \frac{d}{2} + \frac{s}{2} + 2$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\lambda,\sigma,M,s}^2 &\lesssim \left(\|u\|_{W^{1,\infty}} + 1\right) \|f\|_{\lambda,\sigma,M,s}^2 + \|u\|_\sigma \|f\|_{\sigma,M} \|f\|_{\lambda,\sigma,M,s} \\ &\quad + \left(\dot{\lambda} + \lambda(1 + \|u\|_\sigma) + \lambda^2 \|u\|_{\lambda,\sigma,s}\right) \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}^2 \\ &\quad + \left(\lambda \|f\|_{\sigma,M} + \lambda^2 \|f\|_{\lambda,\sigma,M,s}\right) \|u\|_{\lambda,\sigma+\frac{s}{2},s} \|f\|_{\lambda,\sigma+\frac{s}{2},M,s}. \end{aligned} \tag{5.3.28}$$

This completes the proof of Proposition 5.3.3. □

5.3.3 Gevrey estimates for Navier-Stokes

The purpose of this subsection is to prove the inequality of the following Proposition:

Proposition 5.3.7. *Assume that the radius of regularity $t \mapsto \lambda(t)$ depends smoothly on time. Let $s \in (0,1]$, $M > \frac{d}{2} + 1$ and $\sigma > \frac{d}{2} + \frac{s}{2} + 2$. For $f \in \mathcal{G}^{\lambda,\sigma+\frac{s}{2},M,\frac{1}{s}}(\mathbb{T}^d \times \mathbb{R}^d)$ and*

$u \in \mathcal{G}^{\lambda,\sigma+\frac{s}{2},\frac{1}{s}}(\mathbb{T}^d)$ *satisfying* (5.1.1), *the following estimate holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s}^2 + \|u\|_{\lambda,\sigma+1,s}^2 \lesssim \left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|_{\lambda,\sigma,s}^2 + \|u\|_\sigma^2\|u\|_{\lambda,\sigma,s} + \|f\|_{\lambda,\sigma,M,s}\|u\|_{\lambda,\sigma,s} + \left(\dot{\lambda} + \lambda\|u\|_\sigma + \lambda^2(\|u\|_\sigma + \|u\|_{\lambda,\sigma,s})\right)\|u\|_{\lambda,\sigma+\frac{s}{2},s}^2. \tag{5.3.29}$$

As in Proposition 5.3.3 for Vlasov, we will choose a suitable λ to absorb the Gevrey norm of Sobolev regularity which overflows $\|u\|_{\lambda,\sigma+\frac{s}{2},s}$. Since we have two constraints now, we have to take a λ which satisfies both at the same time, but here the same λ is good and it gives :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s}^2 + \|u\|_{\lambda,\sigma+1,s}^2 \lesssim \left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|_{\lambda,\sigma,s}^2 + \|u\|_\sigma^2\|u\|_{\lambda,\sigma,s}.$$

Therefore, for $\|f\|_{\lambda,\sigma,M,s}$ finite, Gronwall's Lemma gives us a majoration of the norm $\|u\|_{\lambda,\sigma,s}$.

Proof of Proposition 5.3.7. Recall that $\Lambda := (\mathrm{Id} - \Delta_x)^{\frac{1}{2}}$ and that

$$\|u\|_{\lambda,\sigma,s}^2 := \int_{\mathbb{T}^d} \left|\Lambda^\sigma e^{\lambda\Lambda^s} u\right|^2 \mathrm{d}x = \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma} e^{2\lambda\langle k\rangle^s}|\hat{u}_k|^2.$$

Then, applying $\Lambda^\sigma e^{\lambda\Lambda^s}$ to the equation satisfied by u and taking the scalar product with $\Lambda^\sigma e^{\lambda\Lambda^s}u$, we obtain:

$$\begin{aligned}&\left\langle \Lambda^\sigma e^{\lambda\Lambda^s}\partial_t u, \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2} + \|\Lambda^{\sigma+1}e^{\lambda\Lambda^s}u\|_{L^2}^2 + \left\langle \Lambda^\sigma e^{\lambda\Lambda^s}(u\cdot\nabla_x u), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}\\ &= \left\langle \Lambda^\sigma e^{\lambda\Lambda^s} u, \Lambda^\sigma e^{\lambda\Lambda^s} j_f\right\rangle_{L^2} - \left\langle \Lambda^\sigma e^{\lambda\Lambda^s}(\rho_f u), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}.\end{aligned}$$

We have $\|\Lambda^{\sigma+1}e^{\lambda\Lambda^s}u\|_{L^2}^2 = \|u\|_{\lambda,\sigma+1,s}^2$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s}^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^\sigma e^{\lambda\Lambda^s}u\|_{L^2}^2 = \dot{\lambda}\|u\|_{\lambda,\sigma+\frac{s}{2},s}^2 + \left\langle \Lambda^\sigma e^{\lambda\Lambda^s}\partial_t u, \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}.$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s}^2 + \|u\|_{\lambda,\sigma+1,s}^2 \leqslant \dot{\lambda}\|u\|_{\lambda,\sigma+\frac{s}{2},s}^2 + \left|\left\langle \Lambda^\sigma e^{\lambda\Lambda^s}(u\cdot\nabla_x u), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}\right| + \|j\|_{\lambda,\sigma,s}\|u\|_{\lambda,\sigma,s} + \|\rho u\|_{\lambda,\sigma,s}\|u\|_{\lambda,\sigma,s}. \tag{5.3.30}$$

We claim that for $\sigma > \frac{d}{2} + \frac{s}{2} + 2$,

$$\left|\left\langle \Lambda^\sigma e^{\lambda\Lambda^s}(u\cdot\nabla_x u), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}\right| \lesssim \|\nabla_x u\|_\infty\|u\|_{\lambda,\sigma,s}^2 + \|u\|_\sigma^2\|u\|_{\lambda,\sigma,s} + \left((\lambda+\lambda^2)\|u\|_\sigma + \lambda^2\|u\|_{\lambda,\sigma,s}\right)\|u\|_{\lambda,\sigma+\frac{s}{2},s}^2. \tag{5.3.31}$$

