Existence of SRB measures for hyperbolic maps with weak regularity

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Résumé

L'un des objectifs des systèmes dynamiques est de comprendre le comportement d'une transformation. Plus précisément, si $f : X \to X$ est une application ayant des propriétés raisonnables sur un espace X, nous voulons décrire les orbites de $f; \mathcal{O}^+(x) = \{x, fx, f^2x, \cdots\}$ ou $\mathcal{O}(x) = \{\cdots, f^{-1}x, x, fx, \cdots\}$ si f est inversible. Si nous supposons que X est un espace métrique et que $x, y \in X$ sont suffisamment proches, que pouvons-nous dire des orbites de x et y? La réponse tient en un mot : "Chaos". Cette notion a été introduite par Poincaré lors de l'étude du problème des trois corps. Il s'avère que certains systèmes sont sensibles aux conditions initiales, c'est-à-dire que même si $x, y \in X$ sont proches, le comportement de leurs orbites peut différer l'un de l'autre. Pour décrire le comportement asymptotique d'une ortbite $\mathcal{O}^+(x)$, on considère une fonction continue $\phi : X \to \mathbb{R}$, et la suite $\big(S_n\phi(x) = \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x)\big)_n$. S'il existe une mesure de probabilité μ tel que pour toute fonction continue ϕ, la suite $\big(S_n(\phi)(x)\big)_{n\in\mathbb{N}}$ converge vers $\int \phi \, d\mu$, on dit que x est dans le basin $B(\mu)$ de la mesure μ. Si la mesure de Lebesgue de $B(\mu)$ (quand ça a du sens) est positive, on dit que la mésure μ est physique.

Dans cette thèse, nous nous intéresserons à deux classes de systèmes dynamiques. La première est celle des applications dilatantes (voir la section 3). La seconde est celle des difféomorphismes hyperboliques (voir la section 4).

Soit M une variété riemannienne compacte, et $f : M \to M$ un difféomorphisme hyperbolique de classe C^1. Une telle application possède de nombreuses mesures invariantes, mais il existe deux classes intéressantes appelées mesures physiques et mesures SRB (voir définition 2.27 et définition 2.33). En gros, une mesure physique nous donne le comportement asymptotique de l'orbite d'un point typique par rapport à la mesure de Lebesgue sur M. Si f est $C^{1+\alpha}$, c'est un résultat classique que f a une mesure SRB unique qui est aussi physique (pour un article de synthèse voir [You02]). Dans cette thèse, nous nous intéresserons à l'existence de mesures physiques et SRB en régularité faible. Si $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ est un module de continuité, on dit que f est $C^{1+\omega}$ si f est C^1 et que le module de continuité de df est au plus un multiple de ω, c'est-à-dire qu'il existe $C > 0$ tel que

$$\|df_x - df_y\| \le C\omega\big(d(x,y)\big), \forall x, y \in M.$$

Nous disons qu'un module ω est Dini sommable si

$$\int_0^1 \frac{\omega(t)}{t} \, dt < +\infty. \tag{1}$$

Cette condition était connue depuis les travaux d'Anosov en 1967 [Ano67a, Ano67b], mais il ne lui avait pas donné de nom spécifique. Plus tard, en 1975, Walters a donné une formulation équivalente de cette condition dans le cas d'un espace symbolique, qui est une sommabilité de variation [Wal75].

Ensuite, en 1994, Gora a nommé cette condition la condition de Schmitt, et a prouvé qu'il s'agit de la condition la plus faible qui garantit l'existence d'une

mesure de probabilité invariante absolument continue (ACIP) pour une application dilatante par morceaux [Gór94a]. Peu après, en 2000, Li et Zhang ont appelé cette condition la condition de Dini, et ont montré qu'elle est suffisante pour obtenir la convergence de l'opérateur de transfert [LZ00]. Séparément, en 2001, Fan et Jiang ont prouvé les mêmes résultats (voir [FJ01a]), de plus, ils ont donné une vitesse de convergence de l'opérateur de transfert défini par un potentiel qui a un module de continuité de Dini ([FJ01b]) (Pour une introduction plus classique à l'opérateur de transfert, voir [PP90, Bal00]).

Si T est une application dilatante de classe C^1, avec une dérivée dont le module satisfait la condition (1), alors T a une ACIP ergodique unique. Gora et Schmitt ont donné un exemple explicite d'une application dilatante de classe C^1 qui n'admet pas d'ACIP [GS89]. Ensuite, Quas a prouvé que, génériquement, les applications dilatantes de classe C^1 du cercle n'ont pas d'ACIP [Qua99]. Il a donné aussi un exemple d'une application dilatante de classe C^1 avec un ACIP qui n'est pas ergodique. Peu après, Avila et Bochi ont prouvé que, génériquement, les applications de classe C^1 d'une variété riemanniene compacte n'ont pas d'ACIP [AB06a].

Une question naturelle est de savoir si la condition (2) est suffisante pour avoir une mesure SRB ou physique invariante pour les applications hyperboliques? La réponse est positive, et c'est l'objectif principal de cette thèse.

Une autre question ouverte intéressante est de savoir si un difféomorphisme hyperbolique générique C^1 a une mesure SRB. Un exemple d'un attracteur hyperbolique C^1 qui n'a pas de mesure SRB est donné dans 4.4.

D'autres exemples sont construits dans le cas $C^1 \setminus C^{1+\mathrm{Dini}}$, (voir [Bow75b], exercice 3.2 [Man12]). Plus récemment, dans [Qiu11], il est prouvé que pour un attracteur hyperbolique générique C^1, il existe une mesure d'équilibre unique pour le potentiel géométrique, qui est physique.

Supposons maintenant que f est $C^{1+\mathrm{Dini}}$, ce qui signifie que f est C^1 et que df a un module de continuité qui satisfait la condition (2). Le point crucial de la preuve de l'existence de mesure SRB est la régularité de la distribution instable. Anosov a prouvé dans [Ano67b] que si f est $C^{1+\alpha}$ alors E^u est hölderienne. Nous prouverons que si f est C^{1+Dini} alors E^u a un module de continuité qui satisfait la condition de Dini. En utilisant ce fait, nous prouvons que f a une distorsion, ce qui est la propriété principale pour prouver l'existence d'une mesure SRB. La condition de Dini est également suffisante pour avoir la continuité absolue des holonomies stables (ACH), ce qui implique que la mesure SRB est une mesure physique.

Une autre approche pour prouver l'existence d'une mesure physique consiste à utiliser le fait que l'application hyperbolique $f : M \to M$ possède une partition de Markov (voir [Bow75a, BR75]). Ceci implique que f est semi-conjugué à un sous-décalage de type fini. Plus précisément, il existe (Σ_A, σ) et une application surjective hölderienne $\pi : \Sigma_A \to M$ telle que $\pi \circ \sigma = f \circ \pi$. Donc si nous prenons un potentiel avec un module de continuité Dini sommable $\phi : \Lambda \to \mathbb{R}$, alors $\pi \circ \sigma$ a un module de Dini sommable. Pour obtenir un état d'équilibre pour $(\Sigma_A, \sigma, \pi \circ \phi)$, on peut réduire le problème à un sous-décalage de type fini unilatéral $(\Sigma_A^+, \sigma, \tilde{\phi})$, (où $\tilde{\phi}$ est un potentiel qui ne dépend que du futur), et cohomologue à $\pi \circ \phi$ [PP90]) qui est une application dilatante [FJ01a]. On pousse cette mesure par π pour obtenir une mesure d'équilibre pour $\big(f, \phi\big)$. Enfin, si nous considérons le potentiel géométrique $\phi^{(u)} = -\log J^u f = -\log \det df_{|E^u}$, alors $\phi^{(u)}$ a un module de continuité Dini sommable à condition que f soit $C^{1+\text{Dini}}$. Dans ce cas la mesure d'équilibre $\mu_{\phi^{(u)}}$ de ce potentiel est la mesure physique.

Le théorème principal de cette thèse est le suivant

Theorem 0.1 ([Bou22]). *Si $f : M \to M$ est un difféomorphisme hyperbolique C^{1+Dini}, alors le potentiel géométrique $\phi^{(u)} = -\log \det df_{|E^u}$ a un module de continuité de Dini. En particulier, nous avons*

i. f admet une mesure SRB invariante,

ii. les holonomies locales stables sont absolument continues,

iii. si f est transitif, alors la mesure SRB est ergodique et physique, et c'est l'unique mesure d'équilibre pour le potentiel géométrique.

Cette thèse est organisée comme suit ;

Dans la section 2, nous donnons des notions générales sur les systèmes dynamiques, quelques exemples et une brève introduction à la théorie ergodique.

Dans la section 3, nous présentons quelques résultats classiques sur les applications dilatantes. Nous commençons par la preuve de l'existence d'un ACIP pour une application dilatante $C^{1+\text{Dini}}$ due à [FJ01a]. Nous rappelons ensuite la preuve de Quas sur la non-existence [Qua99]. Ensuite, en utilisant la propriété géométrique de l'opérateur de transfert d'une application dilatante de classe $C^{1+\alpha}$, nous prouvons la décroissance des corrélations. A la fin de la section, nous prouvons un résultat simple et nouveau sur les applications dilatantes du cercle qui préserve la mesure de Lebesgue.

Dans la section 4, nous rappelons quelques définitions de base sur les applications hyperboliques.

Dans la section 5, nous donnons la preuve de la régularité de la distribution stable lorsque l'application hyperbolique en question est $C^{1+\text{Dini}}$. Ceci implique que f possède une propriété très importante appelée distorsion. Cette propriété sera utilisée pour prouver l'existence d'une mesure SRB et la continuité absolue des holonomies stable. Ceci implique en utilisant de l'argument de Hopf que la mesure SRB est physique.

La section 6 est consacrée à la preuve de la décroissance des correlations d'un difféomorphisme d'Anosov de classe $C^{1+\alpha}$. Une nouvelle approche de la preuve a été employée, utilisant le concept de transport optimal.

Dans l'annexe, nous rappelons un théorème de couverture, nécessaire pour prouver la continuité absolue des holonomies stables. Ensuite, nous introduisons brièvement le concept de transport optimal.

Mots clés

Systèmes dynamiques, dynamique uniformément hyperbolique, applications dilatantes, décroissance des corélations, connexité par arc, faible régularité, mesure absolument continue, mesure SRB, mesure physique, transport optimal.

1 Introduction

One of the goals of dynamical systems is to understand the behavior of a transformation. More precisely, if $f : X \to X$ is a map with reasonable properties on a space X, we want to describe the orbits of $f; \mathcal{O}^+(x) = \{x, fx, f^2x, \cdots\}$ or $\mathcal{O}(x) = \{\cdots, f^{-1}x, x, fx, \cdots\}$ if f is invertible. If we suppose that X is a metric space and $x, y \in X$ are close enough, what can we say about the orbits of x and y? The answer is one word "Chaos". This notion was introduced by Poincaré while studying the three-body problem. It turns out that some systems are sensible to initial conditions, in other words even if $x, y \in X$ are close, the behavior of their orbits may differ from one another.

In this thesis we will be interested in two classes of dynamical systems. The first one is expanding maps (see section 3). The second one is hyperbolic diffeomorphisms (see section 5).

Let M be a compact Riemannian manifold, and $f : M \to M$ a C^1 hyperbolic diffeomorphism. Such a map has a lot of invariant measures, however, there are two interesting classes called physical and SRB measures (see Definition 2.27 and Definition 2.33). Roughly speaking, a physical measure gives us the asymptotic behavior of the orbit of a typical point with respect to the Lebesgue measure on M. If f is $C^{1+\alpha}$, it is a classical result that f has a unique SRB measure which is also physical (for a survey article see [You02]). In this thesis, we will be interested in the existence of physical and SRB measures in weak regularity. If $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity, we say that f is $C^{1+\omega}$ if f is C^1 and the modulus of continuity of df is at most a multiple of ω, i.e there is $C > 0$ such that

$$\|df_x - df_y\| \leq C\omega\big(d(x, y)\big), \forall x, y \in M.$$

We say that a modulus ω is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t}\, dt < +\infty. \tag{2}$$

This condition was known since the work of Anosov in 1967 [Ano67a, Ano67b], but he didn't give it a specific name. Later in 1975, Walters gave an equivalent formulation of this condition in the case of a symbolic space, which is a summability of variation [Wal75]. Then, in 1994, Gora named this condition Schmitt's condition, and proved that it is the weakest condition that ensures the existence of an absolutely continuous invariant probability measure (ACIP) for a piecewise expanding map [Gór94a]. Soon after, in 2000, Li and Zhang named this condition Dini condition, and showed that it is sufficient to get the convergence of the transfer operator [LZ00]. Separately, in 2001 Fan and Jiang proved the same results (see [FJ01a]), moreover, they gave a speed of convergence of the transfer operator defined by a potential that has a Dini modulus of continuity ([FJ01b]). (for a more classical introduction to transfer operator see [PP90, Bal00]).

If T is a C^1 expanding map, with a derivative that has a modulus that satisfies condition (2), then T has a unique ergodic ACIP. Gora and Schmitt gave an explicit example of a C^1 expanding map that does not admit an ACIP [GS89]. Then, Quas proved that generically C^1 expanding maps of the circle do not have an ACIP [Qua99], and he gave an example of a C^1 expanding map with an ACIP which is not ergodic. Soon after, Avila and Bochi proved that generically C^1 maps of a smooth compact Riemannian manifold do not have an ACIP [AB06a].

A natural question to ask is whether condition (2) is sufficient to have a SRB and physical invariant measures for hyperbolic maps. The answer is positive, and it is the main goal of this thesis.

Another interesting open question is whether a generic C^1 hyperbolic map has a SRB measure. An example of a C^1 hyperbolic attractor that does not have a SRB measure is given in 4.4.

Other examples are constructed in $C^1 \setminus C^{1+\mathrm{Dini}}$ case, (see [Bow75b], exercise 3.2 [Man12]), and in [Qiu11], it is proven that for a generic C^1 hyperbolic attractor, there is a unique equilibrium measure for the geometric potential, which is physical.

Now, assume that f is $C^{1+\mathrm{Dini}}$, meaning that f is C^1 and df has a modulus of continuity that satisfies condition (2). The crucial point in the proof is the regularity of the unstable distribution. Anosov proved in [Ano67b] that if f is $C^{1+\alpha}$ then E^u is Hölder continuous. We will prove that if f is C^{1+Dini} then E^u has a modulus of continuity that satisfies Dini condition. Then using this fact, we prove that the unstable Jacobian of f has distortion, which is the main property to prove the existence of a SRB measure. The Dini condition is also sufficient to have the absolute continuity of the holonomy maps (ACH), which implies that the SRB measure is a physical measure.

Another approach to prove the existence of a physical measure, is to use the fact that the hyperbolic map $f : M \to M$ has a Markov partition (see [Bow75a]), which implies that f is semi-conjugated to a subshift of finite type. More precisely, there are (Σ_A, σ) and a surjective Hölder map $\pi : \Sigma_A \to M$ such that $\pi \circ \sigma = f \circ \pi$, so if we take a potential with Dini summable modulus $\phi : \Lambda \to \mathbb{R}$, then $\pi \circ \sigma$ has a Dini summable modulus. To get an equilibrium state for $(\Sigma_A, \sigma, \pi \circ \phi)$ one can reduce the problem to one-sided shift $(\Sigma_A^+, \sigma, \tilde{\phi})$, (where $\tilde{\phi}$ is a potential depending only on the future, and cohomologous to $\pi \circ \phi$ [PP90]) which is an expanding map, then we apply the adapted Ruelle-Perron-Frobenius theorem [FJ01a] to get an equilibrium measure for $(\sigma, \pi \circ \phi)$. We push this measure by π to get an equilibrium measure for $\big(f, \phi\big)$. Finally, if we consider the geometric potential $\phi^{(u)} = -\log J^u f = -\log \det df_{|E^u}$, then $\phi^{(u)}$ has a Dini modulus of continuity provided that f is $C^{1+\mathrm{Dini}}$, and the equilibrium measure $\mu_{\phi^{(u)}}$ of this potential is the physical measure. [Bow75a, BR75]

The main theorem of this thesis is

Theorem 1.1 ([Bou22]). *Let $f : M \to M$ be a C^{1+Dini} hyperbolic diffeomorphism, then the geometric potential $\phi^{(u)} = -\log df_{|E^u}$ has a Dini modulus of continuity. In particular we have* *i. f admits an invariant SRB measure,*

ii. the local holonomy maps are absolutely continuous,

iii. if f is transitive, then the SRB measure is ergodic and physical, and it is the unique equilibrium measure for the geometric potential.

This thesis is organized as follows;

In section 2 we give general notions about dynamical systems, some examples, and a brief introduction to ergodic theory.

In section 3, we present some classical results about expanding maps. We start with the proof of existence of an ACIP for a $C^{1+textDini}$ expanding map due to [FJ01a]. Then we recall Quas' proof about non-existence [Qua99]. After that, using the geometric property of the transfer operator of a $C^{1+\alpha}$ expanding map, we prove decay of correlations. At the end of the section, we prove a simple and new result about expanding maps of the circle that preserves Lebesgue measure.

In section 4, we recall some basic definitions of hyperbolic maps.

In section 5, we give the proof of the regularity of the stable distribution when the hyperbolic map in question is $C^{1+\mathrm{Dini}}$. This implies that f has a very important property called distortion. This property will be used to prove the existence of a SRB measure and the absolute continuity of the holonomy, which implies using the Hopf argument that the SRB measure is physical.

Section 6 is devoted to prove decay of correlations of a $C^{1+\alpha}$ Anosov diffeomorphism. A novel approach to the proof has been employed, utilizing the concept of optimal transport.

In the appendix, we recall a covering theorem, needed to prove the absolute continuity of the stable holonomy. Next, we briefly introduce the concept of optimal transport.

Keywords

Dynamical systems, Uniformly hyperbolic maps, weak regularity, SRB and physical measure, Decay of correlations, Optimal transport.

2 Preliminaries

2.1 Introduction to ergodic theory

What is a dynamical system? Roughly speaking, it is a collection of of maps $\{f_i\}_{i\in I}$ defined on some parts of a space X with values in X, and the collection $\{f_i\}_{i\in I}$ form a pseudo-group. This definition is informal, in this dissertation, we will be interested in discrete dynamical systems, i.e we consider a single map $f : X \to X$ with reasonable properties, and the collection $\{f^k\}_{k\in\mathbb{Z}}$ if f is invertible and $\{f^k\}_{k\in\mathbb{N}}$ if it is not.

Suppose that $(X, \mathfrak{B})$ is measurable space, and $f : X \to X$ is a measurable map. We denote by $\mathcal{M}(X)$ the space of probability measures on X. Let $\mu \in \mathcal{M}(X)$, then the pushforward of the measure μ by f is the measure given by

$$(f_*\mu)(E) = \mu(f^{-1}E), \ \forall E \in \mathfrak{B}. \tag{3}$$

Definition 2.1. *A probability measure μ is f-*invariant *if $f_*\mu = \mu$. We denote by $\mathcal{M}_f(X)$ the space of f-invariant probability measures.*

Remarks 2.2.

- *If X is a compact metric space and f is continuous, then the set $\mathcal{M}_f(X)$ is a nonempty, compact and convex space.*
- *If f is not continuous, $\mathcal{M}_f(X)$ can be empty. For example, consider the map $f : [0, 1] \to [0, 1]$, which maps 0 to $\frac{1}{2}$, and for $x \neq 0, f(x) = \frac{1}{3}x$.*
- *If $\sharp\mathcal{M}_f(X) = 1$, then f is called* uniquely ergodic.

The following theorem is one of the most classical theorems in dynamical systems.

Theorem 2.3 (Poincaré recurrence)**.** *Let $(X, \mathfrak{B}, \mu, f)$ be a dynamical system, then for all $E \in \mathfrak{B}$, μ-a-e $x \in E$ is recurrent, i.e $\forall n \in \mathbb{N}, \exists k \geq n$, such that $f^k(x) \in E$.*

Proof. Assume by contradiction that this is not the case, i.e there is $E \in \mathfrak{B}$ with $\mu(E) > 0$, such that μ-a-e $x \in E$, there is n such that for all $k \geq n$ we have $f^k(x) \notin E$. Denote by $B_n = \{x \in E \mid \forall k \geq n, f^k(x) \notin E\}$, then we have by assumption

$$E = \bigcup_{n\geq 0} B_n \bmod 0. \tag{4}$$

Since $\mu(E) > 0$, there is $n_1 \in \mathbb{N}$ such that $\mu(B_{n_1}) > 0$. So we have for all $k, k' \geq 1$ and $k \neq k'$, $f^{-kn_1}(B_{n_1}) \cap f^{-k'n_1}(B_{n_1}) = \emptyset$.

Using the fact that μ is f-invariant, we have $\mu(f^{-kn_1}(B_{n_1})) = \mu(B_{n_1})$. We deduce that $\mu\left(\bigcup_{k\geq 1} f^{-kn_1}(B_{n_1})\right) = \sum_{k=1}^{+\infty} \mu(B_{n_1}) = +\infty$, which contradicts the fact that μ is a probability measure. □

Poincaré recurrence theorem does not tell us how often we return to a set E. A more powerful theorem that tell us how often we return is the Birkhoff ergodic theorem. Denote by $\mathcal{I} := \{E \in \mathcal{B} \mid \mu(E \triangle f^{-1}E) = 0\}$ the set of invariant elements of $\mathcal{B}$, then the Birkhoff theorem states that

Theorem 2.4 (Birkhoff ergodic theorem)**.** *Let $\varphi \in L^1(X, \mu)$, then there is $\varphi^* \in L^1(X, \mu)$ such that for μ-a.e $x \in X$ we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \varphi^*(x), \tag{5}$$

moreover $\varphi^* = E_\mu[\varphi \mid \mathcal{I}]$.

Now we give some classical examples of discrete dynamical systems.

- **Circle rotations**: Consider the rotation $R_\theta : S^1 \to S^1$, defined by $z \mapsto e^{i\theta}z$. If $\theta \in \mathbb{Q}$, then R_θ is periodic , i.e there is $n \in \mathbb{N}$ such that $R_\theta^n = Id$. If $\theta \notin \mathbb{Q}$, then R_θ is uniquely ergodic, and the unique invariant probability measure is the normalized Lebesgue measure of S^1. To unique ergodicity, it is enough to prove that for any invariant measure μ and a small interval I_{ϵ_0}, we have $\mu(I_{\epsilon_0}) = |I_{\epsilon_0}|$.