Let denote by H the term

$$H := \left\langle \Lambda^\sigma e^{\lambda\Lambda^s}(u\cdot\nabla_x u), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}.$$

Since $\nabla_x \cdot u = 0$ then,

$$H = \left\langle u\cdot\nabla_x u, \Lambda^{2\sigma} e^{2\lambda\Lambda^s} u\right\rangle_{L^2} - \left\langle u\cdot\nabla_x\big(\Lambda^\sigma e^{\lambda\Lambda^s} u\big), \Lambda^\sigma e^{\lambda\Lambda^s} u\right\rangle_{L^2}.$$

Thus, in Fourier variable, we write

$$\begin{aligned} H &= \mathrm{i}\sum_{k\in\mathbb{Z}^d}\left[\mathcal{F}[u\cdot\nabla_x u]_k \langle k\rangle^{2\sigma} e^{2\lambda\langle k\rangle^s}\overline{\hat{u}_k} - \mathcal{F}\big[u\cdot\nabla_x\big(\Lambda^\sigma e^{\lambda\Lambda^s} u\big)\big]_k \langle k\rangle^\sigma e^{\lambda\langle k\rangle^s}\overline{\hat{u}_k}\right] \\ &= \mathrm{i}\sum_{k,l\in\mathbb{Z}^d}\left[\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s}\overline{\hat{u}_k}\left(\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s} - \langle k-l\rangle^\sigma e^{\lambda\langle k-l\rangle^s}\right)\hat{u}_l\ (k-l)\cdot\hat{u}_{k-l}\right]. \end{aligned}$$

Observe that $\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s} = A_k^\sigma(0)$. Therefore,

$$H = \mathrm{i}\sum_{k,l} A_k^\sigma(0)\overline{\hat{u}_k}\left(A_k^\sigma(0) - A_{k-l}^\sigma(0)\right)\hat{u}_l\ (k-l)\cdot\hat{u}_{k-l}.$$

Thus, proceeding exactly the same way as for $\hat{G}_1$ for $\eta = 0$ (see formula (5.3.10)), we obtain for $\sigma > \frac{d}{2} + \frac{s}{2} + 2$:

$$|H| \lesssim \|\nabla_x u\|_\infty \|u\|^2_{\lambda,\sigma,s} + \|u\|^2_\sigma \|u\|_{\lambda,\sigma,s} + \left((\lambda+\lambda^2)\|u\|_\sigma + \lambda^2\|u\|_{\lambda,\sigma,s}\right)\|u\|^2_{\lambda,\sigma+\frac{s}{2},s}.$$

In order to continue these estimates, we need the following lemma:

Lemma 5.3.8. *The following estimates hold for $\sigma > \frac{d}{2} + 2s$ and $M > \frac{d}{2}$,*

$$\|\rho u\|_{\lambda,\sigma,s} \lesssim \|u\|_\sigma \|f\|_{\lambda,\sigma,M,s} + \|f\|_{\sigma,M}\|u\|_{\lambda,\sigma,s} + \lambda^2\|u\|_{\lambda,\sigma,s}\|f\|_{\lambda,\sigma,M,s}. \tag{5.3.32}$$

Proof of Lemma 5.3.8. We have

$$\|\rho u\|^2_{\lambda,\sigma,s} = \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2\sigma} e^{2\lambda\langle k\rangle^s}\left|\widehat{\rho u}_k\right|^2 = \sum_{k\in\mathbb{Z}^d}\left|\sum_{l\in\mathbb{Z}^d}\langle k\rangle^\sigma e^{\lambda\langle k\rangle^s}\hat{\rho}_l\hat{u}_{k-l}\right|^2.$$

By using inequalities 1 and 2 of Lemma 5.3.2 and the inequality $e^x \leqslant e + x^2 e^x$ for $x \geqslant 0$, we write:

$$\begin{aligned} \|\rho u\|^2_{\lambda,\sigma,s} &\lesssim \sum_k\left|\sum_l[\langle l\rangle^\sigma + \langle k-l\rangle^\sigma]e^{\lambda\langle l\rangle^s}e^{\lambda\langle k-l\rangle^s}|\hat{\rho}_l||\hat{u}_{k-l}|\right|^2 \\ &\lesssim \sum_k\left|\sum_l\langle l\rangle^\sigma e^{\lambda\langle l\rangle^s}|\hat{\rho}_l|\left(e + \lambda^2\langle k-l\rangle^{2s}e^{\lambda\langle k-l\rangle^s}\right)|\hat{u}_{k-l}|\right. \\ &\qquad\left. + \left(e + \lambda^2\langle l\rangle^{2s}e^{\lambda\langle l\rangle^s}\right)|\hat{\rho}_l|\langle k-l\rangle^\sigma e^{\lambda\langle k-l\rangle^s}|\hat{u}_{k-l}|\right|^2. \end{aligned}$$

Then, by applying inequalities (5.3.1) and (5.3.2), we obtain for $\sigma > \frac{d}{2} + 2s$

$$\|\rho u\|^2_{\lambda,\sigma,s} \lesssim \|\rho\|^2_{\lambda,\sigma,s}\|u\|^2_{\sigma} + \|\rho\|^2_{\sigma}\|u\|^2_{\lambda,\sigma,s} + \lambda^2\|\rho\|^2_{\lambda,\sigma,s}\|u\|^2_{\lambda,\sigma,s}.$$

Finally, we get inequality (5.3.32) by using inequality (5.2.3), $\|\rho\|_{\lambda,\sigma,s} \lesssim \|f\|_{\lambda,\sigma,M,s}$ for $M > \frac{d}{2}$. □

Returning now to the estimate of $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s}$. By using Lemma 5.3.8 and inequality (5.2.3) then, for $\sigma > \frac{d}{2} + \frac{s}{2} + 2$ and $M > \frac{d}{2} + 1$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_{\lambda,\sigma+1,s} \lesssim{}& \left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|^2_{\lambda,\sigma,s}\\ &+ \|u\|^2_{\sigma}\|u\|_{\lambda,\sigma,s} + \|f\|_{\lambda,\sigma,M,s}\|u\|_{\lambda,\sigma,s}\\ &+ \left(\dot{\lambda} + (\lambda+\lambda^2)\|u\|_\sigma + \lambda^2\|u\|_{\lambda,\sigma,s}\right)\|u\|^2_{\lambda,\sigma+\frac{s}{2},s}.\end{aligned}$$

□

Remark 5.3.9. We used the inequalities $e^x \leqslant 1 + xe^x$ and $e^x \leqslant e + x^2e^x$ for $x \geqslant 0$ in order to reduce the power of the Gevrey norms and apply Gronwall instead of having a power greater than 2, which gives results for a time that depends on the initial data. More precisely, without using the last two inequalities we will have terms like $\|u\|_{\lambda,\sigma,s}\|f\|^2_{\lambda,\sigma,M,s}$ and an inequality of the type

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\lambda,\sigma,s} + \|f\|^2_{\lambda,\sigma,M,s}\right) \lesssim \phi(t, \|u\|_\sigma, \|f\|_{\sigma,M})\left(\|u\|^2_{\lambda,\sigma,s} + \|f\|^2_{\lambda,\sigma,M,s}\right)^{3/2}.$$

5.3.4 Proof of the main results

Proof of Theorem 5.1.3. By Proposition 5.3.3 and Proposition 5.3.7, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} \lesssim{}& \|f\|_{\lambda,\sigma,M,s}\Big(\left(\|u\|_{W^{1,\infty}} + 1\right)\|f\|_{\lambda,\sigma,M,s} + \|u\|_\sigma\|f\|_{\sigma,M}\Big)\\ &+ \Big(\dot{\lambda} + \lambda(1 + \|u\|_\sigma + \|f\|_{\sigma,M}) + \lambda^2(\|u\|_{\lambda,\sigma,s} + \|f\|_{\lambda,\sigma,M,s})\Big)\left(\|f\|^2_{\lambda,\sigma+\frac{s}{2},M,s} + \|u\|^2_{\lambda,\sigma+\frac{s}{2},s}\right).\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_{\lambda,\sigma+1,s} \lesssim{}& \left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M} + \lambda^2\|f\|_{\lambda,\sigma,M,s}\right)\|u\|^2_{\lambda,\sigma,s} + \|u\|^2_\sigma\|u\|_{\lambda,\sigma,s}\\ &+ \|f\|_{\lambda,\sigma,M,s}\|u\|_{\lambda,\sigma,s} + \Big(\dot{\lambda} + \lambda\|u\|_\sigma + \lambda^2(\|u\|_\sigma + \|u\|_{\lambda,\sigma,s})\Big)\|u\|^2_{\lambda,\sigma+\frac{s}{2},s}.\end{aligned}$$

In particular, by choosing λ such that

$$\dot{\lambda} + \lambda(1 + \|f\|_{\sigma,M} + \|u\|_\sigma) + \lambda^2(\|u\|_\sigma + \|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}) \leqslant 0, \tag{5.3.33}$$

we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\lambda,\sigma,M,s} \lesssim \|f\|_{\lambda,\sigma,M,s}\Big(\left(\|u\|_{W^{1,\infty}}+1\right)\|f\|_{\lambda,\sigma,M,s} + \|u\|_\sigma \|f\|_{\sigma,M}\Big) \tag{5.3.34}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\lambda,\sigma,s} \lesssim \|u\|_{\lambda,\sigma,s}\Big(\left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M}\right)\|u\|_{\lambda,\sigma,s} + \|u\|^2_\sigma + \|f\|_{\lambda,\sigma,M,s}\Big). \tag{5.3.35}$$

Note that we will look for λ such that

$$\dot{\lambda} + \lambda(1+\|f\|_{\sigma,M}+\|u\|_\sigma) + \lambda^2(\|u\|_\sigma + \|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}) = 0. \tag{5.3.36}$$

The last two inequalities on $\|f\|_{\lambda,\sigma,M,s}$ and $\|u\|_{\lambda,\sigma,s}$ lead to the following two

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\lambda,\sigma,M,s} \leqslant C\left(\|u\|_{W^{1,\infty}}+1\right)\|f\|_{\lambda,\sigma,M,s} + C\|u\|_\sigma\|f\|_{\sigma,M} \tag{5.3.37}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s} \leqslant C\left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M}\right)\|u\|_{\lambda,\sigma,s} + C\left(\|u\|^2_\sigma + \|f\|_{\lambda,\sigma,M,s}\right), \tag{5.3.38}$$

Remark that in (5.3.37) there is no Gevrey norm of u. So, we deal first with inequality (5.3.37), we integrate it in order to have a bound on the Gevrey norm of f. Once we have a bound on f, we inject it into inequality (5.3.38) in order to obtain a bound on the Gevrey norm of u. By Proposition 5.2.1, we have: $\|f\|^2_{\sigma,M} + \|u\|^2_\sigma \leqslant \left(\|f_0\|^2_{\sigma,M} + \|u_0\|^2_\sigma\right) g(t)$, with $g(t) := \exp\big[C_0\int_0^t \left(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(\tau)+1\right)\mathrm{d}\tau\big]$.

Therefore, (5.3.37) leads us to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\lambda,\sigma,M,s} \leqslant C\frac{g'(t)}{g(t)}\|f\|_{\lambda,\sigma,M,s} + Cg(t).$$

Then, by integrating the last inequality, we obtain:

$$\|f\|_{\lambda,\sigma,M,s} \leqslant F(t) := C_1\,(t+1)\,g(t), \tag{5.3.39}$$

where C_1 is a positive constant which depends on the initial data (f_0, u_0), the radius of regularity λ_0, the Sobolev correction σ, the weight M and the dimension d.