- **South-North dynamic**: Consider the map $f_{S,N} : S^1 \to S^1$ given by $z \mapsto \frac{-3z-i}{iz-3}$. Notice that any invariant measure must be supported on $-i$ and i. Moreover, if $f : S^1 \to S^1$ is a C^1 orientation preserving diffeomorphism, with exactly two fixed point, and the derivative of f at each fixed point has absolute value different than 1, then f is topologically conjugate to $f_{S,N}$, i.e there is a homeomorphism $h : S^1 \to S^1$, such that $f = h \circ f_{S,N} \circ h^{-1}$.

- **Parabolic homeomorphism of S^1:** A homeomorphism of S^1 preserving the orientation and with a unique fixed point, is said to be parabolic.

- **Action of $PSL_2(\mathbb{R})$ on $\partial\mathbb{H}^2$**: Consider the group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\langle Id, -Id\rangle$, where $\mathbb{H}^2$ is the upper half plane, and $SL_2(\mathbb{R})$ are two by two matrices with determinant 1. This group acts on $\mathbb{H}^2$ by homography, i.e for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}), z \in \mathbb{H}^2$, the action is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. Let $\gamma \in PSL_2(\mathbb{R})$. If $|tr(\gamma)| > 2$, then the action of γ on $\partial\mathbb{H}^2$ (the boundary of $\mathbb{H}^2$ which can be identified with S^1) is topologically conjugate to $f_{S,N}$, and in this case γ is called hyperbolic. If $|tr(\gamma)| < 2$, then the action on $\partial\mathbb{H}^2$ is conjugate to a rotation, and γ is called elliptic. If $|tr(\gamma)| = 2$, then the action on $\partial\mathbb{H}^2$ is parabolic.

- **Translation on the n-dim torus**: Consider $\tau_\theta : \mathbb{T}^n \to \mathbb{T}^n, x \mapsto x + \theta$, where $\theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \mathbb{T}^n$. Then τ_θ is uniquely ergodic if and only if $p_0 + \sum\limits_{k=1}^{n} p_k\theta_k \neq 0$ for all $(p_k) \in \mathbb{Q}^{n+1} \setminus \{0\}$.

- **Toral automorphisms**: Consider the linear action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$. This action preserves the lattice $\mathbb{Z}^2$, so the action passes to $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. As in the case of $PSL_2(\mathbb{R})$ we can classify elements of $SL_2(\mathbb{Z})$ into three types depending on the trace. If $\gamma \in SL_2(\mathbb{Z})$ then the characteristic polynomial has the form $X^2 - tr(\gamma)X + 1$, so $\delta_\gamma = tr(\gamma)^2 - 4$.

  If $|tr(\gamma)| > 2$, then $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism (see Definition 4.1).

  If $|tr(\gamma)| < 2$, then γ has two conjugate eigenvalues in S^1, and since we have $P \begin{pmatrix} \lambda^k & \\ & \lambda^{-k} \end{pmatrix} P^{-1} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in SL_2(\mathbb{Z})$, so there is some $M > 0$ such that for all k, we have $|a_k|, |b_k|, |c_k|, |d_k| \leq M$, which implies that γ has finite order.

  If $|tr(\gamma)| = 2$, then γ has 1 as eigenvalue, γ fixes the direction of this eigenvalue, and since γ is continuous and different from the identity, this direction is a closed loop in $\mathbb{T}^2$.

  In fact for each homeomorphism of $\mathbb{T}^2$, there is a unique $\gamma \in SL_2(\mathbb{Z})$, such that f is homotopic to γ, so the set $MCG(\mathbb{T}^2) := Homeo^+(\mathbb{T}^2)/Homotopy$ can be identified with $SL_2(\mathbb{Z})$. This set is called the mapping class group of $\mathbb{T}^2$. For a closed surface S of higher genus, there is a similar classification of $MCG(S)$ due to Thurston.

2.1.1 Entropy

The concept of entropy was introduced in 1958 by Kolmogorov, it is an isomorphism invariant, which roughly speaking measures the complexity of a system.

To define the entropy of a measure preserving transformation f of $(X, \mathfrak{B}, \mu)$, we introduce the entropy of a finite partition of X, then entropy of f with respect to a finite partition.

Definition 2.5. *A partition of X is a collection of elements $\{P_i\}_{i \in I}$ of $\mathfrak{B}$ such that $\bigcup_{i \in I} P_i = X$ and $\mu(P_i \cap P_j) = 0$, for $i \neq j$.*

Definition 2.6. *Let $\mathcal{P}_1, \mathcal{P}_2$ two partitions of X. We say that $\mathcal{P}_1$ is a refinement of $\mathcal{P}_2$ and write $\mathcal{P}_1 \leq \mathcal{P}_2$, if each element of $\mathcal{P}_2$ is a union of elements of $\mathcal{P}_1$. The joint partition of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the partition*

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \{P_1 \cap P_2 \mid P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2\}.$$

For $n \in \mathbb{N}$, we denote by $f^{-n}\mathcal{P}_1$ the partition $\{f^{-n}P \mid P \in \mathcal{P}_1\}$.

Definition 2.7 (Entropy of a partition)**.** *Let $\mathcal{P}$ be a finite partition of X. The entropy of the partition $\mathcal{P}$ is the real number*

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P), \tag{6}$$

with the convention $0 \log 0 = 0$. *If* $\mathcal{A}$ *is another partition, then the entropy of* $\mathcal{P}$ *knowing* $\mathcal{A}$ *(or conditional entropy) is defined as*

$$H(\mathcal{P}|\mathcal{A}) = -\sum_{P \in \mathcal{P}, A \in \mathcal{A}} \mu(P|A) \log \mu(P|A), \tag{7}$$

where $\mu(P|A) = \frac{\mu(P \cap A)}{\mu(A)}$ *is the conditional probability of* P *knowing* A.

Using the convexity of $\phi : [0, \infty) \to \mathbb{R}, x \mapsto x \log x$, we have the following properties of the entropy of a partition.

Proposition 2.8. *Let* $\mathcal{P}, \mathcal{A}, \mathcal{C}$ *be three partitions of* $(X, \mathfrak{B}, \mu)$ *then we have*

- $H(\mathcal{P}) \geq 0$,
- $H(\mathcal{A} \vee \mathcal{P}) \leq H(\mathcal{A}) + H(\mathcal{P})$,
- $H(f^{-1}\mathcal{P}) = H(\mathcal{P})$,
- *If* $\mathcal{A} \leq \mathcal{P}$ *then* $H(\mathcal{A}) \leq H(\mathcal{P})$ *and* $H(\mathcal{A}|\mathcal{C}) \leq H(\mathcal{P}|\mathcal{C})$.

For a proof and more properties see theorem 4.3 in [Wal00] or [Cou16].

The definition of entropy of a partition does not involve the map f. We define entropy of f with respect to a partition.

Definition 2.9. *Let* $\mathcal{P}$ *be a partition, then the entropy of* f *with respect to this partition is given by*

$$h_\mu(f, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H(\vee_{i=0}^{n-1} f^{-i} \mathcal{P}). \tag{8}$$

This limit exists, because the sequence $(u_n)_n$ defined by $u_n = H(\vee_{i=0}^{n-1} f^{-i} \mathcal{P})$ satisfies for all $k, l \in \mathbb{N}, u_{k+l} \leq u_k + u_l$. Now we give the definition of entropy.

Definition 2.10. *The entropy of* f *with respect to the invariant measure* μ *is*

$$h_\mu(f) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P}). \tag{9}$$

Proposition 2.11. *The entropy is an isomorphism invariant.*

Examples 2.12.

- *Let* $X = \Sigma_2 := \{0, 1\}^{\mathbb{Z}}$, *with the product topology, and let* σ *be the shift map, and* μ *the* $(\frac{1}{2}, \frac{1}{2})$ *Bernoulli measure, then* $h_\mu(\sigma) = \log 2$. *For* Σ_3 *and the shift map in this space the entropy is* $\log 3$, *so by the previous proposition there is no measurable isomorphism between* Σ_2 *equipped with* $(\frac{1}{2}, \frac{1}{2})$ *Bernoulli measure, and* Σ_3 *equipped with* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *Bernoulli measure.*

- *If* X *is a metric space, and* f *is an isometry, then for any* f *invariant measure* μ, *we have* $h_\mu(f) = 0$.

- *The entropy of the doubling map on the circle has entropy* $\log 2$ *with respect to the Lebesgue measure.*

Topological entropy

Let (X, d) be a compact metric space, and $f : X \to X$ be a continuous map. For each $n \in \mathbb{N}$, let $d_n : X \times X \to \mathbb{R}$ be given by $d_n(x, y) = \max\limits_{0 \leq k < n} d(f^k x, f^k y)$. It is easy to check that d_n is well defined, and it is a distance on X. We denote by $B_n(x, r)$ the ball of center x and radius r with respect to the distance d_n. We call $B_n(x, r)$ a dynamical ball, and d_n a dynamical distance. Notice that if $n \leq m$ then $d_m \leq d_n$, and if we know the orbit of length n of a point x, then $\forall y \in B_n(x, r)$, the orbit of length n of y is close to the orbit of x. We say that a finite set $\mathcal{C}$ is (ϵ_0, n)-covering if

$$X = \bigcup_{x \in \mathcal{C}} B_n(x, \epsilon_0),$$

and a finite set $\mathcal{S}$ is (ϵ_0, n)-separating if for all $x, y \in \mathcal{S}$ with $x \neq y$, we have

$$B_n(x, \epsilon_0) \cap B_n(y, \epsilon_0) = \emptyset.$$

Consider the following numbers

$$q(n, \epsilon_0) = \inf \sharp \mathcal{C};\ p(n, \epsilon_0) = \sup \sharp \mathcal{S}, \tag{10}$$

where the infimum is taken over (ϵ_0, n)-covering sets, and the supremum is taken over (ϵ_0, n)-separating sets. These two numbers measure the complexity of f and they are related by the inequalities

$$q(n, 2\epsilon_0) \leq p(n, \epsilon_0) \leq q(n, \frac{\epsilon_0}{2}).$$

Proposition 2.13. *The limit* $\lim\limits_{n \to +\infty} \frac{1}{n} \log q(n, \epsilon_0)$ *exist.*

Definition 2.14. *We call the limit* $h(f) = \lim\limits_{n \to +\infty} \frac{1}{n} \log q(n, \epsilon)$ *the topological entropy of* f*.*

Remarks 2.15.

- *The topological entropy* $h(f)$ *depends only on the topology induced by* d*.*
- *If* f *is conjugated to* $g : Y \to Y$ *by a homeomorphism then* $h(f) = h(g)$*.*
- *We will define later a more general notion, which is the pressure* $P(\phi)$ *with respect to a potential* ϕ*. In this case* $h(f) = P(0)$*.*

Examples 2.16.

- *If* $f : X \to X$ *is an isometry, then* $\forall n \in \mathbb{N}^*, d_n = d$*, which implies that* $h(f) = 0$*.*
- *If* $f : S^1 \to S^1, z \mapsto z^2$ *then we have* $d_n = \frac{d}{2^n}$*, which gives us* $h(f) = \log 2$*.*

- *For any expanding map g of the circle of degree d, g is conjugated to the map $z \mapsto z^d$, so we have $h(g) = \log d$.*

Let (X, d, μ) be a metric space with a probability measure, and $f : X \to X$ a continuous map that preserves μ, then we have $h_\mu(f) \leq h(f)$. In fact we have

Theorem 2.17 (Variational principle). $h(f) = \sup_\mu h_\mu(f)$, *where the supremum is taken over the space of invariant probability measures of f.*

If an invariant measure μ satisfies $h_\mu(f) = h(f)$, then we call μ a measure of maximal entropy (MME for abbreviation). For example the Lebesgue measure on S^1 is a MME for the doubling map. In general the MME does not always exist. A sufficient condition for the existence is upper-semi continuity of the entropy function, i.e the map $h_\cdot(f) : \mathcal{M}_f(X) \to \mathbb{R}, \mu \mapsto h_\mu(f)$ is upper-semi continuous.

2.1.2 Ergodicity

Let $(X, \mathcal{B}, f, \mu)$ be a dynamical system. A natural question to ask is whether we can decompose X into a disjoint union of sets in $\mathcal{B}$ with positive measure. The notion of ergodicity tell us that we can't do this, more precisely

Definition 2.18. *A probability measure μ is ergodic if for all $E \in \mathcal{B}$ satisfying $f^{-1}E = E$, we have $\mu(E) = 0$ or 1.*

Examples 2.19.

- *Any irrational rotation of the circle is ergodic with respect to the Lebesgue measure.*
- *A toral translation is ergodic with respect to the Lebesgue measure if and only if it is transitive.*
- *The geodesic flow in negatively curved closed surface is ergodic with respect to the normalized Liouville measure.*
- *If μ is a unique equilibrium measure for a potential ϕ, then μ is ergodic.*

Proposition 2.20 (Equivalent conditions for ergodicity). *A dynamical system $(X, f, \mathcal{B}, \mu)$ is ergodic if and only if*

- *For all $\phi \in L^1$, if $\phi \circ f = \phi$, then ϕ is constant μ-a.e,*
- *For all $\phi \in L^1$, and μ-a.e x, the limit $\frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x)$ exists and is independent of x.*

2.1.3 Subshift of finite type

The subshift of finite type is one of the fundamental dynamical systems that is well understood from ergodic point of view, we will see later that uniformly expanding maps and Axiom A attractors are factors of a subshift of finite type.

Let $\mathcal{A}$ be the set (called the alphabet)

$$\mathcal{A} = \{1, 2, \cdots, n\}. \tag{11}$$

Denote by Σ_n (resp Σ_n^+) the set $\mathcal{A}^{\mathbb{Z}}$ (resp $\mathcal{A}^{\mathbb{N}}$) then define the transformation $\sigma : \Sigma_n \to \Sigma_n$ which associates to $(x_n)_{n\in\mathbb{Z}}$ the sequence $(y_n)_{n\in\mathbb{Z}}$, where for all $n, y_{n+1} = x_n$. This transformation is continuous with respect to the product topology, it is called the shift map. Consider the distance d on Σ_n given by

$$d\left((x_n)_n, (x'_n)_n\right) = \left(\frac{1}{2}\right)^{\inf\limits_{k\in\mathbb{Z}}\{|k| \ | \ x_k \neq x'_k\}}, \tag{12}$$

this distance is an ultrametric distance, and it induces the product topology on Σ_n. In the case of Σ_n^+, this map is expanding with respect to this distance.

Let $A \in M_n(\mathbb{R})$ be a square matrix with entries $A_{ij} \in \{0, 1\}$, and each line or column contains a 1, then the set

$$\sigma_A = \{(\cdots, x_{-1}, x_0, x_1, \cdots) \in \Sigma_n \mid A_{x_i x_{i+1}} = 1\} \tag{13}$$

is closed and σ invariant, we say that $(\Sigma_A, \sigma_{|A})$ is *a subshift of finite type.*

2.2 Thermodynamical formalism and equilibrium states

Let $f : X \to X$ be a continuous map of a compact metric space, and $\phi : X \to \mathbb{R}$ a map (often called potential).

Consider the two following numbers that depend on f, ϕ, ϵ and $n \geq 1$

$$Q_n(f, \phi, \epsilon) = \inf \left\{ \sum_{x\in A} e^{(S_n f)(x)} : A \text{ is a } (n,\epsilon)\text{-spanning set for } X \right\},$$
$$P_n(f, \phi, \epsilon) = \sup \left\{ \sum_{x\in A} e^{(S_n f)(x)} : A \text{ is a } (n,\epsilon)\text{-separated set for } X \right\},$$

where $(S_n f)(x) = \sum\limits_{k=0}^{n-1} \phi \circ f^k(x)$. If ϕ is continuous, then the limit

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \big(Q_n(f, \phi, \epsilon)\big)$$

exists, is finite and is equal to

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \big(P_n(f, \phi, \epsilon)\big).$$

The limit is called the topological pressure of f with respect to the potential ϕ, we denote it by $P(\phi)$.

Denote by $mathcal M_f(X)$ the space of $f-$invariant probabilities measures on X, and let $\mu \in \mathcal{M}_f(X)$. The pressure with respect to this measure is defined by:

$$P_\mu(\phi) = h_\mu(f) + \int_X \phi \, d\mu,$$

The variational principle (see theorem 9.10 in [Wal00]) gives the following formula:

$$P(\phi) = \sup\left\{h_\mu(f) + \int \phi \, d\mu : \mu \in \mathcal{M}_f(X)\right\}.$$

If a measure $\mu \in \mathcal{M}_f(X)$ satisfies $P_\mu(\phi) = P(\phi)$, then μ is called an equilibrium measure for the potential ϕ.

Examples 2.21.

- *If $\phi = 0$, then the pressure equals to the topological entropy, and an equilibrium measure is a MME.*
- *If $f : M \to M$ is a $C^{1+\alpha}$ expanding map (see 3.1), and $\phi^u = -\log Jac(f) := -\log|\det df|$, then there is a unique equilibrium measure μ_g, which is equivalent to the Lebesgue measure. In this case the topological pressure is 0, so we have $h_{\mu_g}(f) = \int \log Jac(f) \; dLeb$.*

The proof of the second example is a corollary of Ruelle's theorem and the fact that an expanding map is a factor of a subshift of finite type.

Consider Σ_A a topologically mixing subshift of finite type, and $\phi \in C^\alpha(\Sigma_A)$ a Hölder potential (or more generally a potential with summable variations), and let $\mathcal{L}_\phi : C^\alpha(\Sigma_A) \to C^\alpha(\Sigma_A)$, given by

$$\mathcal{L}_\phi(\psi)(\overline{x}) = \sum_{\overline{y} \in \sigma^{-1}\overline{x}} e^{\phi(\overline{y})}\psi(\overline{y}), \;\; \forall \overline{x} \in \Sigma_A.$$

This operator is a linear operator, and called a transfer (or Ruelle) operator. In the following theorem we denote this operator by $\mathcal{L}$, and its adjoint operator by $\mathcal{L}^*$.

Theorem 2.22 (Ruelle theorem)**.** *There are $\lambda > 0, h \in C^\alpha(\Sigma_A)$ with $h > 0$ and a measure ν on Σ_A for which we have*

1. *$\mathcal{L}h = \lambda h$ and $\mathcal{L}^*\nu = \lambda\nu$,*
2. *$\nu(h) = 1$, and the measure $\mu = h\nu$ is invariant by the shift map,*
3. *The probability measure μ satisfies Gibbs property, in other words, for all $r > 0$, there is $C = C(r)$ such that $\forall n \in \mathbb{N}$ we have*

$$C^{-1} \leq \frac{\mu\big(B_n(x,r)\big)}{e^{-nP+S_n\phi(x)}} \leq C, \tag{14}$$

where P is the topological pressure with respect to ϕ, and

$$S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k x),$$

4. $\lim_{n\to\infty} \|\lambda^{-m}\mathcal{L}^m\psi - \nu(\psi)h\|_{C^\alpha} = 0, \forall\psi \in C^\alpha(\Sigma_A)$,

5. *λ is a simple eigenvalue for $\mathcal{L}$, and it is equal to its spectral radius.*

For a proof see theorem 1.7 in [Bow75a], and for further spectral properties (for instance spectral gap) see [Bal00].

2.3 Modulus of continuity

Definition 2.23. A modulus of continuity *is a continuous, increasing and concave map $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\omega(0) = 0$.*

We say that the modulus ω is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty. \tag{15}$$

For instance, for any $\alpha \in (0,1)$, the map $\omega(t) = t^\alpha$ is a modulus of continuity which is Dini summable.

The following proposition gives an equivalent condition on a modulus ω to be Dini summable.

Proposition 2.24. *The following conditions are equivalent:*

- *ω is Dini summable.*
- $\forall c \in (0,1)$ *and* $\forall t \geq 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$,
- $\exists c \in (0,1)$ *and* $\exists t > 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$.

Proof. Since ω is concave, the map $t \mapsto \frac{\omega(t)}{t}$ is decreasing, hence we have the following inequalities for all n and small t:

$$\sum_{k=0}^{n-1}(c^k - c^{k+1})\frac{\omega(c^k t)}{c^k t} \leq \int_{c^n t}^{t} \frac{\omega(x)}{x} dx \leq \sum_{k=0}^{n-1}(c^k - c^{k+1})\frac{\omega(c^{k+1} t)}{c^{k+1} t}. \tag{16}$$

We deduce the proposition from these inequalities. □

Let ω be a Dini summable modulus, and for $c \in (0,1)$ define

$$\tilde{\omega}_c(t) = \sum_{k=0}^{+\infty} \omega(c^k t), \forall t \geq 0. \tag{17}$$

It follows immediately that $\tilde{\omega}_c$ is a modulus of continuity.

Remark 2.25. *The Dini summability condition is the weakest known regularity to have a nice invariant space for the transfer operator (see [FJ01a]), it implies also bounded distortion which will be crucial to prove existence of ACIP (resp. SRB) measure for expanding maps (resp. hyperbolic maps).*

Examples:

- For $\alpha \in (0,1]$, $\omega(t) = t^\alpha$ is Dini summable.
- The modulus $\omega_{\beta \log}(t) = \frac{1}{\left(\log(\frac{1}{t})\right)^\beta}$ is Dini summable if and only if $\beta > 1$. In this example ω is defined only for small t, then we extend it by an affine map.

Definition 2.26. *Let X, Y be two metric spaces, and ω a modulus of continuity, we say that a map $f : X \to Y$ is $C^{0+\omega}$ if there is a $C > 0$ such that:*

$$d\big(f(x), f(y)\big) \leq C\omega\big(d(x,y)\big), \quad \forall x, y \in X. \tag{18}$$

- If $\omega(t) = t$, then $C^{0+\omega}$ is the set of Lipschitz maps.
- If $\omega(t) = t^\alpha$, where $0 < \alpha < 1$, then $C^{0+\omega}$ is the set of Hölder maps with exponent α.

Given a continuous map $g : M \to M$ of a compact manifold, a natural way to define the modulus of continuity of g would be to take:

$$\tilde{\omega}_g(t) = \sup_{\substack{x,y \in M \\ d(x,y) \leq t}} d(gx, gy), \tag{19}$$

but $\tilde{\omega}_g$ is not concave. To get concavity, we take:

$$\omega_g = \inf\{h \mid h \text{ continuous, concave and increasing and } h \geq \tilde{\omega}_g\}. \tag{20}$$

It is clear that ω_g is a modulus of continuity, and it satisfies the inequality (18) with constant $C = 1$.

For equivalent formulation of the Dini summability condition see [Gór94b].