Returning to the inequality (5.3.38), by the previous estimate, $\|f\|_{\lambda,\sigma,M,s} \leqslant F(t)$, the Sobolev embedding, $\|\nabla_x u\|_\infty \lesssim \|u\|_\sigma$ for $\sigma > \frac{d}{2}+1$, and the inequality $g(t) \lesssim F(t)$, we write:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s} &\leqslant C\left(\|\nabla_x u\|_\infty + \|f\|_{\sigma,M}\right)\|u\|_{\lambda,\sigma,s} + C\left(\|u\|_\sigma^2 + \|f\|_{\lambda,\sigma,M,s}\right)\\ &\leqslant C\big(\|u\|_\sigma + \|f\|_{\sigma,M}\big)\|u\|_{\lambda,\sigma,s} + C\big(g(t)+F(t)\big)\\ &\leqslant C_2 g(t)\|u\|_{\lambda,\sigma,s} + C_2F(t).\end{aligned}$$

Hence, by integrating the last inequality, we get:

$$\|u\|_{\lambda,\sigma,s} \leqslant U(t) := \left(\|u_0\|_{\lambda_0,\sigma,s} + C_2\int_0^t F(\zeta)e^{-C_2\int_0^\zeta g(\tau)\mathrm{d}\tau}\mathrm{d}\zeta\right)e^{C_2\int_0^t g(\tau)\mathrm{d}\tau}. \tag{5.3.40}$$

Remark 5.3.10. Note that we can estimate $\|f\|^2_{\lambda,\sigma,M,s}+\|u\|^2_{\lambda,\sigma,s}$ directly, by summing the two inequalities (5.3.34) and (5.3.35), using $\|u\|_{W^{1,\infty}} \lesssim \|u\|_\sigma$ for $\sigma > \frac{d}{2}+1$, $\|u\|_\sigma \leqslant \|u\|_{\lambda,\sigma,s}$ and $\|f\|_{\sigma,M} \leqslant \|f\|_{\lambda,\sigma,M,s}$, in order to get:

$$\|f\|^2_{\lambda,\sigma,M,s} + \|u\|^2_{\lambda,\sigma,s} \leqslant \left(\|f_0\|^2_{\lambda_0,\sigma,M,s} + \|u_0\|^2_{\lambda_0,\sigma,s}\right)\exp\Big[\tilde{C}_1\int_0^t\big(1+\|f\|_{\sigma,M}+\|u\|_\sigma\big)\mathrm{d}\tau\Big]. \tag{5.3.41}$$

Finally, we need to find a positive continuous function λ such that inequality (5.3.33) holds.

Lemma 5.3.11. *The function λ defined by*

$$\lambda(t) := \frac{1}{G(t)}\left(\lambda_0^{-1} + \int_0^t \frac{\|u\|_\sigma + \|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}}{G(\tau)}\mathrm{d}\tau\right)^{-1} \tag{5.3.42}$$

satisfies the condition (5.3.33) and where the function G in the previous formula is given by

$$G(t) := \exp\left[\int_0^t\left(1+\|u\|_\sigma + \|f\|_{\sigma,M}\right)\mathrm{d}\tau\right]. \tag{5.3.43}$$

Moreover, λ is positive and non-increasing and satisfying

$$\lambda(t) \geqslant \left(2C_3t+\lambda_0^{-1}\right)^{-1}\exp\left[-C_3\int_0^t\left(1+\|u\|_\sigma+\|f\|_{\sigma,M}\right)\mathrm{d}\tau\right] > 0, \tag{5.3.44}$$

where C_3 is a constant depending on the initial data (f_0,u_0) , the radius of regularity λ_0, the Sobolev correction σ, the weight M and the dimension d.

Proof of Lemma 5.3.11. The function λ given by the formula (5.3.42) is solution to the differential equation (5.3.36). Also, thanks to the inequality (5.3.41)

$$\|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s} \leqslant \tilde{C}_2\exp\Big[\tilde{C}_2\int_0^t\big(1+\|f\|_{\sigma,M}+\|u\|_\sigma\big)\mathrm{d}\tau\Big],$$

and since the function G, given by (5.3.43), is nondecreasing then, from (5.3.42) we get:

$$\begin{aligned}\lambda(t) &= \left(\lambda_0^{-1}G(t) + \int_0^t \left(\|u\|_\sigma + \|f\|_{\lambda,\sigma,M,s} + \|u\|_{\lambda,\sigma,s}\right)\frac{G(t)}{G(\tau)}\mathrm{d}\tau\right)^{-1} \\ &\geqslant \left(\lambda_0^{-1}G(t) + 2\tilde{C}_2\int_0^t G^{\tilde{C}_2}(\tau)\frac{G(t)}{G(\tau)}\mathrm{d}\tau\right)^{-1} \\ &\geqslant \left(\lambda_0^{-1} + 2Ct\right)^{-1} G^{-C}(t) \\ &\geqslant \left(2Ct + \lambda_0^{-1}\right)^{-1} e^{-C\int_0^t(1+\|f\|_{\sigma,M}+\|u\|_\sigma)\mathrm{d}\tau}\end{aligned}$$

where $C := \max\{1; \tilde{C}_2\}$. □

For a Gevrey initial data (f_0, u_0), we have in particular,

$$\|u_0\|_\sigma^2 + \|f_0\|_{\sigma,M}^2 \leqslant \|f_0\|_{\lambda_0,\sigma,M,s}^2 + \|u_0\|_{\lambda_0,\sigma,s}^2 < +\infty.$$

Then, by Proposition 5.2.1,

$$\|f(t)\|_{\sigma,M}^2 + \|u(t)\|_\sigma^2 \leqslant \left(\|u_0\|_\sigma^2 + \|f_0\|_{\sigma,M}^2\right)g(t) < +\infty.$$

Which implies that

$$\|f(t)\|_{\lambda,\sigma,M,s}^2 + \|u(t)\|_{\lambda,\sigma,s}^2 < +\infty.$$

□

Proof of Theorem 5.1.5. By using the Sobolev embeddings $\|u\|_{W^{1,\infty}} \lesssim \|u\|_{\sigma+1}$ and $\|\rho_f\|_\infty \lesssim \|\rho_f\|_\sigma \lesssim \|f\|_{\sigma,M}$ for $\sigma > \frac{d}{2}$, and since $\|u\|_\sigma \leqslant \|u\|_{\sigma+1}$ then, the two inequalities (5.3.4) and (5.2.5) become

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\sigma,M}^2 \lesssim \left(\|u\|_{\sigma+1} + 1\right)\|f\|_{\sigma,M}^2$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_\sigma^2 + \|u\|_{\sigma+1}^2 \lesssim \left(\|u\|_{\sigma+1} + 1\right)\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right)$$

respectively. Thus, by summing these last two inequalities and using Young, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right) + \|u\|_{\sigma+1}^2 \leqslant \frac{1}{2}\|u\|_{\sigma+1}^2 + C'\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right)\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2 + 1\right).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right) + \|u\|_{\sigma+1}^2 \leqslant C\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2\right)\left(\|u\|_\sigma^2 + \|f\|_{\sigma,M}^2 + 1\right).$$

Now, by integrating this last inequation with the notations $Y(t) := \|u\|_\sigma^2 + \|f\|_{\sigma,M}^2$ and

$Y_0 := Y(0)$, we obtain

$$Y(t) := \|u(t)\|_\sigma^2 + \|f(t)\|_{\sigma,M}^2 \leqslant \left(1 - \frac{Y_0}{1+Y_0} e^{Ct}\right)^{-1}$$

for

$$t \leqslant T_0 := T_0(u_0, f_0) = \frac{1}{C} \ln\left(\frac{1+Y_0}{Y_0}\right).$$

Now if we know that

$$\|u(T_0)\|_\sigma^2 + \|f(T_0)\|_{\sigma,M}^2 < \infty,$$

then we can repeat the argument above. For the propagation of Analyticity, we proceed as in the Gevrey case. Indeed, by the inequality (5.3.37), we write for $\sigma > \frac{d}{2} + 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\lambda,\sigma,M,s} \lesssim \big(\|u\|_{W^{1,\infty}} + 1\big)\|f\|_{\lambda,\sigma,M,s} + \|u\|_\sigma \|f\|_{\sigma,M} \lesssim \big(\|u\|_\sigma + 1\big)\|f\|_{\lambda,\sigma,M,s}.$$

Hence,

$$\|f(t)\|_{\lambda,\sigma,M,s} \leqslant \|f_0\|_{\lambda_0,\sigma,M,s} \exp\big[C\int_0^t (1 + \|u(\tau)\|_\sigma)\mathrm{d}\tau\big].$$

For u, from the inequality (5.3.38), we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\lambda,\sigma,s} &\lesssim (\|\nabla_x u\|_\infty + \|f\|_{\sigma,M})\,\|u\|_{\lambda,\sigma,s} + \big(\|u\|_\sigma^2 + \|f\|_{\lambda,\sigma,M,s}\big)\\ &\lesssim (\|u\|_\sigma + \|f\|_{\sigma,M})\,\|u\|_{\lambda,\sigma,s} + \|f\|_{\lambda,\sigma,M,s}.\end{aligned}$$

Hence,

$$\|u(t)\|_{\lambda,\sigma,s} \leqslant \left(\|u_0\|_{\lambda_0,\sigma,s} + C\int_0^t \|f(\zeta)\|_{\lambda,\sigma,M,s} e^{-C\int_0^\zeta Y(\tau)\mathrm{d}\tau}\mathrm{d}\zeta\right) e^{C\int_0^t Y(\tau)\mathrm{d}\tau}.$$

□

Proof of Corollary 5.1.8. By Theorem 2.2 of [HKMM20], for initial modulated energy $\mathcal{E}(0)$ small enough, in the sense of Theorem 2.1 in [HKMM20], we get

$$\int_0^\infty \|\nabla_x(\tau)\|_\infty \mathrm{d}\tau + \|\rho_f\|_{L^\infty((0,\infty)\times\mathbb{T}^3)} < \infty.$$

Then, for initial data $(f_0, u_0) \in \mathcal{G}^{\lambda_0,\sigma,M,\frac{1}{s}}(\mathbb{T}^3\times\mathbb{R}^3)\times\mathcal{G}^{\lambda_0,\sigma,\frac{1}{s}}(\mathbb{T}^3)$ such that f_0 has a compact support in velocity, the quantity

$$\exp\left[C_1\int_0^t \big(\|u\|_{W^{1,\infty}} + \|\rho\|_\infty + V^M(\tau) + 1\big)\,\mathrm{d}\tau\right]$$

is finite for every $t \geqslant 0$. This implies that the Sobolev norms, and consequently the Gevrey norms, are finite for all $t \geqslant 0$. Hence the end of the proof. □

Appendices

APPENDIX A

Complement to Part II: some classical Theorems

A.1 The Implicit Function Theorem in Banach Spaces

We give in this section a general version of the Implicit Function Theorem, the case of functions defined and with values in Banach spaces.

Theorem A.1.1 (Implicit Function Theorem: Banach version)**.**

Let E, F and G three Banach spaces and f a function of class C^p defined on an open set U of $E \times F$ and with values in G. Let (x_0, y_0) be a point of U such that $f(x_0, y_0) = 0$ and such that the partial differential $\mathrm{D}_y f(x_0, y_0)$ is invertible. There exists a function φ of class C^p with values in F, defined on an open neighborhood V of x_0, and an open neighborhood Ω of (x_0, y_0) in U such that, for all $(x, y) \in E \times F$:

$$((x, y) \in \Omega \text{ and } f(x, y) = 0) \Leftrightarrow (x \in V \text{ and } y = \varphi(x)) .$$

Remark A.1.2.