2.4 SRB and Physical measures

There is a confusion in the literature about the definition of SRB and physical measure, in this thesis, we follow the convention used by [You02] to define SRB and physical measures. An example of a physical measure that is not SRB was given by Bowen (see Example 2.35).

Let M be a compact Riemannian manifold, and denote by λ its normalized volume measure. Let $f : U \to M$ be a C^1 diffeomorphism, where U in an open subset of M, and μ be an invariant probability measure of f.

First, we begin by defining whats a physical measure. Consider the set

$$B(\mu) = \left\{ x \in M \mid \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \xrightarrow[n \to +\infty]{} \mu \right\}, \tag{21}$$

where the convergence is in the weak* topology.

Definition 2.27. *The measure μ is called physical if $\lambda\big(B(\mu)\big) > 0$.*

Remark 2.28. *If the measure μ is absolutely continuous with respect to λ and ergodic, then using Birkhoff ergodic theorem, μ is physical.*

Example 2.29. *$f : S^1 \to S^1, z \mapsto z^2$ preserves the Lebesgue measure λ, and since λ is ergodic, it is a physical measure for f.*

Example 2.30. *If $f : M \to M$ is a C^1 map, then any Dirac measure δ_x of an attracting fixed point x of f (i.e $\|D_x f v\| < \|v\|, \forall v \in T_x M$) is a physical measure.*

Now, to define an SRB measure, we need to introduce the multiplicative ergodic theorem and Lyapunov exponents.

Theorem 2.31 ([Ose68])**.** *Let $f : U \to M$ a C^1 diffeomorphism preserving a probability measure μ. Then, for μ almost every $x \in M$, there exist a real numbers*

$$\lambda_1(x) < \cdots < \lambda_{k(x)}(x), \tag{22}$$

and a filtration of subspaces

$$\{0\} = E_0(x) \subset E_1(x) \subset \cdots \subset E_{k(x)}(x) = T_x M, \tag{23}$$

such that for all $v \in E_i(x) \setminus E_{i-1}(x), i = 1, \cdots k(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \|df_x^n(v)\| = \lambda_i(x). \tag{24}$$

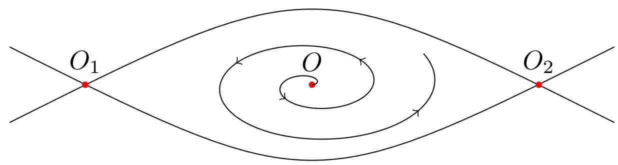
Remark 2.32. *If f is ergodic then λ_i are constant, and called Lyapunov exponents of f. For a proof see [Fil19] for example.*

Assume that f has non-zero Lyapunov exponents, then by Pesin theory, ([Pes77]) there exist local unstable (corresponding to positive Lyapunov exponents) and local stable manifolds.

Definition 2.33. *The measure μ is called a SRB measure, if it has absolutely continuous conditionals along local unstable leaves.*

Remarks 2.34.

- *If f has only positive Lyapunov exponents, then a SRB measure is equivalent to the Lebesgue measure.*
- *For $C^{1+\alpha}$ diffeomorphism, Ruelle gave a upper bound for the entropy of a measure μ, and Pesin proved that equality holds if and only if μ is a SRB measure (see Theorem 2 in [You02]).*
- *We will see later that $C^{1+\alpha}$ expanding maps have an SRB (absolutely continuous with respect to Lebesgue) which is also a physical measure.*



Example 2.35. *The Bowen example: consider the vector field on* $\mathbb{R}^2$ *given by the picture. Denote by* U *the connected open set bounded by the lines* l_1 *and* l_2*. Consider the time 1 map* f *of this flow, then the orbit of any point* $x \in U \setminus \{O\}$ *will converge to the boundary, furthermore,* $\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \delta_x$ *converges in the weak* topology to* $\frac{1}{2}\delta_{O_1} + \frac{1}{2}\delta_{O_2}$*. Clearly the latter measure is not SRB, but it is physical since its basin contains* $U \setminus \{O\}$*.*

3 Uniformly expanding maps

Definition 3.1. *Let $f : M \to M$ be a C^1 map of a compact Riemannian manifold M. We say that f is expanding, if f is a local diffeomorphism, and there exist $C > 0$ and $\lambda \in (0,1)$ such that for all $n \in \mathbb{N}, x \in M$ and $v \in T_xM$,*

$$\|df_x^n(v)\| \geq C\lambda^{-n}\|v\|. \tag{25}$$

An alternative definition which work in metric spaces, is that f asymptotically uniformly expand the distance.

Remark 3.2.

- *The notion of expanding map does not depend on the Riemannian metric on M, i.e if f is expanding for a Riemannian metric, then it is expanding for any equivalent Riemannian metric.*
- *We can choose a Riemannian metric such that $C = 1$, this metric is called a Lyapunov metric.*
- *Notice that not all manifolds have expanding maps, for instance, S^2 does not have an expanding map, because otherwise, the expanding map would be a non trivial covering map, which is impossible because S^2 is simply connected.*
- *It is proved by [Gro81] that if the manifold M admits an expanding map, then M is homeomorphic to an infra-nil-manifold.*

Examples 3.3. *Some of the following examples have singularities, or the expanding map is defined on a metric space.*

- *Let $\Sigma^+ = \{0,1\}^{\mathbb{N}}$, $\sigma : \Sigma^+ \to \Sigma^+$ be the shift map defined by*

$$\sigma\big((x_0, x_1, \cdots)\big) = (x_1, x_2, \cdots),$$

and the distance d given by $d\big((x_n)_n, (y_n)_n\big) = (1/2)^{\max\{n \ \mid \ x_n = y_n\}}$. In this case we have

$$d\Big(\sigma\big((x_n)_n\big), \sigma\big((y_n)_n\big)\Big) \geq 2d\Big((x_n)_n, (y_n)_n\Big),$$

so the shift map is expanding with respect to the defined distance.

- *For $k \in \mathbb{N}\backslash\{0,1\}$, the map : $E_k : S^1 \to S^1$ is expanding. Any C^1-expanding map f of the circle is topologically conjugated to $E_{degree(f)}$.*
- *The Gauss map given by*

$$\begin{aligned} G : [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

G is not uniformly expanding, and it has points of discontinuities, but it is locally expanding, and preserves an ACIP, it has the Gibbs-Markov structure (see [Alv20]). For $x \in [0,1]$, let $G^{-1}(x) = \{x_1, x_2, \cdots\}$ such that $\frac{1}{1+n} < x_n \leq \frac{1}{n}$, using the formula of G we find that $x_n = \frac{1}{x+n}$. Let $h \in L^1[0,1]$, and consider the transfer operator associated to G given by

$$\mathcal{L}_G[h](x) = \sum_{n\geq 1} \frac{h(x_n)}{|G'(x_n)|} = \sum_{n\geq 1} \frac{h(\frac{1}{x+n})}{(x+n)^2}. \tag{26}$$

We will prove later that G has an ACIP if and only if the equation $\mathcal{L}_G[h] = h$ has a solution in L^1. It turns out that $\rho(x) = \frac{1}{ln(2)}\frac{1}{(1+x)}$ is the unique fixed point for $\mathcal{L}_G$.

3.1 Existence of ACIP for expanding maps

Existence of absolutely continuous invariant measure for $C^{1+\alpha}$ expanding maps was originally proved by [KS69]. In this subsection, we present briefly the theorem and some of its consequences.

Let $f : M \to M$ be a $C^{1+\alpha}$ expanding map. To prove that f has an ACIP, we introduce an operator called the transfer operator. Denote by λ the normalised volume on M. Since f is C^1, the measure $f_*\lambda$ is absolutely continuous with respect to λ, and a candidate of an ACIP is to consider a weak* limit μ of the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k\lambda, \tag{27}$$

but μ is not necessarily absolutely continuous. We can in fact compute the Radon-Nikodym derivative of $f_*\lambda$ with respect to λ. Indeed, since f is a covering map, and a local diffeomorphism, we have for a small open ball B

$$f_*\lambda(B) = \lambda(f^{-1}B) = \sum_i \lambda(U_i), \tag{28}$$

where the U_i are the disjoint connected subsets sent to B by f. Since $f_i : B \to U_i$ is a diffeomorphism, we apply the change of variable formula and get that

$$\lambda(U_i) = \int_B Jacf^{-1} d\lambda = \int_B \frac{1}{Jac\ f \circ f_i^{-1}}\ d\lambda, \tag{29}$$

hence we have

$$f_*\lambda(B) = \sum_i \lambda(U_i) = \sum_i \int_B \frac{1}{Jac\ f \circ f_i^{-1}} d\lambda,$$

which gives the required Radon-Nikodym derivative. Using the same computation it is easy to prove for $h \in L^1(M)$ that for λ-a.e x,

$$\frac{df_*(h\lambda)}{d\lambda}(x) = \sum_{y\in f^{-1}(x)} \frac{h(y)}{Jac\ f(y)}.$$

Definition 3.4. *The transfer operator associated to f is the map $\mathcal{L}_f$ defined by*

$$\begin{aligned} \mathcal{L}_f : L^1(M) &\to L^1(M) \\ h &\mapsto \sum_i \frac{h}{Jac\ f} \circ f_i^{-1}. \end{aligned}$$

We proved that

Proposition 3.5. *A probability measure of the form $h\lambda$ is f invariant if and only if $\mathcal{L}_f(h) = h$.*

In other words, to determine an ACIP for f, we have to determine a fixed point of the transfer operator $\mathcal{L}_f$ (this operator was called Perron-Frobenius-Ruelle operator). For further properties of this operator, see [Bal00, PP90].

A folklore theorem is

Theorem 3.6. *If $f : M \to M$ is $C^{1+\alpha}$ expanding map, then $\mathcal{L}_f$ admits a unique invariant positive $L^1(M)$ function ρ_f. Furthermore, ρ_f is $C^{r-1+\alpha}$ if f is $C^{r+\alpha}$, and $\mathcal{L}_f^n(1)$ converges exponentially to ρ_f in the C^α topology.*

One of the consequences of the existence of an ACIP is rigidity of conjugacy.

Corollary 3.7 (Rigidity)**.** *Let f, g be two C^r ($r \geq 2$) expanding maps of the circle, which are conjugated by a homeomorphism h that is absolutely continuous, then h is C^r.*

Proof. Using the previous theorem, f preserves a measure $\mu = \rho\lambda$, where λ is the Lebesgue measure and ρ is a C^{r-1} density. Consider the map $\varphi : S^1 \to S^1, x \mapsto \mu\big([0, x]\big)$. The map φ is well defined, and is a C^r diffeomorphism of S^1, and it conjugates f to an expanding map that preserves the Lebesgue measure. Indeed, put $f_0 = \varphi \circ f \circ \varphi^{-1}$, then this map is a local diffeomorphism, and it preserves the Lebesgue measure (because $\varphi_*^{-1}\lambda = \mu$ and $f_*\mu = \mu$), in particular, we have for almost every $x \in S^1$, $\mathcal{L}_{f_0}[1](x) = 1$, which implies that f_0 is expanding. Using this argument, we may assume without loss of generality, that f and g preserve the Lebesgue measure, in this case, h also preserves the Lebesgue measure (because $h_*\lambda$ is g invariant, and g is ergodic with respect to λ), so h is an isometry, which finishes the proof. $\square$

Remark 3.8. *In [SS85], Shub and Sullivan proved that this corollary is true if we assume only that h is an absolutely continuous measurable bijection.*

Corollary 3.9 (Decay of correlations)**.** *For a $C^{1+\alpha}$ expanding map of a compact Riemannian manifold M, the unique ACIP $\mu = \rho\lambda$ is mixing. In fact we have exponential decay of correlations, i.e there are $C, \tau > 0$ such that for all $\varphi \in C^\alpha(M)$ and $\psi \in L^1(S^1)$ we have for all $n \in \mathbb{N}$*

$$Cor_\mu(\varphi, \psi \circ f^n) := \frac{1}{\|\varphi\|_{C^\alpha}\|\psi\|_1} \left| \int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq Ce^{-\tau n}. \tag{30}$$

Proof. Let $\varphi \in C^\alpha(M), \psi \in L^1$, we may assume without loss of generality that $\|\varphi\|_{C^\alpha} = \|\psi\|_1 = 1$. We have

$$\begin{aligned}\operatorname{Cor}_\mu(\varphi, \psi \circ f^n) &:= \left| \int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right| \\ &= \left| \int \mathcal{L}_f^n[\rho_f \varphi] \cdot \psi dLeb - \int \varphi d\mu \int \psi d\mu \right|,\end{aligned}$$

and since $\mathcal{L}^n(\rho_f\varphi)$ goes to $\left(\int \rho_f \varphi \, d\lambda\right)\rho$ exponentially fast, we have decay of correlations, in particular the measure μ is mixing. □

Remark 3.10. *Another approach to prove decay of correlations is to use optimal transport, this approach was done in [KLS15].*

3.2 Existence in weaker regularity

In [FJ01a], Fan and Jiang, proved that if $f : M \to M$ is a $C^{1+\text{Dini}}$ expanding map of a compact Riemannian manifold, i.e f is C^1 and df has a modulus of continuity ω that is Dini summable, then f admits an ACIP. We stress that $C^{1+\text{Dini}}$ map is not necessarily $C^{1+\alpha}$.

Theorem 3.11 (Fan and Jiang)**.** *Let $f : M \to M$ be a $C^{1+\omega}$ expanding map, where ω is a modulus of continuity that satisfies Dini condition, assume also that f is topologically mixing, then f has a unique ergodic ACIP $\mu = \rho\lambda$ and ρ has $\tilde{\omega}$ as a modulus of continuity.*

To prove this theorem, we give a suitable space of function $\mathcal{H}^{\tilde{\omega}}_{K,s}$ that depends on the modulus ω and two constants $K, s \in \mathbb{R}$, which is invariant by the transfer operator associated to f. Then using some properties of functions of $\mathcal{H}^{\tilde{\omega}}_{K,s}$, we prove that $\mathcal{L}_f$ has a fixed point.

For a modulus ω let $\mathcal{H}^\omega$ be the space of real valued continuous functions over M which have ω as a modulus of continuity.

In the following lemmas, let $a > 0, D > 1$ such that whenever we have $d(x, y) \le a$, then $d(fx, fy) \ge Dd(x, y)$. Then, consider a $K > 0$ such that for all $x, y \in M$ with $d(x, y) \le a$ we have $\frac{|Jacf(x)|}{|Jacf(y)|} \le \exp K\omega\big(d(x, y)\big)$. Now, for $s > 0$ define the set $\mathcal{H}^{\tilde{\omega}}_{K,s}$ by

$$\mathcal{H}^{\tilde{\omega}}_{K,s} = \left\{ \phi \in \mathcal{H}^{\tilde{\omega}} \mid \forall x, y \in M, d(x, y) \le a : \phi(x) \ge s, \frac{\phi(x)}{\phi(y)} \le e^{K\tilde{\omega}\left(d(x,y)\right)} \right\}, \tag{31}$$

and let $\mathcal{L} = \Big(\sup_{x \in M} |Jacf(x)|\Big)\mathcal{L}_f$, then we have

Lemma 3.12. *The set $\mathcal{H}^{\tilde{\omega}}_{K,s}$ is $\mathcal{L}$-invariant, in other words $\mathcal{L}\mathcal{H}^{\tilde{\omega}}_{K,s} \subset \mathcal{H}^{\tilde{\omega}}_{K,s}$.*

Proof. Consider $\phi \in \mathcal{H}^{\tilde{\omega}}_{K,s}, x, y \in M$ with $d(x,y) \leq a$. Let $T^{-1}(x) = \{x_1, \cdots, x_k\}$ and $T^{-1}(y) = \{y_1, \cdots, y_k\}$ such that $d(x_i, y_i) \leq \lambda d(x,y)$. Then we have

$$\begin{aligned}
\mathcal{L}_f[\phi](x) &= \sum_{i=1}^{k} \frac{1}{|Jacf(x_i)|}\phi(x_i) \\
&\leq \sum_{i=1}^{k} \frac{e^{K\omega\left(d(x_i,y_i)\right)}}{|Jacf(y_i|)}\phi(y_i)e^{K\tilde{\omega}(d(x_i,y_i))} \\
&\leq \mathcal{L}_f[\phi](y)e^{K\omega\left(\lambda d(x,y)\right)+K\tilde{\omega}\left(\lambda d(x,y)\right)},
\end{aligned}$$

and since $\tilde{\omega}(t) = \sum\limits_{k\geq 0} \omega(\lambda^k t)$,

$$\mathcal{L}[\phi](x) \leq \mathcal{L}_f[\phi](y)e^{K\tilde{\omega}\left(d(x,y)\right)}.$$

We deduce that $\mathcal{L}[\phi]$ has $\tilde{\omega}$ as a modulus of continuity. Also, by definition of $\mathcal{L}$, we have $\mathcal{L}[\phi] \geq s$, which proves the lemma.

□

We need two more technical lemmas to prove the theorem,

Lemma 3.13. *Let (h_n) be a uniformly bounded sequence in $\mathcal{H}^{\tilde{\omega}}_{K,s}$, then it has a subsequence that converges uniformly to some $h \in \mathcal{H}^{\tilde{\omega}}_{K,s}$.*

Proof. Since h_n are equicontinuous, we apply Ascoli theorem to find a subsequence that converges uniformly to a continuous map h. We have also for all $n \in \mathbb{N}$

$$h_n(x) \geq s, \ \frac{h_n(x)}{h_n(y)} \leq e^{K\tilde{\omega}\left((d(x,y)\right)},$$

which implies that $h \in \mathcal{H}^{\tilde{\omega}}_{K,s}$. □

Lemma 3.14. *There exist $A, B > 0$ such that for all $\phi \in \mathcal{H}^{\tilde{\omega}}_{K,s}$ we have*

$$B\phi \leq \mathcal{L}[\phi] \leq A\phi. \tag{32}$$

Proof. Let $\phi \in \mathcal{H}^{\tilde{\omega}}_{K,s}$ and $x \in M$ then we have

$$\mathcal{L}_f[\phi](x) = \sum_{y \in f^{-1}x} \frac{\phi(y)}{|Jacf(y)|} = \Big(\sum_{y \in f^{-1}x} \frac{\phi(y)}{\phi(x)}\frac{1}{|Jacf(y)|}\Big) \cdot \phi(x),$$

which implies that

$$\mathcal{L}_f[\phi](x) \geq \frac{ks}{\|\phi\|_\infty \sup\limits_{y\in M}|Jacf(y)|}\phi(x),$$

and since $\|\phi\|_\infty \leq s \cdot e^{K\tilde{\omega}\big(diam(M)\big)}$ we get the first inequality. For the other inequality we use the fact that

$$\mathcal{L}_f[\phi](x) = \Big(\sum_{y \in f^{-1}x} \frac{\phi(y)}{\phi(x)} \frac{1}{|Jacf(y)|} \Big) \cdot \phi(x) \leq e^{K\tilde{\omega}\big(diam(M)\big)} \frac{k}{\min |Jacf|} \phi(x).$$

□

Proof of theorem 3.11. Let $b = \sup\{t > 0 \mid \exists \phi \in \mathcal{H}^{\tilde{\omega}}_{K,s} \text{ such that } \mathcal{L}\phi \geq t\phi\}$. It is clear using the previous lemma that $b \in (0, \infty)$. Let $(b_n)_n$ an increasing sequence that converges to b, and let $(\phi_n)_n$ a sequence in $\mathcal{H}^{\tilde{\omega}}_{K,s}$ such that $\mathcal{L}\phi_n \geq b_n \phi_n$. Using lemma 3.13 we can find $\rho \in \mathcal{H}^{\tilde{\omega}}_{K,s}$ such that $\mathcal{L}\rho \geq b\rho$. If we prove that we have equality, ρ will be a fixed point for $\mathcal{L}_f$. Assume that we don't have equality, then there is some $y \in M$ such that $\mathcal{L}\rho(y) > b\rho(y)$, and by continuity we can find a neighborhood $\mathcal{U}$ of y such that

$$\mathcal{L}\rho(x) > b\rho(x), \ \forall x \in \mathcal{U}.$$

Since f is mixing, there is $n \in \mathbb{N}^*$ such that $f^n(\mathcal{U}) = M$, so we have $\mathcal{L}^n(\mathcal{L}\rho - b\rho) > 0$, which gives by linearity of $\mathcal{L}$ that

$$\mathcal{L}(\mathcal{L}^n\rho) > b\mathcal{L}^n\rho,$$

so we can choose some $c > b$ such that $\mathcal{L}(\mathcal{L}^n\rho) \geq c\mathcal{L}^n\rho$, which is a contradiction. Hence $\mathcal{L}_f[\rho] = \rho$. □

Remark 3.15. *To prove uniqueness of the ACIP μ of f, it is enough to prove that μ is ergodic. Note that being expanding and topologically mixing does not imply that μ is ergodic, Quas [Qua96] gave an example of a C^1 expanding map of the circle which is not ergodic. In the proof of existence we used a crucial property which is called distortion. The distortion also implies uniqueness of the ACIP in dimension 1.*

Lemma 3.16. *Let $f : S^1 \to S^1$ be a C^1 expanding map of the circle preserving an absolutely continuous probability μ, and assume that f' has distortion, i.e $\forall \epsilon > 0, \exists C_\epsilon > 1$ with C_ϵ goes to 0 when ϵ goes to 0, such that $\forall x \in S^1, n \in \mathbb{N}$ and $y \in B_n(x, \epsilon)$ we have*

$$C_\epsilon^{-1} \leq \left| \frac{f^{n'}(x)}{f^{n'}(y)} \right| \leq C_\epsilon, \tag{33}$$

then the measure μ is ergodic.

Proof. We can assume without loss of generality that μ is the Lebesgue measure. To prove ergodicity of f, it is enough to prove that for each measurable set A of positive measure, and $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $\mu(f^n A) > 1 - \epsilon$. Let A be a set of positive measure, and fix $\epsilon > 0$, then we can find a small interval I

and $n \in \mathbb{N}$ such that $\frac{\mu(A\cap I)}{\mu(I)} \geq 1-\epsilon^2$, and $1 > f^n(I) > 1-\epsilon^2$. Using distortion we have for any $a \in A \cap I$

$$C_\epsilon^{-1}\mu(A\cap I)|f^{n\prime}(a)| \leq \mu\big(f^n(A\cap I)\big) \leq C_\epsilon \mu(A\cap I)|f^{n\prime}(a)|,$$

and

$$C_\epsilon^{-1}\mu(I)|f^{n\prime}(a)| \leq \mu\big(f^n(I)\big) \leq C_\epsilon \mu(I)|f^{n\prime}(a)|,$$

so we deduce that

$$\frac{\mu\big(f^n(A\cap I)\big)}{\mu(f^n I)} \geq (1-\epsilon)^2 C_\epsilon^{-2}.$$

Since C_ϵ is arbitrary close to 1 when ϵ is arbitrary small, we deduce that $\mu\big(f^n(A)\big) \geq 1-\epsilon$, which finishes the proof of ergodicity. □

Remark 3.17. *If f is a C^{1+Dini} expanding map of the circle, then f' satisfies* (33)*, in particular f has a unique ACIP.*

3.3 A new proof of Decay of correlations

In this subsection, we give a new approach to prove exponential decay of correlations for a $C^{1+\alpha}$ expanding map f preserving the Lebesgue measure λ. The proof is based on controlling the total variation of the pushforward of absolutely continuous measures with reasonable density.