1. The existence of such an open Ω means that the graph of φ is an open of the set of zeros of f, which automatically guarantees the uniqueness of φ, in the following sense:

   On any connected part C of V containing x_0, φ is the only continuous application satisfying

$$\varphi(x_0) = y_0 \quad \text{and} \quad \forall x \in C \quad (x, \varphi(x)) \in U \text{ and } f(x, \varphi(x)) = 0.$$

2. Uniqueness on any connected open (even convex) containing x_0 is trivially false if f is defined on $\mathbb{R}^2$ by $f(x, y) = x^2 - y^2$ and if $x_0 = y_0 = 1$, $\varphi_1(x) = x$ and $\varphi_2(x) = |x|$ are two distinct solutions defined on $\mathbb{R}$.

3. The open Ω can be chosen of the form $V \times W$.

A.2 Some existence theorems in Hilbert spaces

We collect the results of this section from [Kho72] (pages 234-237).

Surjectivity of the adjoint of an injective application with continuous inverse

Theorem A.2.1 ([Kho72], page 234)**.** *Let X and Y be two pre-Hilbertian spaces, let T be a linear map (continuous or not) defined on X with values in Y such that*

$$\|Tx\| \geqslant \alpha\|x\|, \quad \forall x \in X,$$

where α is a positive constant. Then, T^* is a surjection from $D(T^*)$ to $\overline{X'}$, admitting a continuous lift.

Lions existence lemma

Theorem A.2.2 ([Kho72], page 235)**.** *Let F be a Hilbert space and Φ a pre-Hilbertian space such that $\Phi \subset F$. Let S be a sesquilinear form on $F \times \Phi$ having the following properties:*

1. *for all $\phi \in \Phi$, the linear form $u \mapsto S(u, \phi)$ is continuous on F;*
2. *there exists a constant $\alpha > 0$ such that, for all $\phi \in \Phi$, we have $|S(\phi, \phi)| \geqslant \alpha\|\phi\|_F\|\phi\|_\Phi$.*

Then, for any continuous semilinear form L on Φ, there exists at least one $u \in F$ satisfying the equation $S(u, \phi) = L(\phi)$ for any $\phi \in \Phi$.

Lax-Milgram isomorphism lemma

Theorem A.2.3 (First statement)**.**

Let V be a pre-Hilbertian space; let w be a continuous semilinear form on V, a a continuous sesquilinear form on $V \times V$ and possessing the following property (called V-ellipticity): there exists a constant $\alpha > 0$ such that $|a(u, u)| \geqslant \alpha\|u\|_V^2$, for all $u \in V$.
Then, there exists an element $u \in V$ and only one such that $a(u, v) = w(v)$, for all $v \in V$.

Proof. Let u_1 and u_2 be two solutions of the equation $a(u, v) = w(v)$, $v \in V$. Put $u = u_1 - u_2$ and let $v = u$, we get $a(u, u) = 0$; the V-ellipticity of the form a shows then $\|u\|_V^2 = 0$; hence the uniqueness.
The existence of the solution u follows from Lions existence lemma with $F = \Phi = V$. □

Remark A.2.4. We can prove directly the Lax-Milgram Lemma (without using the Lions existence Lemma). Indeed, the data of the sesquilinear form a continuous on $V \times V$ is equivalent to the data of the linear map $\mathcal{A}$ from V to V defined by

$$a(u,v) = (\mathcal{A}u|v)_V, \quad u \in V, v \in V.$$

The V-ellipticity of a gives $\alpha\|u\|_V \leqslant \|\mathcal{A}u\|_V$, which shows that $\mathcal{A}$ is injective with continuous inverse, so it is a topological isomorphism from V onto $\mathrm{Im}\mathcal{A}$; in particular $\mathrm{Im}\mathcal{A}$ is complete and therefore closed in V. To show that $\mathcal{A}$ is surjective, it suffices to prove that $\mathrm{Im}\mathcal{A}$ is dense; for this, let $u_0 \in V$ such that $(\mathcal{A}v|u_0) = 0$ for all $v \in V$; taking $v = u_0$ we get $a(u_0, u_0) = 0$, which gives $u_0 = 0$.

Theorem A.2.5 (Second statement)**.**

Let A be a continuous linear map from a Hilbert space V to its dual $\overline{V'}$, having the following property: there exists a number $\alpha > 0$ such that $|(Au,u)| \geqslant \alpha\|u\|_V^2$ for all $u \in V$. Then A is a topological isomorphism from V to $\overline{V'}$.

In this statement, (w,u) denotes the value taken by $w \in \overline{V'}$ in $u \in V$.

Proof. The given of a continuous linear map A is equivalent to the continuous sesquilinear form a defined by $a(u,v) = (Au,v)$, $u \in V$ and $v \in V$. Applying the first statement of the Lax-Milgram lemma shows that A is an algebraic isomorphism from V into $\overline{V'}$. Since $\alpha\|u\|_V \leqslant \|Au\|_{\bar{V}}$ we see that the reciprocal bijection of A sends continuously $\overline{V'}$ into V; hence the topological isomorphism. ☐

Remark A.2.6. By Riesz's theorem on the isomorphism between a Hilbert space and its antidual, there exists an isometric isomorphism Λ from V onto $\overline{V'}$ such that

$$(u|v)_V = (\Lambda u, v), \quad u, v \in V.$$

Between $\Lambda, \mathcal{A}$ and A, we have the relation

$$\Lambda\mathcal{A} = A.$$

Another use of the Lax-Milgram lemma

Let H be a Hilbert space such that $V \hookrightarrow H$ (continuous embedding) and V dense in H. According to a general theorem on the injection of duals (chapter AD, IV, 1^o) $\overline{H'}$ is continuously injected into $\overline{V'}$. Identifying H to its antidual $\overline{H'}$, we can also write $V \hookrightarrow H \hookrightarrow \overline{V'}$.
If $(\cdot|\cdot)$ is the scalar product in H, then we have $(f|g) = (f,g)$, $h, g \in H$. Let

$$D(A) = \{u \in V;\ Au \in H\}.$$

Then the restriction of A (considered as a continuous linear map from V to $\overline{V'}$) to $D(A)$ defines a linear map (not continuous in general) in H, of domain $D(A)$.
Note that $D(A)$ is nothing else than the set of $u \in V$ such that the form $v \mapsto a(u,v)$ is continuous on V for the topology induced by H and we have

$$a(u,v) = (Au|v), \quad u \in D(A) \text{ and } v \in V.$$

From the Lax-Milgram lemma (second statement), we deduce the following proposition.