Fix $s \in (0, \frac{2}{3})$, and let $\rho_1, \rho_2 \in \mathcal{H}^\alpha_{C_f,s}$ where $C_f > 0$ is the constant such that $\frac{f'(x)}{f'(y)} \leq e^{C_f d(x,y)^\alpha}$ and $\mathcal{H}^\alpha_{C_f,s}$ is given by

$$\mathcal{H}^\alpha_{C_f,s} = \left\{ \phi \in C^\alpha(S^1) \;\middle|\; \phi(x) \geq s, \frac{\phi(x)}{\phi(y)} \leq e^{C_f d(x,y)^\alpha}, \forall x,y \in S^1 \right\}. \tag{34}$$

Denote by μ_1 (resp. μ_2) the measure with density ρ_1 (resp. ρ_2). We prove in the following that the Wasserstein distance between $f^n_*\mu_1$ and $f^n_*\mu_2$ converges exponentially to 0 by controlling the total variation between the two measures (see the appendix), then this exponential convergence will give us decay of correlations. We will need the following lemma to construct a coupling between $f^n_*\mu_1$ and $f^n_*\mu_2$.

Lemma 3.18 (The transfer operator regularizes)**.** *For all $K > C_f$ there is $n_0 = n_0(K,f)$ such that*

$$\mathcal{L}^{n_0}\left(\mathcal{H}^\alpha_{K,\frac{s}{2}}\right) \subset \mathcal{H}^\alpha_{C_f,s}.$$

Proof. First, we prove that there is $n_1 \in \mathbb{N}$ such that $\mathcal{L}^{n_1}\left(\mathcal{H}^\alpha_{K,\frac{s}{2}}\right) \subset \mathcal{H}^\alpha_{K,s}$. Let $\phi \in \mathcal{H}^\alpha_{K,\frac{s}{2}}$ such that $\int \phi \, d\lambda = 1$, then we can find an open interval $I \subset S^1$ of length $l = l(\alpha, K, s)$ (the length does not depend on ϕ) such that $\forall x \in I, \phi(x) \geq \frac{3}{2}s$ (otherwise $\int \phi \, d\lambda$ can not be 1). Since f is expanding, we can find $n_2 \in \mathbb{N}$

such that $f^{n_2}I = S^1$, this implies that $\forall x \in S^1$, the set $I \cap f^{-n_2}(x) \neq \emptyset$. For $x \in S^1$ let y_x be an element of $I \cap f^{-n_2}$, then we have

$$\begin{aligned}
\mathcal{L}^{n_2}(\phi)(x) &= \sum_{y \in f^{-n_2}x} \frac{\phi(y)}{(f^{n_2})'(y)} = \frac{\phi(y_x)}{(f^{n_2})'(y_x)} + \sum_{y \neq y_x} \frac{\phi(y)}{(f^{n_2})'(y)} \\
&\geq \frac{1}{(f^{n_2})'(y_x)} \frac{3}{2} s + \sum_{y \neq y_x} \frac{s}{2(f^{n_2})'(y)} \\
&= \left(\frac{1}{(f^{n_2})'(y_x)} \cdot 3 + \left(1 - \frac{1}{(f^{n_2})'(y_x)} \right) \cdot 1 \right) \frac{s}{2},
\end{aligned}$$

where the last equality comes from the fact that f preserves the Lebesgue measure. Denote by $t = \frac{1}{\sup (f^{n_2})'}$, then have

$$\mathcal{L}^{n_2}(\phi) \geq \big(t \cdot 3 + (1-t) \cdot 1 \big) \frac{s}{2}. \tag{35}$$

Using the same argument for $\mathcal{L}^{n_2}(\phi)$ we find that

$$\mathcal{L}^{2n_2}(\phi) \geq \big(t \cdot 3 + (1-t) \cdot a_1 \big) \frac{s}{2}, \tag{36}$$

where $a_1 = t \cdot 3 + (1-t) \cdot 1$. Using induction we have for $k \in \mathbb{N}^*$

$$\mathcal{L}^{(k+1)n_2}(\phi) \geq \big(t \cdot 3 + (1-t) \cdot a_k \big) \frac{s}{2}, \tag{37}$$

where $(a_n)_n$ is defined by $a_{n+1} = t \cdot 3 + (1-t) \cdot a_n$, and $a_0 = 1$. This sequence converges to 3, so there is $k \in \mathbb{N}$ such that $a_k > 2$. Set $n_1 = (k+1)n_2$, then n_1 satisfies $\mathcal{L}^{n_1}\left(\mathcal{H}^{\alpha}_{K, \frac{s}{2}} \right) \subset \mathcal{H}^{\alpha}_{K,s}$.

Now it remains to prove that we have some $n_0 \in \mathbb{N}$ such that $\mathcal{L}\left(\mathcal{H}^{\alpha}_{K,s} \right) \subset \mathcal{H}^{\alpha}_{C_f,s}$. Consider $\phi \in \mathcal{H}^{\alpha}_{K,s}$ then we have for $n \in \mathbb{N}$ and $x, y \in S^1$

$$\begin{aligned}
\mathcal{L}^n(\phi)(x) &= \sum_{x_i \in f^{-n}(x)} \frac{\phi(x_i)}{(f^n)'(x_i)} \\
&\leq \sum_{y_i \in f^{-n}(y)} \frac{\phi(y_i)}{f^{n'}(y_i)} e^{C_f d(x_i,y_i)^\alpha + K d(x_i,y_i)^\alpha} \\
&\leq \mathcal{L}^n(\phi)(y) e^{(C_f \lambda^n + K \lambda^n) d(x,y)^\alpha},
\end{aligned}$$

where $\lambda = \sup_{x \in S^1} \frac{1}{|f'(x)|} \in (0,1)$. Taking $n_0 \geq n_1$ and such that $C_f \lambda^{n_0} + K \lambda^{n_0} \leq C_f$ finishes the proof. □

Proposition 3.19. *There is $C > 0$ and $\delta \in (0,1)$ such that for all $\rho, \varrho \in \mathcal{H}^{\alpha}_{C_f,s}$, and associated μ, ν measures respectively, we have*

$$W_1(f^n_* \mu, f^n_* \nu) \leq C \delta^n. \tag{38}$$

Proof. Let $\tau = \frac{s}{2}$. Since $\rho, \varrho \in \mathcal{H}^{\alpha}_{C_f,s}$, the measure μ and ν has at least τ as common mass, in other words

$$\|\mu - \nu\|_{TV} \leq 1 - \tau.$$

Define the operator

$$T : \mathcal{M}^s(S^1) \to \mathcal{M}^{\tau}(S^1)$$
$$\mu \mapsto \frac{1}{1-\tau}(\mu - \tau\lambda),$$

where λ is the Lebesgue measure on S^1, and $\mathcal{M}^c(S^1)$ is the space of absolutely continuous probability measures with density bigger than c.

Write the measures μ and ν in the following way

$$\mu = \tau\lambda + (1-\tau)T\mu,$$
$$\nu = \tau\lambda + (1-\tau)T\nu.$$

Notice that the densities of $T\mu$ and $T\nu \in$ are in $\mathcal{H}^{\alpha}_{2K,\frac{s}{2}}$, so using Lemma 3.18, the measures $f_*^{n_0}T\mu$ and $f_*^{n_0}T\nu$ has at least a common mass 2τ, so we have in particular

$$\begin{aligned} f_*^{n_0}\mu &= \tau f^{n_0}\lambda + (1-\tau)\tau\lambda + (1-\tau)(1-\tau)T f_*^{n_0}T\mu \\ &= \big(\tau + \tau(1-\tau)\big)\lambda + (1-\tau)(1-\tau)T f_*^{n_0}T\mu, \end{aligned}$$

and a similar formula for $f_*^{n_0}\nu$. This implies that $f_*^{n_0}\mu$ and $f_*^{n_0}\nu$ have at least $\big(\tau + \tau(1-\tau)\big)$ as common mass, which implies using Corollary 7.16 that

$$W_1(f_*^{n_0}\mu, f_*^{n_0}\nu) \leq 1 - \big(\tau + \tau(1-\tau)\big).$$

Using induction we get for all $k \in \mathbb{N}$:

$$W_1(f_*^{kn_0}\mu, f_*^{kn_0}\nu) \leq 1 - \tau\sum_{i=0}^{k}(1-\tau)^i = (1-\tau)^{k+1}. \tag{39}$$

Using Proposition 7.11 in the appendix, we deduce that there is $C > 0$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} W_1(f_*^n\mu, f_*^n\nu) &= W_1(f_*^{kn_0+r}\mu, f_*^{kn_0+r}\nu) = W_1(f_*^r f_*^{kn_0}\mu, f_*^r f_*^{kn_0}\nu) \\ &\leq C W_1(f_*^{kn_0}\mu, f_*^{kn_0}\nu) \\ &\leq C(1-\tau)^k. \end{aligned}$$

Taking $\delta = \sqrt[n_0]{1-\tau}$ finishes the proof. □

Remark 3.20. *It is possible to prove decay of correlations using only Lemma 3.18. If ϕ, ψ are two densities in $\mathcal{H}^{\alpha}_{C_f}$, then we can write them as $\tau \times 1 + (1 -$*

$\tau) \times \phi_1$ and $\tau \times 1 + (1-\tau)\psi_1$ where ϕ_1, ψ_1 are two densities in $\mathcal{H}^\alpha_{2C_f}$, then applying the transfer operator to the difference we get

$$\mathcal{L}^{n_0}(\phi - \psi) = (1-\tau)(\phi_2 - \psi_2),$$

where $\phi_2 = \mathcal{L}^{n_0}(\phi_1)$ and $\psi_2 = \mathcal{L}^{n_0}(\psi_1)$, iterating this argument finishes the proof.

The Wasserstein distance is used to illustrate the method for Anosov diffeomorphisms (see Section 6).

Corollary 3.21 (exponential decay of correlations). *If f is a $C^{1+\alpha}$ expanding map of the circle preserving the Lebesgue measure λ, then it has exponential decay of correlations for Hölder observables, in other words there is $\theta \in (0,1)$ such that for all α-Hölder maps φ, ψ we have:*

$$\left| \int \varphi \ \psi \circ f^n \ d\mu - \int \varphi \ d\mu \int \psi \ d\mu \right| \le C_{\varphi,\psi} \theta^n. \tag{40}$$

The proof is similar to the proof of Corollary 6.11.

3.4 Non Existence of ACIP

Now, if we assume that our expanding map is only C^1, what can we say about the existence of an ACIP? To answer that, Gora and Schmitt gave an explicit example of a C^1 expanding map that does not admit an ACIP [GS89]. Then, Quas proved that generically C^1 expanding maps of the circle do not have an ACIP [Qua99]. Soon after, Avila and Bochi proved that generically C^1 maps of a smooth compact Riemannian manifold do not have an ACIP [AB06a]. In this subsection, we recall briefly Quas argument.

In Theorem 3.6, $\mathcal{L}^n_f[1]$ converges to the density of the invariant measure, so to find an expanding maps without an ACIP, one would consider the set

$$\mathcal{S} = \left\{ f \in E^1(S^1) \mid \liminf_{n\to+\infty} \lambda\big(\{x : \mathcal{L}^n_f[1](x) > \frac{1}{2}\}\big) = 0 \right\}, \tag{41}$$

where $E^1(S^1)$ is the set of C^1 expanding maps of the circle, and λ is the normalized Lebesgue measure of S^1. Clearly if f is $C^{1+\alpha}$, then it is not in $\mathcal{S}$, and this set is not empty, in fact we have

Lemma 3.22 ([Qua99]). *The set $\mathcal{S}$ contains a dense G_δ.*

From the previous lemma, we expect that for a generic expanding map f, the sequence $(\mathcal{L}_f[1])_n$ does not converge in the uniform convergence topology. What happens if we consider instead the sequence $\big(\frac{\mathcal{L}^n_f[h]}{\mathcal{L}^n_f[1]}\big)_{n\ge 0}$, where h is a continuous function? The following lemma answers this question.

Lemma 3.23 (Quas). *The set*

$$\mathcal{R} = \left\{ f \in E^1(S^1) \mid \forall h \in C^0(S^1), \frac{\mathcal{L}^n_f[h]}{\mathcal{L}^n_f[1]} \xrightarrow[n\to+\infty]{c.u} \int h \ d\lambda \right\} \tag{42}$$

contains a dense G_δ-set, and for all $f \in \mathcal{R}$, f is ergodic and conservative with respect to the Lebesgue measure.

Using these two lemmas, we can prove the genericity of non existence of ACIP for expanding maps.

Theorem 3.24. *The set of f in $E^1(S^1)$ which have no absolutely continuous invariant probability measure contains a dense G_δ set.*

Proof. Let $f \in \mathcal{R} \cap \mathcal{S}$, and assume that it has an ACIP $\mu = \rho\lambda$. Fix $\epsilon > 0$ and $h \in C^0(M)$ such that

$$\|f - \rho\| < \epsilon \text{ and } \int h \, d\lambda = 1.$$

We have for all $n \in \mathbb{N}$

$$\|\mathcal{L}_f^n[h] - \rho\|_1 = \|\mathcal{L}_f^n[h] - \mathcal{L}_f^n[\rho]\| \leq \|f - \rho\|_1 < \epsilon. \tag{43}$$

Since $f \in \mathcal{R}$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\left| \frac{\mathcal{L}_f^n[h]}{\mathcal{L}_f^n[1]} - 1 \right| < \epsilon,$$

which implies

$$\left|\mathcal{L}_f^n[h](x) - \mathcal{L}_f^n[1]\right| < \epsilon \mathcal{L}_f^n[1](x), \forall x \in S^1,$$

integrating the previous inequality we get

$$\|\mathcal{L}_f^n[h] - \mathcal{L}_f[1]\|_1 < 2\epsilon, \tag{44}$$

in other words $(\mathcal{L}_f^n[1])_n$ converges to ρ in the L^1 norm. On the other hand, since $f \in \mathcal{R}$, we can take a sequence $n_i \to +\infty$ such that

$$\left(\lambda(\{x \mid \mathcal{L}_f^{n_i}(x) > \frac{1}{2}\}) \right) \to 0, \tag{45}$$

we take another subsequence of n_i so that

$$\mathcal{L}_f^{n_{i_j}}[1] \xrightarrow{\text{a.e}} \rho. \tag{46}$$

The equations (45) and (46) imply that for λ-a.e x in S^1, we have $\rho(x) \leq \frac{1}{2}$, but $\int_{S^1} \rho \, d\lambda = 1$, which gives the contradiction, so the map f does not have an ACIP. □

Remarks 3.25.

- *In [AB06a], Avila and Bochi generalised this result for C^1 maps of a compact Riemannian manifold, i.e for a generic $C^1(M, M)$ map f, it does not have an ACIP. They also proved in [AB06b] that a generic expanding map of the circle does not have a σ-finite ACIM.*

- *Campbell and Quas proved in [CQ01] that a generic C^1 expanding map f has a unique equilibrium measure for its geometric potential, then using Keller argument (see [Kel98]), they proved that the equilibrium measure is physical. So we can have an expanding map such that the physical measure is not absolutely continuous with respect to Lebesgue.*

The following question might be investigated in the future.

Let ω be a modulus of continuity that does not satisfy Dini condition, then the set of $C^{1+\omega}$ expanding maps of the circle contains a G_δ dense set whose elements have no ACIP.

Remark 3.26. *In chapter 2, we took a manifold with a C^1 expanding map, a natural question to ask is whether any manifold M admit an expanding map? For instance if $M = S^2$ the two dimensional sphere, then M does not have an expanding map, because otherwise, we can find a covering map (which is the expanding map) from S^2 to itself, and since S^2 is simply connected, this is not possible. In fact, Gromov proved that if a compact manifold has an expanding map, then the expanding map is topologically conjugate to a infra-nilmanidold endomorphism. The proof uses several results. First, Franks proved in [Fra70] that if a compact manifold M admits an expanding map, then the fundamental group $\pi_1(M)$ has polynomial growth. Then the result in [Gro81] which states that if a finitely generated group Γ has polynomial growth, then Γ is virtually nilpotent (i.e it contains a nilpotent subgroup of finite index). Finally, using the result in [Shu70], which states that if $\pi_1(M)$ is virtually nilpotent, then any expanding map of M is topologically conjugate to an expanding infra-nil-endomorphism.*

3.5 Topology of Lebesgue preserving expanding maps

Denote by Λ_{Leb} the set of degree 2, C^1 expanding maps of the circle, which preserve the Lebesgue measure and the orientation. In this part, we prove that Λ_{Leb} is arc connected (we equip Λ_{Leb} with the C^1 topology).

Lemma 3.27 ([BO23])**.** *Let $a \in (0,1)$ and $f_1 : [0,a] \to [0,1]$ be a C^2 expanding diffeomorphism fixing 0, then there exists a unique extension of f_1 to a Lebesgue preserving full branch expanding transformation of the unit interval. Moreover, the extension depends continuously on the C^1 topology.*

Proof. Consider the differential equation

$$\frac{1}{f_1'\big(f_1^{-1}(x)\big)} + \frac{1}{f_2'\big(f_2^{-1}(x)\big)} = 1,\ x \in [0,\epsilon], \tag{47}$$

with initial condition $f_2(a) = 0$. We want to find a solution in the class of diffeomorphisms from $[a, a+\epsilon']$ to $[0,\epsilon]$, where ϵ, ϵ' are small positive numbers. Equation (47) is equivalent to

$$f_2'(x) = \frac{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big)}{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big) - 1},\ x \in [a, a+\epsilon]. \tag{48}$$

Since f_1 is C^2, the Cauchy problem with the initial condition $f_2(a) = 0$ admits a solution. Let f_2 be a maximal solution for (48), denote by $[0, x_{max}]$ the maximal interval of definition, and $y_{max} = f_2(x_{max})$.

Let's show that $x_{\max} = y_{\max} = 1$. Denote by $f : [0, x_{\max}] \to \mathbb{R}$ the map given by f_1 on $[0, a]$ and f_2 on $[a, x_{\max}]$.

Using (47), we have $f_2'(x) > 1$ for all $x \in [0, x_{max}]$, so f_2 is an expanding diffeomorphism into its image.

We have for all continuous function h on $[0, 1]$

$$\begin{aligned}\int_0^{x_{\max}} h \circ f \; dLeb &= \int_0^a h \circ f_1 \; dLeb + \int_a^{x_{\max}} h \circ f_2 \; dLeb \\ &= \int_0^1 \frac{h}{f_1' \circ f_1^{-1}} \; dLeb + \int_0^{y_{\max}} \frac{h}{f_2' \circ f_2^{-1}} \; dLeb.\end{aligned}$$

If $y_{\max} = 1$, then taking $h = 1$ shows that $x_{\max} = 1$. If $y_{\max} < 1$, then we have

$$\int_0^{x_{\max}} h \circ f \; dLeb = \int_0^{y_{\max}} h \; dLeb + \int_{y_{\max}}^1 \frac{h}{f_1' \circ f_1^{-1}} \; dLeb.$$

Taking $h = 1$ proves that $x_{\max} = 1$, then taking $h = 0$ on $[0, y_{\max}]$, affine and positive on $(y_{\max}, 1]$ gives $\int_{y_{\max}}^1 h \; dLeb = \int_{y_{\max}}^1 \frac{h}{f' \circ f^{-1}} \; dLeb < \frac{1}{1+\epsilon} \int_{y_{\max}}^1 h \; dLeb$, which gives a contradiction. $y_{\max}$ can not be bigger than 1, because of the equation (47). This proves that $x_{\max} = y_{\max} = 1$. Using equation (47) we deduce that f preserves the Lebesgue measure. □

Remarks 3.28.

- *The map f_1 can be assumed to be C^1 only, in this case to solve equation (47), we use Peano existence theorem. Uniqueness comes from the fact that the extension preserves the Lebesgue measure. Indeed, assume first that $h_1, g_1 : [a, 1] \to [0, 1]$ are two extension of f_1 to a Lebesgue preserving maps h and g respectively, then we have for all $x \in [0, 1]$*

  $$Leb(h^{-1}[0, x]) = Leb([0, x]) = Leb(g^{-1}[0, x]),$$

  which implies that $Leb([a, h_1^{-1}(x)]) = Leb([a, g_1^{-1}(x)])$, hence $h_1 = g_1$.

- *If $f \in \Lambda_{Leb}$, then f has a unique fixed point x_1, and there is $x_f \in S^1$ such that $\int_{x_1}^{x_f} f'(t) \; dt = 1$, and $f'(x_f) = \frac{f'(x_1)}{f'(x_1)-1}$, which makes a relationship between expanding maps of the circle, and the interval $[0, 1]$ full branches map. This remark implies that Λ_{Leb} is arc connected. Moreover, it gives us a parametrization of Λ_{Leb}, in particular the fundamental group of Λ_{Leb} is $\mathbb{Z}$.*

Theorem 3.29 ([BO23])**.** *The space Λ_{Leb} of C^1 expanding and Lebesgue preserving maps of degree 2 is arc connected.*

Proof. Let f be the doubling map of the circle, and $g \in \Lambda_{Leb}$. Up to composing g with a rotation, we can assume that g and f have the same fixed point 0. Denote by x_g the point in S^1 such that $\int_0^{x_g} g'(t)\, dt = 1$. Assume that $x_g = \frac{1}{2}$, and let $(h_t)_{t\in[0,1]}$ be a homotopy between $f_{|[0,\frac{1}{2}]}$ and $g_{|[0,\frac{1}{2}]}$ such that for all $t \in [0,1]$, h_t is a C^1 expanding map satisfying $h_t(0) = \frac{h_t'(\frac{1}{2})}{h_t'(\frac{1}{2})-1}$. Using lemma 3.27, h_t can be extended to a map H_t in $\Lambda_L eb$. In this case, $(H_t)_t$ is a homotopy between f and g.

Now, assume that $x_g < \frac{1}{2}$. Let $k_1 : [0, x_g] \to [0,1]$ be a C^1 convex diffeomorphism satisfying $k(0) = \frac{k(x_g)}{k(x_g)-1}$, then the previous argument and lemma 3.27 implies that k_1 can be extended to a map $k \in \Lambda_{Leb}$ which is in the arc connected component of g. It remains to prove that k is in the arc connected component of f. For $t \in [0,1]$ let $h_t : [0, (1-t)x_g + \frac{t}{2}]$ be a concave function that can be extended to an expanding and volume preserving transformation of the circle (see figure 1), then $(h_t)_t$ can be extended to $(H_t)_t$, which shows that k is in the connected component of f. □

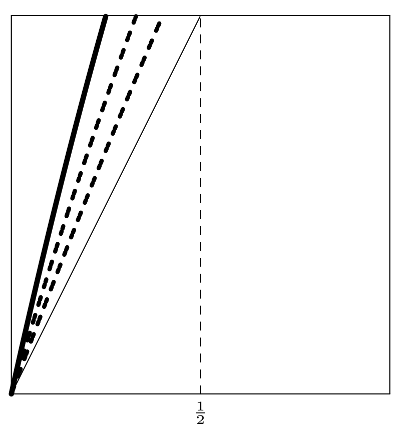


Figure 1: A homotopy between f and h.

4 Uniformly hyperbolic maps

In this section, we introduce the notion of uniform hyperbolicity. One can deduce from this notion a lot of properties on the dynamic of the underlying map, for instance chaos. For a more detailed introduction to this subject see [KH97, BS02].

4.1 Uniform hyperbolicity

Let U be an open subset of a compact Riemannian manifold M, and $f : U \to M$ a C^1 diffeomorphism.

Definition 4.1. *An invariant set $\Lambda \subset U$ is called hyperbolic if there are some $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ we have a splitting of $T_xM = E^u_x \oplus E^s_x$ which is f invariant, i.e $df_x(E^u_x) = E^u_{fx}$ and $df_x(E^s_x) = E^s_{fx}$ and such that*

$$\|df^n(v)\| \leq C\lambda^n\|v\|, \forall n \in \mathbb{N},\ v \in E^s_x, \tag{49}$$

$$\|df^{-n}(v)\| \leq C\lambda^n\|v\|, \forall n \in \mathbb{N},\ v \in E^u_x. \tag{50}$$

In the definition, we didn't assume any continuity on the distributions E^s and E^u. In fact it is not hard to prove the continuity of E^u and E^s starting from the given definition. When $\Lambda = M$, f is called an Anosov diffeomorphism.

Definition 4.2. *A hyperbolic set Λ is called an attractor if there is an open set $U \supset \Lambda$, such that $f(U) \subset U$ and $\bigcup\limits_{n\in\mathbb{N}} f^n U\Lambda$.*

Proposition 4.3. *The distributions E^s and E^u are continuous.*

Proof. We can assume that $C = 1$ by considering another equivalent Riemannian metric on M, and taking $\lambda' \in (\lambda, 1)$ (see Proposition 5.2.2 of [BS02]). Let $(x_n)_n$ be a sequence in Λ that converges to x. If $E^u_x = T_xM$ (resp $E^s_x = T_xM$), then df_{x_n} expands (resp contracts) vectors in the tangent space $T_{x_n}M$, in other words $E^u_{x_n} = T_{x_n}M$ (resp $E^s_{x_n} = T_{x_n}M$). Assume that the splitting $E^u \oplus E^s = T_xM$ is not trivial. By choosing a subsequence of $(x_n)_n$ we can assume that $dim E^s_{x_n} = c$, and since the Grassmannian fiber over M by c-dimensional subvector spaces is compact, there is another subsequence of $(x_n)_n$ such that $(E^s_{x_n})_n$ converges to some $E \subset T_xM$ of dimension c. Since f is C^1, df_x is contracting on E hence $E \subset E^s_x$, the same argument gives $F \subset E^u_x$ and df_x is expanding on F. Since for all $n \in \mathbb{N}, E^u_{x_n} \oplus E^s_{x_n} = T_{x_n}M, E \subset E^s_x, F \subset E^u_x$ and $E^u_x \oplus E^s_x = T_xM$, we deduce that $E = E^s_x$ and $F = E^u_x$.

□

Here are some classical examples of hyperbolic maps.

Examples 4.4.

- *Arnold Cat map: consider the automorphism* $f : \mathbb{T}^2 \to \mathbb{T}^2$ *given by the transformation* $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. *The matrix* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *has two eigenvalues* $\lambda, \frac{1}{\lambda}$, *where* $\lambda = \frac{2}{3+\sqrt{5}}$. *Let* E_0^u *(resp* E_0^s*) be the eigenspace corresponding to* $\frac{1}{\lambda}$ *(resp* λ*). Since* $df_x = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, *the splitting* $T_x\mathbb{T}^2 = E_0^u \oplus E_0^s$ *is* f*-invariant, and for all* $n \in \mathbb{N}$ *and* $v \in E_0^u$ *(resp* $v \in E_0^s$*) we have*

  $$\|df_x^n v\| = \lambda^n \|v\| \text{ (resp } \|df_x^{-n} v\| = \lambda^n \|v\|),$$

  so f *is hyperbolic.*

  In general any matrix in $SL_n(\mathbb{Z})$, *which does not have eigenvalue on the unit circle, induces an Anosov diffeomorphism on* $\mathbb{T}^n$.

  More generally, let G *be a simply connected Lie group, and* Γ *a lattice on* G. *Let* A *be an automorphism of the Lie group* G *such that* $A(\Gamma) = \Gamma$, *and* $d_e A$ *is hyperbolic, then* A *induces an Anosov diffeomorphism on* G/Γ.

  Remarks 4.5.

  - *If a Lie group* G *has an automorphism* A *with* $d_e A$ *hyperbolic, then* G *is necessarily nilpotent. (see [KH97])*
  - *Notice that for any* $A \in SL_n(\mathbb{Z})$, A *preserves the Lebesgue measure on* $\mathbb{T}^n$, *we prove later that any transitive* C^{1+Dini} *Anosov diffeomorphism preserves a SRB measure.*

- *The Smale Horseshoe: to construct it, consider an injective* C^∞ *immersion* $f : R \to \mathbb{R}^2$, *where* $R := [0,1]^2$, *that acts as follows*

Figure 2: Action of the first iterate of f on R.

  So we apply a linear horizontal contraction and vertical expansion, then we bend the new rectangle to a "horseshoe".

  Let H_0^1 *and* H_1^1 *be the two connected component of* $H^1 := f(R) \cap R$, *and* V_0^1, V_1^1 *be the set of points in* R *such that* $f(V_i^1) = H_i$.

  Let $H^n := \bigcap_{i=0}^{n} f^i R$, *which consists of* 2^n *horizontal connected component* H_i^n, *and let* $V^n := \bigcup_{i=0}^{n} V_i^n$. *Then the set* $C = \bigcap_{n \in \mathbb{N}} (H^n \cap V^n)$ *is a Cantor set of zero Lebesgue measure, moreover* $f(C) = C$, *and* $f : C \to C$ *is called the horseshoe map. This map* $f : C \to C$ *is hyperbolic by construction,*

Figure 3: Horizontal and vertical strips.

in this case $E^u(x)$ is the X-axis and $E^s(x)$ is the Y-axis for all $x \in C$. Moreover C is not an attractor, and f does not have a physical measure. In [Bow75b], Bowen constructed a horseshoe of positive measure, and the restriction of the Lebesgue measure on C is an ergodic invariant measure. We emphasize that his example is C^1 but not C^{1+Dini}.

Remark 4.6. *Katok proved in [Kat80] that any $C^{1+\alpha}$ diffeomorphism of a compact surface with positive topological entropy, contains a horseshoe.*

- *Smale-Williams solenoid: consider the solid torus $\mathcal{T} = \mathbb{D}^2 \times S^1$ and define the transformation $f : \mathcal{T} \to \mathcal{T}$ by*

$$f(x, y, \theta) = \big(\frac{1}{4}x + \frac{1}{2}\cos\theta, \frac{1}{4}y + \frac{1}{2}\sin\theta, 2\theta\big).$$

This transformation is contracting by a factor $\frac{1}{4}$ in the $\mathbb{D}^2$ direction, and expanding by a factor 2 in the S^1 direction. It is also injective, and $f(\mathcal{T}) \subset \mathcal{T}$, and moreover the set $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(\mathcal{T})$ is a hyperbolic attractor for f.

Consider the section $S_0 = \mathbb{D}^2 \times \{0\}$, then $C = \bigcap_{n\in\mathbb{N}} f^n(S_0)$ is a Cantor set, and locally, Λ is homeomorphic to $C \times (-\epsilon, \epsilon)$.

Notice that the Lebesgue measure of Λ is 0, because $Jacf_{(x,y,\theta)} = \frac{1}{8}$, and since any invariant measure for f is supported in Λ, f does not have an invariant measure equivalent to Lebesgue. In that case, a natural invariant measures to consider are physical (2.27) and SRB measures (2.33). In this case, f has a unique SRB measure μ, which is also physical, and locally, μ is the product of the Bernoulli measure on the Cantor set C and the Lebesgue measure in the S^1 direction.

A classical approach to deal with uniform hyperbolic maps, is to consider the space of continuous (resp. bounded) sections $\sigma : \Lambda \to T\Lambda$, which is a Banach space, and once we have the first definition of hyperbolicity, we can write this Banach space as the direct sum of two closed subspaces, corresponding to sections with values on E^u or E^s. So we have a natural linear action of f on that Banach space which preserves the closed subspaces. This approach helps us prove a lot of result like shadowing lemma, local stability, etc (see [Yoc95]).

In general, it is hard to check uniform hyperbolicity using Definition 4.1 (for instance we don't know E^u and E^s), to deal with this difficulty we study cones instead of linear subspaces.

4.2 Hyperbolicity via cone techniques

Let $x \in M$ and E be a linear subspace of T_xM, define the cone centered at E by

$$K_\alpha^E(x) = \left\{v \in T_xM : \|v_2\| \leq \alpha\|v_1\| \text{ where } v = v_1 + v_2 \text{ and } v_1 \in E, v_2 \in E^\perp\right\}.$$

For a hyperbolic map f, $K_\alpha^{E^u}$ (resp $K_\alpha^{E^s}$) is called unstable (resp stable) cone field. We say that it has a small angle if α is small.

A cone field K on M is said to be invariant by f if for all $x \in M$

$$df_x\big(K(x)\big) \subset int\big(K(fx)\big) \cup \{0\}.$$

Proposition 4.7. *(Proposition 5.4.3 [BS02]) Let Λ be a compact invariant set of $f : U \to M$. Suppose that there is $\alpha > 0$ and for every $x \in \Lambda$ there are continuous subspaces $\tilde{E}^s$ and $\tilde{E}^u(x)$ such that $\tilde{E}^s(x) \oplus \tilde{E}^u(x) = T_xM$, and the cone $K_\alpha^{\tilde{E}^u}(x)$ and $K_\alpha^{\tilde{E}^s}(x)$ are f invariant and $\|df_xv\| < \|v\|$ for non zero $v \in K_\alpha^{\tilde{E}^s}(x)$, and $\|df_x^{-1}v\| < \|v\|$ for non-zero $v \in K_\alpha^{\tilde{E}^u}(x)$. Then Λ is a hyperbolic set of f.*

4.3 Local manifold theory

In this subsection let $f : \Lambda \to \Lambda$ be a C^r hyperbolic map. Define the local stable and unstable manifolds of $x \in \Lambda$ by

$$W_\epsilon^u(x) = \{y \in M \mid d(f^{-k}x, f^{-k}y) \leq \epsilon, \forall k \geq 0\},$$
$$W_\epsilon^s(x) = \{y \in M \mid d(f^{k}x, f^{k}y) \leq \epsilon, \forall k \geq 0\}.$$

The definition of stable and unstable manifolds is dynamic, and the theorem of Hadamard-Perron says that there is $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and $x \in \Lambda$, $W_\epsilon^u(x)$ (resp. $W_\epsilon^s(x)$) is a C^r submanifold tangent to E^u (resp. E^s). Now, define the global unstable (resp stable) manifold of a point x by

$$W^u(x) = \{y \in M \mid d(f^{-k}x, f^{-k}y) \xrightarrow[k\to+\infty]{} 0\},$$
$$W^s(x) = \{y \in M \mid d(f^{k}x, f^{k}y) \xrightarrow[k\to+\infty]{} 0\},$$

then we have for any $\epsilon > 0$

$$W^u(x) = \bigcup_{n\in\mathbb{N}} f^n\big(W_\epsilon^u(f^{-n}x)\big),\ W^s(x) = \bigcup_{n\in\mathbb{N}} f^{-n}W_\epsilon^s(f^n x)\big).$$

We give the sketch of the proof of Hadamard-Perron (for more details see Theorem 17.4.3 in [KH97]). For simplicity, let x be a fixed point of f. Consider the set of L-Lipschitz submanifolds of dimension $dimE^u_x$ passing through x, more precisely

$$\mathcal{G}_L = \{\exp_x\big(Graph(\varphi)\big) \mid \varphi : B(x_0, r) \cap E^u_{x_0} \to E^s_{x_0}, \text{ with } \varphi \text{ is } L\text{-Lipschitz}\}.$$

The map f acts on $\mathcal{G}_L$ for some $L > 0$, this action is called graph transform, and we will prove that it is contracting for the uniform topology on $\mathcal{G}_L$, hence it has a unique fixed point. This unique fixed point is the local unstable manifold. Then use the fact that it is tangent to the continuous distribution E^u to prove that it is C^1.

Let $\tilde{f} = \exp_x^{-1} \circ f \circ \exp_x$ in a neighborhood $B(0, r)$. For all $\epsilon > 0$ small, we can choose r small enough, so we can write $\tilde{f}(x, y) = \big(Ax + \alpha(x, y), By + \beta(x, y)\big)$, where $A : E^u_{x_0} \to E^u_{x_0}$ (resp $B : E^s_{x_0} \to E^s_{x_0}$) is linear and expanding (resp contracting), and $\|\alpha\|_{C^1}, \|\beta\|_{C^1} \leq \epsilon$.

Claim 4.8. *There is $L > 0$ such that for all L-Lipschitz functions*

$$\varphi : B(x_0, r) \cap E^u_{x_0} \to E^s_{x_0},$$

the set $\tilde{f}(Graph\ \varphi)$ is a graph of a L-Lipschitz function

$$\tilde{\varphi} : B(x_0, r) \cap E^u_{x_0} \to E^s_{x_0}.$$

Proof. For any $\delta > 0$, there is $r > 0$ small such that $\|\alpha\|_{C^1}\|, \|\beta\|_{C^1}\| \leq \delta$, for a given small δ. Choose $L > 0$ such that $\lambda\delta(1 + L) < 1$. Consider the map $G_\varphi : E^u_r(x_0) \to E^u(x_0)$ given by

$$G_\varphi(x) = Ax + \alpha(x, \varphi x), \tag{51}$$

which represents the E^u coordinates of $\tilde{f}(graph\varphi)$.

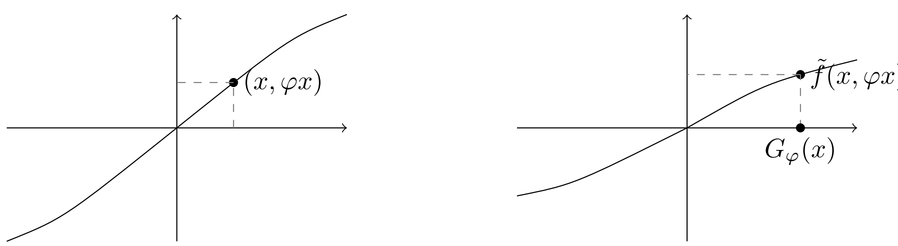


Figure 4: Graph transform

To prove that $\tilde{f}(\varphi)$ is a graph of a function, it is enough to prove that G_φ is injective, and its image covers $E^u_r(x_0) = E^u(x) \cap B(0, r)$. Thus for $y \in E^s_r(x_0)$, we need to find a unique $x \in E^u_r(x_0)$, such that $G_\varphi(x) = y$, or equivalently, the map $F_y(x) = A^{-1}y - A^{-1}\alpha(x, \varphi x)$ has a unique fixed point.

The map F is contracting because

$$\begin{aligned}\|F(x_1) - F(x_2)\| &= \|A^{-1}\alpha(x_1, \varphi x_1) - A^{-1}\alpha(x_2, \varphi x_2)\| \\ &\leq \|A^{-1}\|\|\alpha(x_1, \varphi x_1) - \alpha(x_2, \varphi x_2)\| \\ &\leq \lambda\delta(1 + L)\|x_1 - x_2\|.\end{aligned}$$

So $\tilde{f}(Graph\ \varphi)$ is a graph a a function $\tilde{\varphi}$. It remains to show that $\tilde{\varphi}$ is L-Lipschitz. Suppose that $\tilde{\varphi}(x_1') = y_1'$ and $\tilde{\varphi}(x_2') = y_2'$ and take $(x_1, y_1), (x_2, y_2) \in Graph(\varphi)$ such that

$$(x_i', y_i') = \tilde{f}(x_i, y_i) = \big(Ax_i + \alpha(x_i, \varphi x_i), B\varphi x_i + \beta(x_i, \varphi x_i)\big), \tag{52}$$

then we have

$$\begin{aligned}
\|y_2' - y_1'\| &= \|B(\varphi x_2 - \varphi x_1) + \beta(x_2, \varphi x_2) - \beta(x_1, \varphi x_1)\| \\
&\leq \lambda L \|x_2 - x_1\| + \delta(1+L)\|x_2 - x_1\| \\
&\leq \big(\lambda L + \delta(1+L)\big)\|x_2 - x_1\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_2' - x_1'\| &= \|A(x_2 - x_1) + \alpha(x_2, \varphi x_2) - \alpha(x_1, \varphi x_1)\| \\
&\geq \frac{1}{\lambda}\|x_2 - x_1\| - \delta(1+L)\|x_2 - x_1\| \\
&= \Big(\frac{1}{\lambda} - \delta(1+L)\Big)\|x_2 - x_1\|,
\end{aligned}$$

we deduce that $\|\tilde{\varphi}(x_2') - \tilde{\varphi}(x_1')\| \leq \frac{\lambda L + \delta(1+L)}{\frac{1}{\lambda} - \delta(1+L)}\|x_2' - x_1'\|$, and since

$$\frac{\lambda L + \delta(1+L)}{\frac{1}{\lambda} - \delta(1+L)} \leq L,$$

we deduce that $\tilde{\varphi}$ is L-Lipschitz. Hence, we proved that f acts on $\mathcal{G}_L$ by graph transform. □

Consider the distance on $\mathcal{G}_L$ given by

$$d(\varphi, \psi) := \sup_{x \in E_r^u(x_0)} \frac{\|\varphi(x) - \psi(x)\|}{\|x\|}. \tag{53}$$

It is a well defined distance, $\mathcal{G}_L$ equipped with this distance is a complete metric space, and the action of the graph transform is contracting with respect to this distance, more precisely

Claim 4.9. *For all $\varphi, \psi \in \mathcal{G}_L$, we have*

$$d(\tilde{\varphi}, \tilde{\psi}) \leq \frac{\lambda + \delta(1+L)}{\frac{1}{\lambda} - \delta(1+L)} d(\varphi, \psi). \tag{54}$$

The unique fixed point is a candidate for the local unstable manifold of x_0, using the fact that E^u is the tangent set of the graph of this fixed point (see Definition 6.2.18 in [KH97]), we deduce that the fixed point is C^1, then we can check that points in the graph of the fixed point get exponentially close when we apply f^{-1} to this graph.

4.3.1 Stable and unstable holonomies

Given an Anosov diffeomorphism f, we have two foliations, the stable $\mathcal{F}^s$ and unstable $\mathcal{F}^u$ foliations. Consider two local transversal, τ_1, τ_2 to $\mathcal{F}^s$, then when it make sense, we call a holonomy map h_{τ_1,τ_2} the map defined from τ_1 to τ_2, that associates for each element in τ_1 an element in τ_2 by sliding along stable leaf, in other words $h_{\tau_1,\tau_2}(x) = W^s_{\text{loc}}(x) \cap \tau_2$ (we define similarly the holonomy defined by sliding along unstable leaves).

Since local stable leaves depend continuous in the base point (See The Inclination Lemma in [KH97] page 257), the map h_{τ_1,τ_2} is continuous. One can prove in fact that it is Hölder continuous. This map in not necessarily absolutely continuous, an example of C^1 Anosov diffeomorphism was given by [RY80]. We will prove later that if f is $C^{1+\text{Dini}}$ then the stable and unstable holonomies are absolutely continuous.

4.4 An example of a hyperbolic map without a SRB measure

Fix a C^1 expanding map T of the circle that does not have an absolutely continuous invariant measure, and let $\mathcal{T} = \mathbb{D}^2 \times S^1$ be the open solid torus. Define a map $f : \mathcal{T} \to \mathcal{T}$ by

$$f(x, y, \theta) := \big(\frac{1}{4}x + \frac{1}{2}\cos\theta, \frac{1}{4}y + \frac{1}{2}\sin\theta, T(\theta)\big). \tag{55}$$

The f is a hyperbolic map on $\Lambda = \bigcap_{n\geq 0} f^n(U)$, and any invariant measure for f is supported in Λ. If f has a SRB measure μ, then this measure projects to an invariant measure for T which is absolutely continuous with respect to the Lebesgue measure, which proves that f does not have a SRB measure.

Remark 4.10. *This example is a skew-product over the map T, and in [Klo21], a dictionary between properties for the maps is given by Theorem A.*

Definition 4.11. *Let $f : M \to M$ be a map. A point x in non-wandering for f if for every neighborhood U of x,*

$$U \cap \bigcup_{n>0} f^n(U) \neq \emptyset.$$

The set of non-wandering points is denoted by $\Omega(f)$, and it is a closed set.

Definition 4.12. *A diffeomorphism $f : M \to M$ is said to be Axiom A if $\Omega(f)$ is hyperbolic, and periodic points are dense in $\Omega(f)$.*

In the sequel, f will be assumed to be an Axiom A diffeomorphism.

The main classical theorem in this section is

Theorem 4.13. *[Sinai, Ruelle, Bowen] If $f : \Omega(f) \to \Omega(f)$ is a $C^{1+\alpha}$ transitive Anosov diffeomorphism (Axiom A diffeomorphism) then it has a SRB measure which is physical.*

We prove in Section 5 this theorem in weak regularity using the same approach to prove Theorem 4.13.