Proposition A.2.7 ([Kho72], pages 236-237)**.** *Let a be a continuous sesquilinear form on $V \times V$ and V-elliptic; let A be the (unbounded) operator in H associated with the form a. Then for every $f \in H$, there exists one $u \in D(A)$, and only one, such that $Au = f$.*

Remark A.2.8. Still under the assumption of continuity and V-ellipticity of the sesquilinear form a, the operator A has the following properties:

1. $D(A)$ is dense in H; A is closed (graph);
2. if we set $b(u,v) = \overline{a(v,u)}$ and if B is the operator associated to the form b, then $B = A^*$.

APPENDIX B

Complement to Part III

B.1 Gevrey space

The purpose of this section is to characterize the Gevrey class by fractional Sobolev spaces, which is more convenient than working with the definition based on uniform bounds. This section was taken from [LO97], which we state for completeness.

B.1.1 Definition and notations

A function $w \in C^\infty$ is said to be of *Gevrey class* s for some $s > 0$ if there exist constants $\rho > 0$ and $M < \infty$ such that for every $x \in \mathbb{T}^d$ (or $\mathbb{R}^d$) and every $\alpha \in \mathbb{N}^d$ one has

$$\left|\partial^\alpha w(x)\right| \leqslant M \left(\frac{\alpha!}{\rho^{|\alpha|}}\right)^s. \tag{B.1.1}$$

Here we employ the usual multi-index notation in which

$$|\alpha| := \sum_{j=1}^d \alpha_j, \qquad \alpha! := \prod_{j=1}^d \alpha_j!, \qquad \partial^\alpha := \prod_{j=1}^d \partial_j^{\alpha_j}.$$

The set of all functions of Gevrey class s is a vector space, denoted $\mathcal{G}^s(\mathbb{T}^d)$. It is closed under multiplication and differentiation. Moreover, the composition of two functions of Gevrey class s is again of class s.
It is classical that $\mathcal{G}^1(\mathbb{T}^d)$ is the space of real analytic functions $C^\omega(\mathbb{T}^d)$; a proof can be found, for example, in John [Joh82] (page 65). For $0 < s < 1$ the class $\mathcal{G}^s(\mathbb{T}^d)$ is a subclass of the analytic functions, while for $1 < s < \infty$ it contains the analytic functions. In fact, one has a hierarchy of spaces such that $0 < s_1 < s_2 < \infty$ implies the proper containments

$$\mathcal{G}^{s_1}(\mathbb{T}^d) \subset \mathcal{G}^{s_2}(\mathbb{T}^d) \subset C^\infty(\mathbb{T}^d).$$

Moreover, the union of the $\mathcal{G}^s(\mathbb{T}^d)$ does not exhaust $C^\infty(\mathbb{T}^d)$ because there are quasianalytic functions that are not members of a Gevrey class [LVP+24b].

As we said above, and as in the chapter 5, it is convenient to characterize Gevrey classes in terms of the fractional Sobolev spaces $H^r(\mathbb{T}^d)$ with $r \geqslant 0$, rather than uniform bounds such as (B.1.1). To do so, we will use $\hat{w}_k$ to denote the Fourier coefficients of a function $w \in L^2(\mathbb{T}^d)$, so that

$$w(x) = \sum_{k \in 2\pi\mathbb{Z}^d} \hat{w}_k e^{\mathrm{i}k \cdot x}, \qquad \hat{w}_k = \int_{\mathbb{T}^d} w(x) e^{-\mathrm{i}k \cdot x} \mathrm{d}x.$$

The H^r norm of a function w can be defined as

$$\|w\|_{H^r} := \left(\sum_{k \in 2\pi\mathbb{Z}^d} (1 + |k|^2)^r |\hat{w}_k|^2 \right)^{\frac{1}{2}}. \tag{B.1.2}$$

The space $H^r(\mathbb{T}^d)$ is then the set of all $L^2(\mathbb{T}^d)$ functions for which the norm (B.1.2) is finite.

B.1.2 Characterization of the Gevrey class by fractional Sobolev spaces

Let us start with the following Lemma:

Lemma B.1.1. *Given $s > 0$ and $r \geqslant 0$. Then, $w \in \mathcal{G}^s(\mathbb{T}^d)$ if and only if there are constants $\rho, M \in (0, \infty)$ that may depend on r, s and w such that for every $n \in \mathbb{N}$ one has*

$$\|\nabla^n w\|_{H^r} = \left(\sum_{k \in 2\pi\mathbb{Z}^d} (1 + |k|^2)^r |k|^{2n} |\hat{w}_k|^2 \right)^{\frac{1}{2}} \leqslant M \left(\frac{n!}{\rho^n} \right)^s. \tag{B.1.3}$$

The proof is achieved by a direct application of the Sobolev embedding theorem, see Adams [AF75].

This lemma enables us to characterize functions in $\mathcal{G}^s$ in terms of the decay of their Fourier coefficients. A result of the type was first due to La Vallée Poussin [LVP+24b, Man35] although this attribution has been obscured in the literature. The construction here uses the operator $\Lambda = \sqrt{-\Delta}$ that, like $-\Delta$ itself, is nonnegative and self-adjoint so that arbitrary powers can be defined by spectral theory. For each $s \in (0, \infty)$ we define a family, parameterized by τ, of normed spaces

$$\mathfrak{D}\left(e^{\tau \Lambda^{1/s}} : H^r(\mathbb{T}^d) \right) := \left\{ w \in H^r(\mathbb{T}^d) : \|e^{\tau \Lambda^{1/s}} w\|_{H^r} < \infty \right\}. \tag{B.1.4}$$

The functions in any such space have Fourier coefficients that decay faster than $\exp(-\tau |k|^{\frac{1}{s}})$. The next theorem recovers the Gevrey class $\mathcal{G}^s(\mathbb{T}^d)$ as the union of all such classes.