4.5 Constructing SRB measures using the coding

In this subsection we recall the classical approach to the construction of the SRB measure for Axiom A diffeomorphisms. Consider an Axiom A diffeomorphism $f : \Omega(f) \to \Omega(f)$. Using Smale-decomposition Theorem (See [Bow75a]), we can assume that f transitive in $\Omega(f)$.

Bowen proved in [Bow75a] that $f : \Omega(f) \to \Omega(f)$ has a Markov partition, in particular it is semiconjugated to a subshift of finite type. More precisely, there are (Σ_A, σ) and a surjective Hölder map $\pi : \Sigma_A \to \Omega(f)$ such that $\pi \circ \sigma = f \circ \pi$.

If we take a potential with Dini summable modulus $\phi : \Omega(f) \to \mathbb{R}$, then $\pi \circ \sigma$ has a Dini summable modulus. To get an equilibrium state for $(\Sigma_A, \sigma, \pi \circ \phi)$ one can after several lemmas consider only the one-sided shift $(\Sigma_A^+, \sigma, \tilde{\phi})$, (where $\tilde{\phi}$ is a potential depending only on the future, and cohomologous to $\pi \circ \phi$ [PP90]) which is an expanding map, then we apply the adapted Ruelle-Perron-Frobenius theorem [FJ01a] to get an equilibrium measure for $(\sigma, \pi \circ \phi)$. Pushing this measure by π gives us an equilibrium measure for $\left(f_{\Omega(f)}, \phi\right)$. Finally we can use either Keller argument [Kel98] or the proof in [Bow75a] to prove that the equilibrium measure is physical.

We recall that a set of small diameter $\mathcal{R}$ is called a proper rectangle if $\mathcal{R}$ is invariant by Bowen brackets, i.e for $x, y \in \mathcal{R}$ the point $[x, y] \in \mathcal{R}$, and $\overline{int(\mathcal{R})} = \mathcal{R}$.

Definition 4.14. *A Markov partition for $\Omega(f)$ is a finite cover*

$$\mathcal{C} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m\},$$

such that

i. $int(\mathcal{R}_i) \cap int(\mathcal{R}_j) = \emptyset, \ \forall i \neq j.$

ii. $f\left(W^s_\epsilon(x) \cap R_i\right) \subset W^s_\epsilon(fx) \cap \mathcal{R}_j$ *and* $W^u_\epsilon(fx) \cap \mathcal{R}_j \subset f\left(W^u_\epsilon(x) \cap \mathcal{R}_i\right)$

Theorem 4.15 ([Bow75a])**.** *$\Omega(f)$ has a Markov partition of arbitrarily small diameter.*

Fix a finite Markov partition $\mathcal{C} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m\}$ of diameter ϵ less than the expansivity constant of f, then for $x \in \Omega(f)$, consider the coding $(\cdots, a_{-1}, a_0, a_1, \cdots)$ of the orbit $(f^k)_{k \in \mathbb{Z}}$ with respect to the partition $\mathcal{C}$, where for each k, $a_k \in \{1, \cdots, m\}$ and $f^k x \in \mathcal{R}_{a_k}$. We stress that the coding is not unique, but if $x, y \in \Omega(f)$ have the same coding, then for all $k \in \mathbb{Z}, d(f^k x, f^k y) < \epsilon$, which implies by expansivity that $x = y$.

Consider the set

$$\Sigma_C = \{(a_k)_{k \in \mathbb{Z}}, \text{where } (a_k)_k \text{ is a coding of a point } x \in \Omega(f)\},$$

then Σ_C is invariant by the shift map σ. Since elements of $\mathcal{C}$ are closed, the set Σ_C is closed. Denote by $\pi : \Sigma \to \Omega(f)$ the map which associates to a sequence $(a_k)_k$ the unique point in $\Omega(f)$ whose coding is given by that sequence, then we have by definition $f \circ \pi = \pi \circ \sigma$, moreover the map π is Hölder continuous.

Denote by μ the unique equilibrium measure for the geometric potential $\phi^{(u)} = \log Jac^u f$. In [Kel98] Keller proved, in Theorem 6.1.8, that for Lebesgue almost every point $x \in M$, any weak* limit ν of $(\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x})_n$ satisfies $h_\nu(f) + \int \phi^{(u)}\, d\nu \geq 0$. Since the pressure $P(\phi^{(u)}) := h_\mu(f) + \int \phi^{(u)}\, d\mu = 0$, and by unicity of the equilibrium measure, we deduce that for Lebesgue a.e $x \in M$, $(\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x})_n$ converges to μ, in other words, the measure μ is a physical measure.

Using this argument, [Qiu11] proved that a generic C^1 Axiom A diffeomorphism has a unique physical measure.

Remark 4.16. *It is not known whether a generic C^1 axiom A has a SRB measure or not.*

5 Existence of SRB measure in weak regularity

In this section, we prove existence of SRB measures in low regularity (Theorem 1.1).

Theorem 5.1. *Let $f : \Omega(f) \to \Omega(f)$ be a transitive $C^{1+\omega}$ Axiom A diffeomorphism, and ω satisfies Dini condition, then f has a SRB measure.*

5.1 Regularity of stable and unstable distributions

Anosov proved that the stable and unstable distributions of a C^2 hyperbolic diffeomorphism f are Hölder continuous. The regularity of the stable and unstable distributions is optimal in the sense that; even if f is analytic, the stable and unstable distribution are only Hölder continuous in general (examples are given in the famous paper of Anosov [Ano67a]). Let $f : U \subset\to M$ be a diffeomorphism, and Λ a closed and f-invariant subset of M that is hyperbolic for f, then we have

Theorem 5.2. *If f is C^{1+Dini} hyperbolic map, then the stable E^s and unstable E^u distributions have a modulus of continuity that satisfies Dini condition.*

To prove this theorem, fix a small $\epsilon > 0$, and approach E^s (resp E^u) by a smooth distribution $\tilde{E}^s$ (resp $\tilde{E}^u$) (in other words $\|E^* - \tilde{E}^*\|_\infty < \epsilon$) and extend them to an open neighborhood U of $\Omega(f)$. Consider a distribution Y close to $\tilde{E}^u$, we can view this distribution as a map F defined on M with values on $L(\tilde{E}^u, \tilde{E}^s)$ as follows

$$F(x) : \tilde{E}^u \to \tilde{E}^s, \tag{56}$$

where $F(x)$ is the linear map whose graph is $Y(x)$.

Let $\mathcal{S}^u := C^0\big(M \to L(\tilde{E}^u, \tilde{E}^s)\big)$ the space of such maps, and for $F \in \mathcal{S}^u$ let $\|F\|_\infty = \max\limits_{x\in M} \|F(x)\|$.

Let A, B, C, D the maps defined on M as follows; for $x \in M$, we have $df_x = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$, written with respect to the splitting $\tilde{E}^s \oplus \tilde{E}^u$.

The diffeomorphism f acts on $\mathcal{S}^u$ by pushing forward distributions. If $F \in \mathcal{S}^u$, let $\mathfrak{L}F$ denote the map associated to the push-forward of F. Then we have

$$\mathfrak{L}F(fx) = [A(x)F(x) + B(x)]\,[C(x)F(x) + D(x)]^{-1}. \tag{57}$$

Notice that we have $\|A(x)\|, \|D^{-1}(x)\| < \lambda + \epsilon$, $\|C(x)\|, \|B(x)\| < \epsilon$, and

$$\|(C(x)F(x) + D(x))^{-1}\| \le \lambda + 4\epsilon.$$

5.1.1 Modulus of continuity of a distribution

To talk about the modulus of continuity of a distribution, we can either use local charts on M, so locally a distribution is a collection of matrices, or as in [Ano67a], we use parallel transport to compare between two subspaces in different tangent spaces. To do this we need the following lemma,

Lemma 5.3 ([Ano67b]). *Let D be a smooth distribution on the Riemannian manifold M, then there is a connection on the bundle E (with base M and fiber D) such that local parallel transport along geodesics (with respect to the Riemannian metric on M) is an isometry.*

Consider the quadruplet (E, F, g, h), where E and F are two smooth distributions, and $g, h : M \to M$ are smooth maps. Consider a continuous map ϕ defined on M to linear maps as follows

$$\phi(x) : E_{gx} \to F_{hx}.$$

Let $\Pi^u_{x',x} : E_{x'} \to E_x$ (resp. $\Pi^s_{x',x} : F_{x'} \to F_x$) be the parallel transport of the connection associated to E (resp. F) given by Lemma 5.3, then we define the modulus of continuity of ϕ as follows

$$\omega_\phi(r) = \sup_{x,x' \in M, d(x,x') \leq r} \left\| \Pi^s_{hx',hx} \phi(x') \Pi^u_{gx,gx'} - \phi(x) \right\|, \tag{58}$$

where the norm here is the usual norm on $L\big(E_{gx}, F_{hx}\big)$.

For a distribution F, let $\|F\| = \sup_{x \in M} \|F(x)\|$, and if $\|F\| \leq 1$, then $\|\mathfrak{L}F\| \leq 1$. Using this definition for the modulus of continuity, we have

Proposition 5.4 ([Ano67b]). *Fix (E, F, g_0, h_0) a smooth quadruplet, then we have for all continuous maps $\psi, \phi : M \to Linear(E_{g_0} \to F_{h_0})$*

$$\omega_{\phi+\psi}(r) \leq \omega_\phi(r) + \omega_\psi(r). \tag{59}$$

We also have whenever it makes sense

- $\omega_{\phi \circ g}(r) \leq \omega_\phi(\omega_g(r))$.
- $\omega_{\phi^{-1}}(r) \leq \|\phi^{-1}\|^2 \omega_\phi(r)$.
- $\omega_{\psi\phi}(r) \leq \|\psi\| \omega_\phi(r) + \|\phi\| \omega_\psi(r)$.

Now we give the proof of the main theorem of this thesis.

Proof of Theorem 5.2. Denote by ω the modulus of continuity of Df, which is Dini summable, and assume that $\|F\| \leq 1$. Using the formula (57), we have

$$\begin{aligned}
\omega_{\mathfrak{L}F \circ f}(r) &= \omega_{(AF+B)(CF+D)^{-1}}(r) \\
&\leq \|AF+B\| \omega_{(CF+D)^{-1}}(r) + \|(CF+D)^{-1}\| \omega_{(AF+B)}(r) \\
&\leq \|AF+B\| \|(CF+D)^{-1}\| \omega_{(CF+D)}(r) + \|(CF+D)^{-1}\| \omega_{(AF+B)}(r) \\
&\leq (\lambda + 4\epsilon)^2 \omega_{(CF+D)}(r) + (\lambda + 4\epsilon) \omega_{(AF+B)}(r) \\
&\leq (\lambda + 4\epsilon)^2 \big(\|C\| \omega_F(r) + \|F\| \omega_C(r) + \omega_D(r)\big) \\
&\qquad + (\lambda + 4\epsilon) \big(\|A\| \omega_F(r) + \|F\| \omega_A(r) + \omega_B(r)\big) \\
&\leq \lambda \omega_F(r) + \omega_A(r) + \omega_B(r) + \omega_C(r) + \omega_D(r),
\end{aligned}$$

and since $\omega_A(r), \omega_B(r), \omega_C(r), \omega_D(r) \leq \omega(r)$ because Df has ω as a modulus of continuity, we have for all $r > 0$

$$\omega_{\mathfrak{L}F\circ f}(r) \leq \lambda\omega_F(r) + 4\omega(r). \tag{60}$$

Moreover, if we denote by L the Lipschitz constant of f^{-1} which is bigger than 1, then we have

$$\begin{aligned}\omega_{\mathfrak{L}F}(r) &= \omega_{(\mathfrak{L}F\circ f)\circ f^{-1}}(r) \leq \omega_{\mathfrak{L}F\circ f}\big(\omega_{f^{-1}}(r)\big)\\ &\leq \omega_{\mathfrak{L}F\circ f}(Lr) \leq \lambda\omega_F(Lr) + 4\omega(Lr).\end{aligned}$$

Using the previous inequality we deduce that for all $n \in \mathbb{N}^*$

$$\omega_{\mathfrak{L}^n F}(r) \leq \lambda\omega_{\mathfrak{L}^{n-1}F}(Lr) + 4\omega(Lr). \tag{61}$$

Let F_0 be the map associated to E^u. Since the action of f on distribution transversal to E^s is contracting, we have for all F with norm less than 1

$$\lim_{n\to+\infty} \|\mathfrak{L}^n F - F_0\| = 0. \tag{62}$$

Using (62) and taking n to infinity in (61), we deduce that

$$\omega_{F_0}(r) \leq \lambda\omega_{F_0}(Lr) + 4\omega(Lr). \tag{63}$$

Using induction, we get for $n \in \mathbb{N}$

$$\omega_{F_0}(r) \leq \lambda^n\omega_{F_0}(L^n r) + \sum_{k=1}^{n} \lambda^k\omega(L^k r),$$

equivalently

$$\omega_{F_0}\left(\frac{r}{L^n}\right) \leq \lambda^n\omega_{F_0}(r) + \sum_{k=1}^{n} \lambda^k\omega\left(\frac{r}{L^{n-k}}\right). \tag{64}$$

Since ω is Dini summable, then we have

$$\sum_{n\geq 1}\left(\lambda^n\omega_{F_0}(r) + \sum_{k=1}^{n}\lambda^k\omega\left(\frac{r}{L^{n-k}}\right)\right) < +\infty. \tag{65}$$

It is enough to prove that $\sum\limits_{n\geq 1}\sum\limits_{k=1}^{n}\lambda^k\omega\left(\frac{r}{L^{n-k}}\right)$ is finite. Indeed we have for all $N \in \mathbb{N}$

$$\begin{aligned}\sum_{n=1}^{N}\sum_{k=1}^{n}\lambda^k\omega\Big(\frac{r}{L^{n-k}}\Big) &= \sum_{n=1}^{N}\sum_{k=1}^{N}\lambda^k\omega\Big(\frac{r}{L^{n-k}}\Big)\mathbb{1}\{k \leq n\}\\ &= \sum_{k=1}^{N}\sum_{n=1}^{N}\lambda^k\omega\Big(\frac{r}{L^{n-k}}\Big)\mathbb{1}\{k \leq n\}\\ &= \sum_{k=1}^{N}\lambda^k\sum_{n\geq k}^{N}\omega\Big(\frac{r}{L^{n-k}}\Big) \leq \sum_{k=1}^{N}\lambda^k\Big(\sum_{n\geq 0}\omega\Big(\frac{r}{L^n}\Big)\Big).\end{aligned}$$

We deduce using (64) that $\sum\limits_{n\geq 1} \omega_{F_0}\left(\frac{r}{L^n}\right) < +\infty$, hence the modulus of continuity of E^u satisfies Dini condition. Considering f^{-1} which is $C^{1+\mathrm{Dini}}$ shows that E^s has a modulus of continuity that satisfies Dini condition. □

Remark 5.5. *If f is $C^{1+\alpha}$, this proof gives us an explicit Hölder constants of E^u and E^s.*

5.2 Distortion lemma

In this subsection, we prove that the Dini condition is sufficient to have a distortion lemma for the unstable Jacobian, which is the main tool to prove the existence of a SRB measure, and the absolute continuity of the holonomy maps.

Denote by ω^u the modulus of continuity of the unstable distribution E^u, then we have

Lemma 5.6 (Distortion lemma)**.** *Let $\epsilon > 0$, and let U be an open set of diameter ϵ, then there exist $C_d = C_d(\omega^u, \lambda, \epsilon) > 0$ such that if W^u_{loc} is a piece of unstable manifold in U, then $\forall x, y \in W^u_{loc}$ and $n > 0$ we have*

$$C_d^{-1} \leq \frac{|\det df^{-n}(x)_{|E^u_x}|}{|\det df^{-n}(y)_{|E^u_y}|} \leq C_d. \tag{66}$$

Proof. Using the fact that E^u has a modulus ω^u we have

$$\left| \frac{\det df^{-1}_{|E^u_{\tilde{x}}}}{\det df^{-1}_{|E^u_{\tilde{y}}}} - 1 \right| \leq \omega^u\big(d(\tilde{x}, \tilde{y})\big), \ \forall \tilde{x}, \tilde{y} \in \Lambda,$$

we deduce that

$$\begin{aligned}
\left| \frac{\det df^{-n}_{|E^u_x}}{\det df^{-n}_{|E^u_y}} \right| = \left| \prod_{k=0}^{n-1} \frac{\det df^{-1}_{|E^u_{f^{-k}x}}}{\det df^{-1}_{|E^u_{f^{-k}y}}} \right| &\leq \prod_{k=0}^{n-1} \left(1 + \omega^u\big(d(f^{-k}x, f^{-k}y)\big)\right) \\
&\leq \prod_{k=0}^{+\infty} \left(1 + \omega^u\big(\lambda^k d(x, y)\big)\right),
\end{aligned}$$

and since ω^u satisfies Dini condition, we deduce that there is $C_d = C_d(\omega^u, \lambda, \epsilon)$ such that

$$\prod_{k=0}^{+\infty} \left(1 + \omega^u\big(\lambda^k d(x, y)\big)\right) \leq C_d,$$

which finishes the proof. □

Now we are ready to prove Theorem 5.1

Proof of 5.1. Consider a small piece L of unstable manifold, and let m_L be the normalized Lebesgue measure on L. Define a sequence of measures μ_n by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^n m_L.$$

Let μ be a weak* limit of $(\mu_n)_n$, and fix a rectangle R of small diameter. Let ρ_n be the density of $f_*^n m_L$ with respect to the volume of $f^n(L)$. Using lemma 5.6 we deduce that there is $C_d > 0$ that does not depend on n, such that for any $x, y \in R \cap W^u(x)$ we have

$$C_d^{-1} \leq \frac{\rho_n(x)}{\rho_n(y)} \leq C_d,$$

which implies in particular that

$$C_d^{-1} \leq \frac{\frac{1}{n}\sum\limits_{k=0}^{n-1} \rho_k(x)}{\frac{1}{n}\sum\limits_{k=0}^{n-1} \rho_k(y)} \leq C_d.$$

We disintegrate the measures μ_n in the rectangle R along unstable leaves. Let μ_n^s be the transversal measure of the disintegration. By the previous inequality we deduce that

$$\mu_n^s \times C_d^{-1} m \leq \mu_n \leq \mu_n^s \times C_d m, \tag{67}$$

where m here denote the normalized Lebesgue measure along local unstable leaves in R. Passing to a subsequence of (μ_n) we can assume that μ_n^s converges to a measure μ^s, and since the stable holonomy is continuous, the measure μ^s is the transversal measure of μ. We deduce that

$$C_d^{-1}\mu^s \times m \leq \mu \leq C_d \mu^s \times m,$$

hence μ is a SRB measure. □

Remark 5.7. *In [HY95], the same argument is used to prove existence of SRB measure for Anosov diffeomorphism with a neutral fixed point.*

5.3 Absolute continuity of the holonomy maps

In this section, we will be interested in the absolute continuity of the holonomy, because it implies that the SRB measure is physical.

For simplicity, we consider local unstable manifolds as transversals, and we prove that if f is $C^{1+\mathrm{Dini}}$, then the holonomy is absolutely continuous. Many authors proved the absolute continuity of the holonomy in various settings (Anosov [Ano67a] for C^2 hyperbolic maps, Mané [Man12] for $C^{1+\alpha}$), the proof of Mané can be adapted for $C^{1+\mathrm{Dini}}$ maps. There is also a proof by Abdenur and Viana for $C^{1+\alpha}$ partially hyperbolic diffeomorphism [FH19] Theorem B.7.6, and the proof we give here is inspired by their proof. In [BR75], Bowen and Ruelle proved a weaker version of ACH, which they called volume lemma, that gives an estimation of the volume of a dynamical ball, and is very important to prove the existence of a physical measure using the symbolic dynamics approach.

The proof is as follows: we prove that the holonomy does not change a lot the volume of unstable dynamical balls of the transversal, then using the Besicovitch covering argument (see Section 7.1), we deduce the absolute continuity.

Given $x \in M$, we define the unstable dynamical ball $B_n^u(x, \epsilon)$ to be the ball in $W^u(x)$ of center x and radius ϵ for the distance $d_n(y, z) = d(f^n y, f^n z)$, where d_n is the Riemannian distance on $W^u(f^n x)$ induced by the Riemannian metric on M.

Lemma 5.8 (Volume lemma)**.** *For all $\epsilon > 0$ there is $C_\epsilon > 0$, such that for all $x \in M$ and $n \geq 0$*

$$\frac{C_\epsilon^{-1}}{J^u f^n(x)} \leq m\big(B_n^u(x, \epsilon)\big) \leq \frac{C_\epsilon}{J^u f^n(x)}, \tag{68}$$

where m denotes the volume measure on $W_\epsilon^u(x)$, and $J^u f^n(x) = |\det df(x)_{E_x^u}|$.

Proof. Since the W_ϵ^u are C^1 embedded manifolds, we have for sufficiently small ϵ and for all x

$$m\big(B^u(x, \epsilon)\big) \simeq \epsilon^{\dim E^u}. \tag{69}$$

Let $x \in M$, and consider the the restriction of f to the unstable dynamical ball $_n^u(x, \epsilon)$, $f : B_n^u(x, \epsilon) \to B^u(f^n x, \epsilon)$. Applying the change of variable formulas we get

$$m\big(B^u(f^n x, \epsilon)\big) = \int_{B_n^u(x,\epsilon)} J^u f^n(y) dm(y).$$

By the distortion lemma 5.6, there is C_d such that for all $y \in B_n^u(x, \epsilon)$ we have

$$C_d^{-1} J^u f^n(y) \leq J^u f^n(x) \leq C_d J^u f^n(y).$$

Integrating with respect to y we get

$$C_d^{-1} \epsilon^{\dim E^u} \leq J^u f^n(x) m\big(B_n^u(x, \epsilon)\big) \leq C_d \epsilon^{\dim E^u}, \tag{70}$$

which finishes the proof of the lemma. □

Consider $\tau_1 = W_\epsilon^u(x_1), \tau_2 = W_\epsilon^u(x_2)$, where $x_2 \in W_\epsilon^s(x_1)$, and denote by $h = h_{\tau_1,\tau_2}$ the holonomy from τ_1 to τ_2.

Lemma 5.9. *There is a constant $C'_\epsilon > 0$ such that for any $x \in \tau_1$ and $n \geq 0$ we have*

$$C_\epsilon'^{-1} m\big(B_n^u(x, \epsilon)\big) \leq m\Big(h_{\tau_1,\tau_2}\big(B_n^u(x, \epsilon)\big)\Big) \leq C'_\epsilon m\big(B_n^u(x, \epsilon)\big). \tag{71}$$

Proof. Let $x \in \tau_1$, $y \in B_n^u(x,\epsilon)$ and $x' = h(x), y' = h(y)$. Since the holonomy is α-Hölder for some $\alpha > 0$, and constant $C = C(h,\epsilon)$ we have

$$d_n(x',y') \leq C d_n(x,y)^\alpha,$$

hence $h\big(B_n^u(x,\epsilon)\big) \subset B_n^u(x', C\epsilon^\alpha)$, which implies that

$$\begin{aligned} m\Big(h\big(B_n^u(x,\epsilon)\big)\Big) &\leq m\big(B_n^u(x',C\epsilon^\alpha)\big) \\ &\leq \frac{C_{\epsilon^\alpha}}{J^u f^n(x')} \\ &\leq \frac{C_{\epsilon^\alpha} C_d}{J^u f^n(x)} \\ &\leq C_\epsilon C_\alpha C_d \cdot m\big(B_n^u(x,\epsilon)\big), \end{aligned}$$

where the constant C_d is the constant coming from lemma 5.6, C_ϵ and C_α are given by lemma 5.8. We do the same argument for $h^{-1} = h_{\tau_2,\tau_1}$, to deduce the other inequality, which finishes the proof. □

Since the constant C'_ϵ does not depend on n, and that f restricted to τ_1 is expanding, we can make the dynamical balls sufficiently small by taking n big enough. So the set $\mathcal{B}$ of dynamical balls in τ_1 generates the open sets of τ_1. To prove that the holonomy is absolutely continuous, it is enough to prove that the volume of small balls does not distort much. To do this we need the Besicovitch covering lemma.

Lemma 5.10. *There is $C_l = C_l(\epsilon)$ such that for any ball B in τ_1, we have*

$$C_l^{-1} m(B) \leq m\big(h(B)\big) \leq C_l m(B). \tag{72}$$

Proof. Let B be a ball of small radius $\delta < \epsilon$ in τ_1, and cover it by dynamical balls of depth n, i.e $B \subset \bigcup_{i\in I} B_n^u(x_i,\epsilon)$, and $x_i \in B$. By definition of dynamical balls we have

$$f^n(B) \subset \bigcup_{i\in I} B(f^n x_i, \epsilon). \tag{73}$$

Applying Besicovitch covering lemma (see Appendix) to (73), we can find a subsequence $(x_i^j)_{i,j}$ of $(x_i)_i$ such that $B \subset \bigcup_{i,j} B_n^u(x_i^j,\epsilon)$, the j varies in $\{1,2,...,K\}$, where K is universal and depends only on the dimension of E^u, and for $j \neq j'$, we have for all i, i'

$$\overline{B_n^u}(x_i^j,\epsilon) \cap \overline{B_n^u}(x_{i'}^{j'},\epsilon) = \emptyset.$$

Using subadditivity of the measure m we get

$$m(B) \leq m\big(\bigcup_{i,j} B_n^u(x_i^j,\epsilon)\big) \leq \sum_{i,j} m\big(B_n^u(x_i^j,\epsilon)\big), \tag{74}$$

and

$$m\big(h(B)\big) \le m\Big(\bigcup_{i,j} h\big(B_n^u(x_i^j,\epsilon)\big)\Big) \le \sum_{i,j} m\Big(h\big(B_n^u(x_i^j,\epsilon)\big)\Big). \tag{75}$$

The element of the sequence (x_i^j) that lies in $\partial B_{(\lambda-\epsilon)^n}$ (the $(\lambda-\epsilon)^n$ neighborhood of the boundary of B,) form a set Z_n that does not contribute a lot to the volume of $\bigcup_{i,j} B_n^u(x_i^j,\epsilon)$, because the radius of B is small, so the Riemannian metric on B does not vary a lot. In other words $m\big(\bigcup_{x_i^j\in Z_n} B_n^u(x_i^j,\epsilon)\big) = \epsilon_n$, and ϵ_n converges to 0 as n goes ∞.

So we deduce that

$$\frac{1}{K}\sum_{i,j} m\big(B_n^u(x_i^j,\epsilon)\big) - K\sum_{x_i^j\in Z^n} m\big(B_n^u(x_i^j,\epsilon)\big) \le m(B). \tag{76}$$

Using lemma 5.9, we get also

$$\frac{1}{K\cdot C'_\epsilon} m\Big(h\big(B_n^u(x_i^j)\big)\Big) - K\cdot C'_\epsilon \sum_{x_i^j\in Z^n} m\Big(h\big(B_n^u(x_i^j)\big)\Big) \le m\big(h(B)\big). \tag{77}$$

Using lemma 5.9 again, and (74) and (77) we deduce that

$$m\big(h(B)\big) \ge \frac{1}{\big(KC'_\epsilon\big)^2} m(B) - \big(KC'_\epsilon\big)^2\epsilon_n, \tag{78}$$

taking n to infinity finishes the proof. □

Remark 5.11. *We prove in the same way that if $y \in W_\epsilon^s(x)$, then there is a constant C_d such that for all $n\in\mathbb{N}$ we have*

$$C_d^{-1} \le \frac{J^u f^n(x)}{J^u f^n(y)} \le C_d, \tag{79}$$

where we recall that $J^u f^n(x) = |\det df^n(x)_{|E_x^u}|$.

5.4 Ergodicity of the SRB measure

Let $f : M \to M$ be a $C^{1+\text{Dini}}$ Anosov and μ a SRB measure. If f is transitive, then using Hopf argument, we can prove that the measure μ is ergodic. The Hopf argument consists of two steps; the first one is to prove that if $\phi : M \to \mathbb{R}$ is measurable f- invariant function, then it is constant on stable and unstable leaves $mod_\mu 0$. The second step is to use the absolute continuity of the holonomy maps to deduce that $\phi : M \to \mathbb{R}$ is constant. The following lemma is taken from [BS02] lemma 6.3.2.

Lemma 5.12. *Let $\phi : M \to \mathbb{R}$ be an f-invariant measurable function. Then ϕ is constant modulo μ on stable and unstable sets, i.e there is a null set N for μ such that ϕ is constant on $W^s(x)\setminus N$ and on $W^u(x)\setminus N$ for every $x \in M\setminus N$.*

Remark 5.13. *If f preserves a measure equivalent to the Lebesgue measure, then for all $h \in L^2(M)$, any weak* limit of $(h \circ f^n)_n$ is constant when restricted to global stable (unstable) leaf (see [Cou16] Theorem 4.1).*

Now using the absolute continuity of the holonomy maps we get

Lemma 5.14. *Let $\phi : M \to \mathbb{R}$ be an f-invariant measurable function, then ϕ is constant modulo μ.*

Proof. Since f is transitive, each global stable leaf is dense in M, so it is enough to prove that ϕ is constant in some small rectangle $\mathcal{R}$.

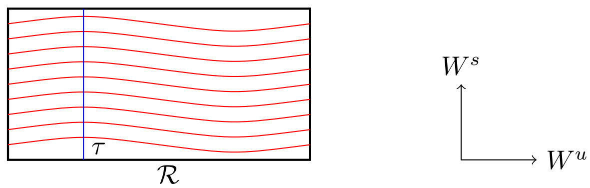


Figure 5: A rectangle foliated by local stable leaves

$\mathcal{R}$ is foliated by local unstable leaves (see figure 5). Let τ be transversal to the unstable foliation in $\mathcal{R}$, then disintegrate the measure μ in this rectangle along unstable leaves, $\mu(E) = \int_\tau \xi_y(E) d\nu(y)$. Since μ is SRB, for ν-a.e $y \in \tau, \xi_y <<$ $Leb_{W^u_{loc}(y)}$. Using the previous lemma, we have for ν-a.e $y \in \tau, \phi$ is constant in $W^u_{loc}(y) \cap \mathcal{R} \setminus N$, moreover, for ν-a.e $y \in \tau, \xi_y\left(N \cap W^u_{loc}(y)\right) = 0$. Consider two generic points y_1, y_2 of ν. Since $\xi_{y_i}(N \cap W^u_{loc}(y_i)) = 0$ there are $x_i \in W^u_{loc}(y_i) \setminus N$, with $x_0 \in W^s_{loc}(x_1) \setminus N$, so we have $\phi(y_0) = \phi(x_0) = \phi(x_1) = \phi(y_1)$, which proves that ϕ is constant on $\mathcal{R}$ modulo μ, hence the measure μ is ergodic. □

Remark 5.15. *If f is a C^1 volume preserving Anosov diffeomorphism, then it is an open problem whether the volume is ergodic or not.*

5.4.1 Physicality of the SRB measure

Using the absolute continuity of the holonomy maps, we prove that in the case of an attractor the SRB measure is physical.

Proposition 5.16. *Let $f : M \to M$ be an Anosov, then the SRB measure is physical.*

Proof. Denote the SRB measure by μ. Using Smale decomposition theorem, we may assume without loss of generality that f is transitive, so the SRB measure is ergodic, in particular, for μ-a-e $x \in M$, $x \in B(\mu)$. Using the fact that μ is SRB, we can find a local unstable leaf $W^u_\epsilon(x)$ such that $W^u_\epsilon(x) \cap B(\mu)$ has positive measure with respect to $Leb_{W^u_\epsilon(x)}$. Notice that if $y \in W^s(x)$ and $x \in B(\mu)$, then $y \in B(\mu)$. Finally, using the absolute continuity of the holonomy map, we deduce that $B(\mu)$ has positive Lebesgue measure. □

6 Decay of correlations for $C^{1+\alpha}$ Anosov diffeomorphisms

The classical approach to prove decay of correlations, is either by using the coding (see [Bow75a]) or the tool developed by Young [You98] now called Young towers. In this dissertation, we use another approach using optimal transport and standard pairs. The idea of the proof is similar to the proof of Proposition A1 page 139 in [CD09]. Roughly speaking, if we consider an Anosov diffeomorphism f and two standard pairs (W_i, ρ_i), then we transport the mass from W_1 to W_2 along stable leaves. This allows us to define a nice distance between measures induced by standard pairs ($\mu_i = \int_{W_i} \rho_i \; dLeb_{W_i}$) which induces the usual weak* topology, and is contracting under the action of f. So we deduce in particular that $(f_*^n \mu_1)_n$ converges exponentially fast to the SRB measure μ, which gives the decay of correlations.

First, we recall the definition of standard pairs introduced by Dolgopyat (see for example [CDP16]). Then, we will define a special distance between measures induced by standard pairs using optimal transport (see the appendix for a brief introduction, or [Vil09] for more detailed introduction to the subject), and prove that it is contracting under the action of f.

In this section we consider a transitive Anosov diffeomorphism f of class $C^{1+\alpha}$, and we fix a Markov partition $\bigcup_i \mathcal{R}_i$ with diameter less than a small $\epsilon > 0$.

6.0.1 Standard pairs

Let W be a local unstable leaf which contains a ball of radius ϵ and is contained in a ball of radius 2ϵ, we say that it fully crosses $\mathcal{R}_i$ (or an admissible manifold) if $\mathcal{R}_i \cap W = W^u_{2\epsilon}(x) \cap \mathcal{R}_i$ for some $x \in \mathcal{R}_i \cap W$.

Assume that $W \subset \mathcal{R}_i$ for some i, and W fully crosses $\mathcal{R}_i$, and let $\rho : W \to (0, +\infty)$ be a L^1 density. We will call (W, ρ) a standard pair.

Now fix $L > 0$, then the space of standard pairs

$$\mathfrak{R}_L := \left\{ (W,\rho) \;\middle|\; (W,\rho) \text{ is a standard pair }, \rho \in C^\alpha(W, [\frac{1}{L}, L]), |\rho|_\alpha \le L \right\} \quad (80)$$

is compact in the natural product topology. A standard pair determines a measure $\Psi(W, \rho)$ on M in the following way:

$$\Psi(W,\rho)(E) := \int_{W \cap E} \rho \; dLeb_W. \quad (81)$$

Moreover, each measure η on $\mathfrak{R}_L$ determines a measure $\Phi(\eta)$ on M given by

$$\Phi(\eta)(E) := \int_{\mathfrak{R}_L} \Psi(W,\rho)(E) \; d\eta(W,\rho). \quad (82)$$

Remark 6.1. *This definition of standard pairs differs from the one given in [CLP17, CDP16], they fixed a cone field around the unstable distribution, and took local manifolds tangent to this cone field, in particular local unstable leaves satisfies that.*

Consider the map $\Phi : \mathcal{M}_{\leq 1}(\mathfrak{R}_L) \to \mathcal{M}(M)$, where $\mathcal{M}_{\leq 1}(\mathfrak{R}_L)$ is the space of measures on $\mathfrak{R}_L$ with total weight less than 1, then $\Phi(\mathcal{M}_{\leq 1}(\mathfrak{R}_L)) := \mathcal{M}_L$ is compact with respect to the weak * topology. A measure μ is called admissible if it has a the form $\Psi(W, \rho)$.

Lemma 6.2. *The space $\mathcal{M}_L$ is compact.*

Proof. Using (81) and (82), we deduce that Φ is continuous, so to prove that $\mathcal{M}_L$ is compact, it is enough to prove that $\mathcal{M}_{\leq 1}(\mathfrak{R}_L)$ is compact. Since the space $C^\alpha(W, [\frac{1}{L}, L])$ with holder constant less than L is compact, and M is compact, we deduce that $\mathcal{M}_{\leq 1}(\mathfrak{R}_L)$ is compact. □

Now, we want f to act on $\mathcal{M}_L$ by pushing forward measures, and a fixed point for this action would be an invariant SRB measure for f. In our setting, f acts on $\mathcal{M}_L$, more precisely we have

Lemma 6.3. *There is $L > 0$, such that for all standard pair $(W, \rho) \in \mathfrak{R}_L$, and $n \in \mathbb{N}^*$ there is $k_n \in \mathbb{N}$ such that*

$$f_*^n \Psi(W, \rho) = \sum_{i=0}^{k_n} a_i \Psi(W_i, \rho_i'), \tag{83}$$

where $(W_i, \rho_i') \in \mathfrak{R}_L$ for all i, and $\sum_{k=0}^{k_n} a_i = 1$.

Proof. Let $n \in \mathbb{N}^*$. Using Markov property (see 4.14) $f^n(W)$ can be covered by local unstable manifolds that intersect only at the boundary (which have 0 Lebesgue measure), so we can write

$$f_*^n \Psi(W, \rho) = \sum_{i=0}^{k_n} a_i \Psi(W_i, \rho_i),$$

where $a_i = f_* \Psi(W, \rho)(W_i)$, and ρ_i are L^1 densities. Now, to finish the proof of the lemma, we have to prove that for all i, ρ_i is α-Hölder continuous with Hölder constant less than L.

Denote by $\tilde{\rho}$ the density of $f_*^n \Psi(W, \rho)$ on $f^n(W)$ with respect to the Lebesgue measure on $f^n(W)$, then using the change of variable formula we get

$$\tilde{\rho}(x) = \frac{\rho \circ f^{-n}(x)}{Jac^u f^n \circ f^{-n}(x)} = Jac^s f^{-n}(x) \cdot \rho \circ f^{-n}(x), \ \forall x \in f^n(W). \tag{84}$$

The density ρ satisfies

$$\frac{\rho(x)}{\rho(y)} \leq e^{Ld(x,y)^\alpha}, \ \forall x, y \in W, \tag{85}$$

and using the fact that the Jacobian is α-Hölder

$$\frac{Jac^s f^{-n}(x)}{Jac^s f^{-n}(y)} \leq e^{C_d d(x,y)^\alpha},\ \forall x, y \in f^n(W), \tag{86}$$

we deduce that

$$\begin{aligned}\frac{\tilde{\rho}(x)}{\tilde{\rho}(y)} &= \frac{Jac^s f^{-n}(x)}{Jac^s f^{-n}(y)} \cdot \frac{\rho(f^{-n}x)}{\rho(f^{-n}y)} \\ &\leq e^{Ld(f^{-n}x, f^{-n}y)^\alpha} \cdot e^{C_d d(x,y)^\alpha} \\ &\leq e^{(L\lambda^n + C_d) d(x,y)^\alpha},\end{aligned} \tag{87}$$

so choosing L large enough, we get for all $n \in \mathbb{N}^*, L\lambda^n + C_d \leq L$ which implies that $\tilde{\rho}$ is α-Hölder with constant less than L. For each i put $\rho_i' = \frac{1}{a_i}\tilde{\rho}_{|W_i}$, it remains to prove that $\rho_i' \in [\frac{1}{L}, L]$. Assume that this is not the case, i.e there is some $x_0 \in W_i$ such that $\rho'(x_0) > L$, which is equivalent to $\tilde{\rho}(x_0) > L \int_{W_i} \tilde{\rho}\ dLeb_{W_i}$. Using (87), we get

$$L \int_{W_i} \tilde{\rho}\ dLeb_{W_i} < \tilde{\rho}(x_0) \leq e^{(L\lambda^n + C_d)\epsilon} \tilde{\rho}(x),\ \forall x \in W_i, \tag{88}$$

integrating with respect to x we get

$$L\epsilon^{dimE^u} \int_{W_i} \tilde{\rho}\ dLeb_{W_i} \lesssim e^{(L\lambda^n + C_d)} \int_{W_i} \tilde{\rho}\ dLeb_{W_i}, \tag{89}$$

but for some $N \in \mathbb{N}$, and for all $n > N$, we have $L\epsilon^{dimE^u} \geq e^{(L\lambda^n + C_d)}$, contradiction. We prove similarly that $\rho_i' \geq \frac{1}{L}$.

□

Now to couple two measures given by standard pairs, we make the standard pair "facing each other" by taking their image by some iterate of f, and wait until part of them land on the same rectangle. Then we use the holonomy map to transport an amount of mass uniformly bounded from below. More precisely, fix a rectangle $\mathcal{R}_j$ then we have

Lemma 6.4. *There is $N \in \mathbb{N}^*$ and $\tau_0 = \tau_0(\epsilon)$ such that for all admissible manifold W_1, W_2 and all $n \geq N$ we have $Leb(W_i \cap f^{-N}(\mathcal{R}_j)) \geq \tau_0$.*

Proof. We prove that if we fix two admissible manifolds W_1, W_2, then we can find such N and τ_0 satisfying the condition of the lemma, then using the fact that admissible manifolds form a compact space, we can choose N and τ_0 uniform, i.e they do not depend on W_1, W_2.

Fix two admissible manifolds W_1, W_2, and $c_i \in W_i$ for $i \in \{0, 1\}$. Denote by $U_i = [W_1, W_\epsilon^s(c_i)]$, then by mixing of the SRB measure μ, we have $\mu\left(U_i \cap f^{-n}(\mathcal{R}_j)\right) \to \mu(U_i)\mu(\mathcal{R}_j)$. We deduce that, there is $N \in \mathbb{N}^*, \tau_0 = \tau_0(\epsilon)$ such that for all $n \geq N$, $\mu(U_i \cap f^{-n}(\mathcal{R}_j)) \geq 2\tau_0$. Since the measure μ is SRB, there is $y_i \in W_\epsilon^s(c_i)$ such that $Leb(W_\epsilon^u(y_i) \cap f^{-n}(\mathcal{R}_j)) > \frac{3}{2}\tau_0$. Finally, using the fact the holonomy is absolutely continuous with Jacobian close to 1, we deduce that $Leb(W_i \cap f^{-n}(\mathcal{R})) \geq \tau_0$. Notice that if W_1' is sufficiently close to W_1 then the same N, τ_0 work with W_1', so using compacity of admissible manifold finishes the proof of the lemma. □

Lemma 6.5. *Fix a rectangle $\mathcal{R}$, then there is $\tau_0 = \tau_0(\epsilon) > 0$ such that for all $(W_1, \rho_1), (W_2, \rho_2) \in \mathfrak{R}_L$ with W_1, W_2 in $\mathcal{R}$ and fully cross it, there exist $\gamma \in \mathcal{M}_1(M \times M)$ and densities ρ_i' on W_i such that*

- *γ is concentrated on $\Delta_\epsilon^s := \{(x,y) \in M \times M \mid d_s(x,y) \leq \epsilon\}$,*
- *$\Psi(W_i, \rho_i) = \tau_0 \pi_{i*}\gamma + (1-\tau_0)\rho_i' Leb_{W_i}$,*
- *$(W_i, \rho_i') \in \mathfrak{R}_{2L}$.*

Proof. Consider the measure $\gamma \in \mathcal{M}(M \times M)$, such that $\pi_{1*}\gamma = \frac{1-\epsilon}{2L} Leb_{W_1}$ and is supported on the graph of the holonomy map $h : W_1 \to W_2$. We have

$$\pi_{2*}\gamma = h_*\Psi(W_1, \rho_1) = \frac{1}{Jac\ h \circ h^{-1}} dLeb_{W_2} = Jac\ h^{-1}. \tag{90}$$

Since $Jac\ h^{-1}$ is sufficiently close to 1 when $\epsilon > 0$ is close to 0, the density of $\pi_{2*}\gamma$ is less than $\frac{1}{2L}$ and Hölder continuous with Hölder constant less than L (L is sufficiently large). We deduce that $\Psi(W, 2\rho_2) - \pi_{2*}\gamma$ is a positive measure with $2L$-Hölder density, with values in $[\frac{1}{2L}, 2L]$, the same is true for $\Psi(W_1, \rho_1) - \frac{1-\epsilon}{2L}$, which finishes the proof. □

In fact we can prove this lemma without taking the admissible manifold to be in the same rectangle, and this is the main tool to prove that the action of f on $\mathcal{M}_L$ is contracting.