Theorem B.1.2. *For any $s > 0$ and $r \geqslant 0$,*

$$\mathcal{G}^s(\mathbb{T}^d) = \bigcup_{\tau>0} \mathfrak{D}\left(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\right). \tag{B.1.5}$$

Remark B.1.3. The use of the more general H^r rather than simply L^2 as the base space does not complicate the structure of the proof, but it is advantageous for the arguments that will follow.

Proof. Let $w \in \mathfrak{D}\left(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\right)$ for some $\tau > 0$ and let $\rho = \frac{\tau}{s}$. Then

$$\begin{aligned}
\|\nabla^n w\|_{H^r}^2 &= \left(\frac{n!}{\rho^n}\right)^{2s} \sum_k (1+|k|^2)^r \left(\frac{\rho^n |k|^{\frac{n}{s}}}{n!}\right)^{2s} |\hat{w}_k|^2 \\
&\leqslant \left(\frac{n!}{\rho^n}\right)^{2s} \sum_k (1+|k|^2)^r e^{2s\rho|k|^{\frac{1}{s}}} |\hat{w}_k|^2 \\
&\leqslant \left(\frac{n!}{\rho^n}\right)^{2s} \|e^{\tau\Lambda^{1/s}} w\|_{H^r}^2.
\end{aligned} \tag{B.1.6}$$

By setting $M = \|e^{\tau\Lambda^{1/s}} w\|_{H^r}$ we obtain (B.1.3), whereby $w \in \mathcal{G}^s(\mathbb{T}^d)$.
On the other hand, let $w \in \mathcal{G}^s(\mathbb{T}^d)$ For an arbitrary $\tau \geqslant 0$ one has

$$\|e^{\tau\Lambda^{1/s}} w\|_{H^r}^2 := \sum_k (1+|k|^2)^r e^{2\tau|k|^{\frac{1}{s}}} |\hat{w}_k|^2 = \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} \sum_k (1+|k|^2)^r |k|^{\frac{m}{s}} |\hat{w}_k|^2. \tag{B.1.7}$$

Now let ρ and M be such that (B.1.3) is satisfied. By interpolating (B.1.3) between $n = 0$ and any integer n such that $\frac{m}{s} \leqslant 2n$, the inner sum appearing in (B.1.7) can be bounded as

$$\sum_k (1+|k|^2)^r |k|^{\frac{m}{s}} |\hat{w}_k|^2 \leqslant M^2 \frac{(n!)^{\frac{m}{n}}}{\rho^m}.$$

This bound is best if we choose $n = n_m := [m/(2s)] + 1$, where $[\cdot]$ denotes the "greatest integer less than" function. Upon making this choice and applying the result in (B.1.7), one arrives at the bound

$$\|e^{\tau\Lambda^{1/s}} w\|_{H^r}^2 \leqslant \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} M^2 \frac{(n_m!)^{m/n_m}}{\rho^m} = M^2 \sum_{m=0}^{\infty} \left(\frac{2\tau}{\rho}\right)^m \frac{(n_m!)^{m/n_m}}{m!}. \tag{B.1.8}$$

By making use of the Stirling formula in the form

$$\lim_{n\to\infty} \frac{e}{n} (n!)^{\frac{1}{n}} = 1,$$

the limit of the mth root of the mth term in the last series of (B.1.8) can be evaluated as

$$\lim_{m\to\infty} \frac{2\tau}{\rho} \frac{(n_m!)^{1/n_m}}{(m!)^{1/m}} = \lim_{m\to\infty} \frac{2\tau}{\rho} \frac{n_m}{m} = \frac{2\tau}{\rho} \frac{1}{2s} = \frac{\tau}{s\rho}.$$

Hence, by the Hadamard root test, the series in (B.1.8) converges for every $\tau < \rho s$, whence $w \in \mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$. □

Remark B.1.4. The above proof gives a sharp relationship between the τ of (B.1.4) and the ρ of (B.1.3). Had we been less careful in our choice of the n_m used in (B.1.8) then this relationship would have been missed.

Remark B.1.5. The theorem also holds over $\mathbb{R}^d$. The proof can proceed in the same way with Fourier integrals in place of Fourier sums.

One reason Gevrey classes are useful in the context of nonlinear partial differential equations is that each $\mathcal{G}^s$ is closed under multiplication. It would be particularly useful if this property extended to each of the approximating (normed!) spaces $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$. An abstract statement of this fact is given in the following theorem, which we state for completeness.

Theorem B.1.6. *If $s \geqslant 1$, $\tau \geqslant 0$, and $r > \frac{d}{2}$ then $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$ is a Banach algebra. This means that it is closed under multiplication and that there exists a finite constant $C(r,d)$ such that any two functions v and w in $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$ satisfy the inequality*

$$\|e^{\tau\Lambda^{1/s}}(vw)\|_{H^r} \leqslant C(r,d)\|e^{\tau\Lambda^{1/s}}v\|_{H^r}\|e^{\tau\Lambda^{1/s}}w\|_{H^r}.$$

The proof is a direct extension of the usual proof that $H^r(\mathbb{T}^d)$ is a Banach algebra when $r > \frac{d}{2}$ [AF75], a result that is recovered above by setting $\tau = 0$. A proof of this theorem for the case when $s = 1$ is given in [FT98]; this proof is easily generalized for any $s > 1$, see [Oli96].

The spaces $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$ are well-suited for application to parabolic partial differential equations. One can identify τ with time so that $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$ evolves from being identical to H^r at $t = 0$ to being a subset of Gevrey class s in an arbitrarily short time. So if one can show that for some $T > 0$ the solutions to the equation with H^r initial data are in $\mathfrak{D}\big(e^{\tau\Lambda^{1/s}} : H^r(\mathbb{T}^d)\big)$ for every $t \in (0,T]$ then one has proved Gevrey regularity of class s over that interval.

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