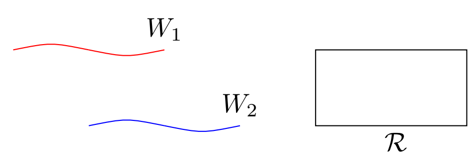


Figure 6:

Fix L and N as in the previous lemmas, then we have

Lemma 6.6. *There is $\tau = \tau(\epsilon, L, N) > 0$ such that for all standard pairs (W_1, ρ_1), $(W_2, \rho_2) \in \mathfrak{R}_L$, there exist $\gamma \in \mathcal{M}_1(M \times M)$ and densities ρ_i' on W_i such that*

- *γ is supported on $\Delta_N^s := \left\{(x,y) \in M \times M \mid d_s(x,y) \leq \frac{1}{\lambda^N}\right\}$,*
- *$\Psi(W_i, \rho_i) = \tau \pi_{i*}\gamma + (1-\tau)\rho_i' Leb_{W_i}$,*
- *$(W_i, \rho_i') \in \mathfrak{R}_{2L}$.*

Proof. Using lemma 6.4, there are τ_0 and $N \in \mathbb{N}^*$ such that $Leb(U_i) > \tau_0$, where $U_i = W_i \cap f^{-N}(\mathcal{R}_j)$, moreover $f^N(U_i)$ consists of admissible manifolds in $\mathcal{R}_j$ that fully cross it (see figure 7). Denote by $(W_{1,k})_{k\in I_1}$ and $(W_{2,k})_{k\in I_2}$ this two families of admissible manifolds. Consider the standard pairs $(W_{i,k}, \rho'_k) \in \mathfrak{R}_L$ given by lemma 6.3. Then we have

$$\sum_{k\in I_i} a_{k,i} \geq \tau_0, \tag{91}$$

where $a_{k,i}$ are given by lemma 6.3. To prove the two first points of the lemma, we couple the measures $\mu_i = \Phi(\eta_i)$ as in lemma 6.5, where η_i is given by

$$\eta_i = \sum_{k\in I_i} a_{k,i}\delta_{\Psi(W_{k,i},\rho'_{k,i})}. \tag{92}$$

□

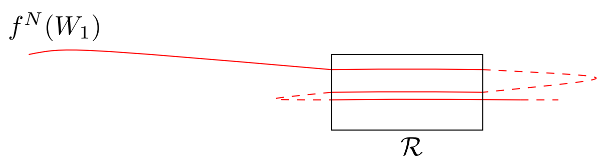


Figure 7: Pushing admissible manifold by f

Lemma 6.7. *Fix τ_0 as in the previous lemmas, then there is $\theta \in (0,1)$ and $C > 0$, such that for any admissibile manifold W and a rectangle $\mathcal{R}$ of radius ϵ we have for all $n \in \mathbb{N}$*

$$Leb_W(W \setminus A_n) \leq C\theta^n, \tag{93}$$

where $A_n = \bigcup_{k=1}^{n} W \cap f^{-n}\mathcal{R}$.

Proof. We proved that there is $n_0 \in \mathbb{N}^*$ such that for any admissible manifold W and a rectangle $\mathcal{R}$, we have $Leb_W(W \cap f^{-n_0}\mathcal{R}) > \tau_0$. Notice that $f^{n_0}(W)$ is a union of disjoint admissible manifolds $(W_j)_j$, and using the previous argument, each W_j contains a mass τ_0 that will be mapped in $\mathcal{R}$ by f^{n_0}, in particular there is a proportion τ_0 of $f^{n_0}(W) \setminus \mathcal{R}$ that will be mapped in $\mathcal{R}$ by f^{n_0}, so we have $\lambda\big(W \cap f^{-2n_0}\mathcal{R}\big) > \tau_0(1-\tau_0)$. Repeating the some argument we get

$$Leb_W\big(W \cap f^{-kn_0}\mathcal{R}\big) > u_k,\ \forall k \in \mathbb{N}^*, \tag{94}$$

where $(u_n)_n$ is the sequence defined by $u_1 = \tau_0$, $u_{n+1} = \tau_0(1-S_n)$ and $S_n = u_1 + \cdots u_n$.

A simple recurrence shows that $S_n = 1-(1-\tau_0)^n$, so we have for all $n \in \mathbb{N}$

$$Leb_W\big(W \setminus A_{n_0 n}\big) \leq 1 - S_n = (1-\tau_0)^n. \tag{95}$$

Taking $\theta = \sqrt[n_0]{1-\tau_0}$ and $C \geq \frac{1}{(1-\tau_0)^{n_0}}$ finishes the proof. □

Remark 6.8. *We proved in particular that*

$$Leb_W\big(\{x \in W \mid d_s(x, \mathcal{R}) \geq \lambda^{-n}\}\big) \leq C\theta^n, \tag{96}$$

where d_s is the distance along the stable leaf containing x.

Now using these lemmas, we deduce that the Wasserstein distance between measures given by standard pairs converges exponentially to the same limit, in other words we have

Proposition 6.9. *There are $\alpha = \alpha(\tau, N) \in (0,1)$ and $C = C(\tau, \lambda, N, \alpha)$ such that for all $\epsilon_0 > 0$, and $\mu, \nu \in \Psi(\mathfrak{R}_L), \forall n \geq 0$ we have*

$$W_1^\alpha(f_*^n\mu, f_*^n\nu) \leq (1-\epsilon_0)C\lambda^{\alpha n} + \epsilon_0 d_M^\alpha, \tag{97}$$

where d_M is the diameter of the manifold M. We have in particular

$$W_1^\alpha(f_*^n\mu, f_*^n\nu) \leq C\lambda^{\alpha n}. \tag{98}$$

Proof. To control the Wasserstein distance, we need to construct for any $\mu, \nu \in \Psi(\mathfrak{R})$ a special coupling $\gamma \in \Gamma(\mu, \nu)$ satisfying

i) $\gamma = (1-\epsilon_0)\gamma_{\epsilon_0}^s + \epsilon_0\gamma_{\epsilon_0}$,

ii) for $\gamma_{\epsilon_0}^s$-a.e (x, y), $y \in W^s(x)$.

Let $(W_1, \rho_1), (W_2, \rho_2) \in \mathcal{R}_L$, and μ_1, μ_2 the corresponding measures. Denote by γ_1 the measure given by lemma 6.6, and by r_1^1, r_1^2 the probability measures such that

$$\mu_i = \tau\pi_{i*}\gamma_1 + (1-\tau)r_1^i.$$

$f_*^N r_1^i$ is a linear combination of elements of $\mathcal{R}_L$, so using lemma 6.6 again, we can find $\tilde{\gamma}_2 \in \mathcal{M}(M \times M)$ such that

$$f_*^N r_1^i = \tau\pi_{i*}\tilde{\gamma}_2 + (1-\tau)r_2^i.$$

We take $\gamma_2 = f_*^{-N}\tilde{\gamma}$, then by definition γ_2 satisfies ii).

We define inductively r_{n+1}^i and $\tilde{\gamma}_{n+1}$($\tilde{\gamma}_{n+1}$ is chosen) by

$$f_*^N r_n^i = \tau\pi_{i*}\tilde{\gamma}_{n+1} + (1-\tau)r_{n+1}^i,$$

and we put $\gamma_{n+1} = f_*^{-nN}\tilde{\gamma}_{n+1}$.

Define the sequence $(u_n)_n$ by $u_1 = \tau$, $u_{n+1} = \tau(1-S_n)$ where $S_n = u_1 + \cdots + u_n$, a simple computation shows that $u_n = \tau(1-\tau)^n$. Consider a large m such that $S_m > 1-\epsilon_0$, and consider the measure

$$\gamma = u_1\gamma_1 + u_2\gamma_2 + \cdots + u_m\gamma_m + \epsilon_0\gamma_{\epsilon_0}$$

where γ_{ϵ_0} is any measure such that $\pi_{i*}\gamma = \mu_i$. The measure $\sum_{i=1}^{m} u_i\gamma_i$ satisfies property ii), so taking $\gamma_{\epsilon_0}^s = \sum_i u_i\gamma_i$ gives us the coupling satisfying *i)* and *ii)*.

Let $\alpha \in (0,1)$, then we have

$$\begin{aligned}
W_1^\alpha(f_*^n\mu, f_*^n\nu) &\leq \int d(x,y)^\alpha \; d\Big(f_*^n\big((1-\epsilon_0)\gamma_{\epsilon_0}^s + \epsilon_0\gamma_{\epsilon_0}\big)\Big) \\
&= (1-\epsilon_0)\int d(x,y)^\alpha \; d(f_*^n\gamma_{\epsilon_0}^s) + \epsilon_0 \int d(x,y)^\alpha \; d(f_*^n\gamma_{\epsilon_0}) \\
&= (1-\epsilon_0)\int d(f^n x, f^n y)^\alpha \; d\gamma_{\epsilon_0}^s + \epsilon_0 \int d(f^n x, f^n y)^\alpha \; d\gamma_{\epsilon_0} \\
&\leq (1-\epsilon_0)\int d_s(f^n x, f^n y)^\alpha \; d\gamma_{\epsilon_0}^s + \epsilon_0 d_M^\alpha \\
&\leq (1-\epsilon_0)\lambda^{\alpha n}\int d_s(x,y)^\alpha \; d\gamma_{\epsilon_0}^s + \epsilon_0 d_M^\alpha.
\end{aligned}$$

It remains to prove that $\int d_s(x,y)^\alpha \; d\gamma_{\epsilon_0}^s$ is finite and does not depend on ϵ_0 and n for a suitable choice of α.

By definition, the measure γ_k is supported on

$$\left\{(x,y) \in M \times M \ , \ d_s(x,y) \leq \frac{1}{\lambda^{kN}}\right\},$$

so we have

$$\begin{aligned}
\int d_s(x,y)^\alpha \; d\gamma_{\epsilon_0}^s &= \sum_k u_k \int d_s(x,y) \; d\gamma_k \\
&\leq \sum_k (1-\tau)^k \lambda^{-kN\alpha} \\
&= \sum_{k=1}^m \left((1-\tau)\lambda^{-N\alpha}\right)^k.
\end{aligned}$$

Choosing α small enough we have $(1-\tau)\lambda^{-N\alpha} < 1$, so there is $C = C(\tau, \lambda, N, \alpha)$ such that $\int d_s(x,y)^\alpha \; d\gamma_{\epsilon_0}^s \leq C$, we have in particular for all n and $\epsilon_0 > 0$

$$W_1^\alpha(f_*^n\mu, f_*^n\nu) \leq (1-\epsilon_0)C\lambda^{\alpha n} + \epsilon_0 d_M^\alpha,$$

which finishes the proof.

□

Now using a standard argument in optimal transport (see theorem 7.15), we deduce that

Theorem 6.10. *For all $\mu, \nu \in \mathcal{M}_L$ and $n \in \mathbb{N}^*$ we have*

$$W_1^\alpha(f_*^n\mu, f_*^n\nu) \leq C\lambda^{\alpha n}. \tag{99}$$

Proof. Let $\mu, \nu \in \mathcal{M}_L$, and let η be a measure on $\mathfrak{R}_L$ such that $\Phi(\eta) = \mu$. Assume that ν is given by a single standard pair, then we have for all $n \in \mathbb{N}$

$$\begin{aligned} W_1^\alpha(f_*^n\mu, f_*^n\nu) &= W_1^\alpha\big(\int f_*^n \mu_z \, d\eta(z), \int f^n \nu_z \, d\eta(z)\big) \\ &\le \int W_1^\alpha(f_*^n\mu_z, f_*^n\nu) \, d\eta(z) \\ &\le \int C\lambda^{\alpha n} \, d\eta(z) \\ &= C\lambda^{\alpha n}, \end{aligned}$$

where $\mu_z : \mathfrak{R}_L \to \mathcal{M}_L$ such that $\mu = \int \mu_z \, d\eta(z)$ and $\nu_z : \mathfrak{R}_L \to \mathcal{M}_L$ is constant and equal to ν. For general ν we use the same argument and the previous inequality, which finishes the proof. □

Corollary 6.11. *[Exponential decay of correlations] If f is a $C^{1+\alpha}$ transitive Anosov, then the SRB measure μ has exponential decay of correlations for Hölder observables, in other words there is $\theta \in (0,1)$ such that for all α-Hölder maps φ, ψ we have:*

$$\left| \int \varphi \ \psi \circ f^n \ d\mu - \int \varphi \ d\mu \int \psi \ d\mu \right| \le C_{\varphi,\psi}\theta^n. \tag{100}$$

Proof. Consider two Hölder observables φ, ψ. Up to adding a large constant to φ and then dividing by a normalizing constant, we reduce to the case when $\varphi > 0$ and $f\mu \in \mathcal{M}_L$, and we get

$$\begin{aligned} Cor_\mu(\varphi, \psi \circ f^n) &:= \left| \int \varphi \ \psi \circ f^n \ d\mu - \int \psi \ d\mu \right| \\ &= \left| \int \psi \ df_*^n(\varphi\mu) - \int \psi \ d\mu \right|. \end{aligned}$$

Using Kantorovich duality (see theorem 7.10) for the Wasserstein distance W_α whose cost function is $d(\cdot,\cdot)^\alpha$, we deduce that

$$\begin{aligned} \left| \int \psi \ df_*^n(\varphi\mu) - \int \psi \ d\mu \right| &\le C_\varphi W_\alpha\big(f_*^n(\varphi\mu), f_*^n\mu\big) \\ &\le C_\varphi W_1\big(f_*^n(\varphi\mu), f_*^n\mu\big)^\alpha \\ &\le C_\varphi d_M^\alpha \lambda^{\alpha n}. \end{aligned}$$

Finally, the dependence on φ comes from its normalization, which proves decay of correlations. □

Remark 6.12. *We should be able to use this approach without assuming that the SRB measure exists, and without use of Markov partitions (which is used here to bypass the technical issue of how to cover the image of a standard pair by standard pairs), and the difficulty in this case is to prove lemma 6.6. The interested reader can find an advanced draft to this direction in [BK23].*

7 Appendix

7.1 Covering theorems

Lemma 7.1 (Besicovitch covering theorem)**.** *Let E be a subset of $\mathbb{R}^N$ and let $\bigcup_{i\in I} \mathcal{B}_i$ an arbitrary cover of E by balls centered in E. Then there is c_N depending only on the dimension N with the following property:*

There are at most c_N disjoint countable subsets $I_1, \cdots, I_{c_N}$ of I such that

- $j, j' \in I_k \Rightarrow \mathcal{B}_j \cap \mathcal{B}_{j'} = \emptyset$,
- $E \subset \bigcup_{i=1}^{c_N} \bigcup_{j\in I_i} \mathcal{B}_j$.

For a proof, see for example [Mat99].

Remark 7.2. *The Besicovitch covering theorem is also true in a C^1 Riemannian manifold. The proof in the case of a C^2 manifold can be found in [Fed69].*

We present another covering theorem, which can be used as an alternative to Besicovitch covering theorem in a C^1 manifold. Let X be a metric space, with a measure μ, which is finite on bounded subsets of X. Consider a family $\mathcal{F}$ of closed subsets of X.

Definition 7.3. *We say that the family $\mathcal{F}$ covers a subset A* finely*, if for all $a \in A$, and all $\epsilon > 0$, there is $B \in \mathcal{F}$ such that*

$$a \in B \subset B(a, \epsilon).$$

Definition 7.4. *The family $\mathcal{F}$ is said to be μ* adequate *for A if and only if for all open set $U \subset X$, there is a countable disjoint subfamily $\mathcal{G}$ of $\mathcal{F}$ such that*

$$\bigcup \mathcal{G} \subset U, \text{ and } \mu\big(U \cap A \setminus \bigcup \mathcal{G}\big) = 0.$$

Definition 7.5. *Let δ be a nonnegative bounded function on $\mathcal{F}$ and $\tau > 1$. For $S \in \mathcal{F}$, we define its δ, τ* enlargement *by*

$$\hat{S} = \bigcup \big\{F : F \in \mathcal{F}, F \cap S \neq \emptyset, \delta(F) < \tau\delta(S)\big\}.$$

Theorem 7.6 (Theorem 2.8.7 in [Fed69])**.** *If $\mathcal{F}$ covers A finely, δ is a nonnegative bounded function on $\mathcal{F}, \tau > 1, \lambda > 1$ and*

$$\mu(\hat{S}) < \lambda\mu(S), \tag{101}$$

whenever $S \in \mathcal{F}$ and $\hat{S}$ is the δ, τ enlargement of S, then $\mathcal{F}$ is μ adequate for A.

Corollary 7.7. *Let f be an expanding diffeomorphism of $\mathbb{R}^N$. Assume that df satisfies distortion, i.e for all $\epsilon > 0$, there is $C_\epsilon > 1$ such that for all $x, y \in B_n(x, \epsilon)$*

$$C_\epsilon^{-1} \leq \frac{\det df_x^n}{\det df_y^n} \leq C_\epsilon.$$

Then for any open set U in $\mathbb{R}^N$, there are $(x_n)_{n\in\mathbb{N}}$ a sequence in U, and a sequence $(m_n)_n$ of integers such that

$$Leb\big(U \setminus \bigcup_{n\in\mathbb{N}} B_{m_n}(x_n, \epsilon)\big) = 0, \tag{102}$$

and for all $n, k \in \mathbb{N}$, and $k \neq n$

$$B_{m_n}(x_n, \epsilon) \cap B_{m_k}(x_k, \epsilon) = \emptyset.$$

Proof. Consider the family

$$\mathcal{F} = \{B_n(x, \epsilon) : x \in \mathbb{R}^N, n \in \mathbb{N}\}.$$

Define the function δ on $\mathcal{F}$, which associates to $B_n(x, \epsilon)$ the value 2^{-n}. Let $\mu = Leb_{\mathbb{R}^N}$, and take $\tau = 2$. Let U be an open set. Since f is an expanding diffeomorphism, $\mathcal{F}$ covers U finely. It remain to verify that equation (101) is true for some $\lambda > 1$.

Fix a dynamical ball $B_n(x, \epsilon)$. Then by definition of δ, $\hat{B}_n(x, \epsilon)$ consists of dynamical balls that intersect $B_n(x, \epsilon)$, and of the form $B_m(\cdot, \epsilon)$ and $m \geq n$. We have $f^n\big(\hat{B}_n(x, \epsilon)\big) \subset B(f^n x, 3\epsilon)$. The distortion property implies that for all $n \in \mathbb{N}$ and $y \in \hat{B}_n(x, \epsilon) \subset B_n(x, 3\epsilon)$

$$C_{3\epsilon}^{-1} \leq \frac{\det df_y^n}{\det df_x^n} \leq C_{3\epsilon}. \tag{103}$$

Integrating the previous equation with respect to y on $\hat{B}_n(x, \epsilon)$ we get

$$C_{3\epsilon}^{-1} \mu\big(\hat{B}_n(x, \epsilon)\big) \det df_x^n \leq \mu\big(f^n \hat{B}_n(x, \epsilon)\big) \leq \mu\big(B(f^n x, 3\epsilon)\big). \tag{104}$$

Using the same argument we get

$$\mu\big(B(f^n x, 3\epsilon)\big) \leq C_{3\epsilon} \mu\big(B_n(x, \epsilon)\big) \det df_x^n. \tag{105}$$

We deduce that

$$\mu\big(\hat{B}_n(x, \epsilon)\big) \leq C_{3\epsilon}^2 \mu\big(B_n(x, \epsilon)\big), \tag{106}$$

so taking $\lambda = C_{3\epsilon}^2$ finishes the proof.

□

7.2 Optimal transport

We recall basic definitions about optimal transport, the interested reader can find more details, motivations and proofs in [Vil09].

Consider a separable completely metrizable topological space X (called a Polish space), and denote by $\mathcal{M}(X)$ the space of probability measures on X. For $\mu, \nu \in \mathcal{M}(X)$, the set $\Gamma(\mu, \nu)$ of coupling of μ and ν is the set of measures on $X \times X$ whose elements project to μ (resp ν) on the first (resp second) coordinate. $\Gamma(\mu, \nu)$ is not empty, because it contains $\mu \otimes \nu$, and is compact (this follows from Prokhorov's Theorem).

Optimal transport provides a way to define a distance on the space of probability measures over X, for $\mu, \nu \in \mathcal{M}(X)$ we put:

$$W_1(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int d(x,y) d\gamma(x,y), \tag{107}$$

Proposition 7.8. *W_1 is a distance on $\mathcal{M}(X)$.*

Proof. To prove that W_1 is distance, we verify the three axioms of a distance

i. Let μ, ν such that $W_1(\mu, \nu) = 0$, then there is a coupling γ such that:

$$\int d(x,y) d\gamma(x,y) = 0,$$

(because the set $\Gamma(\mu, \nu)$ is compact, so the measure γ is concentrated on the diagonal of $X \times X$, which implies that $\mu = \nu$.

ii. W_1 is symmetric because d is symmetric.

iii. Let μ_1, μ_2 and μ_3 be probabilities measures. Let γ_{12} (resp γ_{23}) be a coupling of μ_1 and μ_2 (resp μ_2 and μ_3). Disintegrate γ_{ij} with respect to μ_2, and we define the measure ν on $X \times X$ by:

$$\nu(A) = \int \gamma_{12y}(A) \gamma_{23y}(A) \; d\mu_2(y).$$

By definition ν is a coupling of μ_1 and μ_3, and we have:

$$\begin{aligned} W_1(\mu_1, \mu_3) &\leq \int d(x,z) \; d\nu(x,z) \\ &= \int \Big(\int d(x,z) d\big(\gamma_{12y}(x)\gamma_{23y}(z)\big)\Big) d\mu_2(y) \\ &\leq \int \Big(\int d(x,y) + d(y,z) d\big(\gamma_{12y}(x)\gamma_{23y}(z)\big)\Big) d\mu_2(y) \\ &\leq \int d(\cdot,\cdot) \; d\gamma_{12} + \int d(\cdot,\cdot) \; d\gamma_{23}, \end{aligned}$$

which implies the triangular inequality.

□

Remarks 7.9.

- *For* $p \geq 1$, $W_p(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \left(\int d(x,y)^p d\gamma(x,y)\right)^{1/p}$ *is also a distance on the space of probability measures on* X.
- *When* X *is compact, the topology given by the distance* W_1 *is the weak* topology.*
- *The set* $\Gamma(\mu,\nu)$ *is compact, in particular, the infimum in the definition of Wasserstein metric is achieved by a coupling* γ, *and such a coupling is called an optimal transport plan.*

Theorem 7.10 (Kantorovich duality)**.**

$$W_1(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int d(x,y)\ d\gamma(x,y) = \sup_{\|f\|_{Lip}\leq 1} \mu(f) - \nu(f). \tag{108}$$

Proposition 7.11. *Let* $f : M \to M$ *be a* C^1 *map of a compact Riemannian manifold* M, *then there is* $C > 0$ *such that for all* $\mu,\nu \in \mathcal{M}(M)$

$$W_1(f_*\mu, f_*\nu) \leq CW_1(\mu,\nu). \tag{109}$$

Proof. This follows from the definition, indeed we have

$$\begin{aligned}
W_1(f_*\mu, f_*\nu) &= \inf_{\gamma\in\Gamma(f_*\mu,f_*\nu)} \int d(x,y)d\gamma(x,y) \\
&\leq \inf_{(f,f)_*\gamma,\ \gamma\in\Gamma(\mu,\nu)} \int d(x,y)d(f,f)_*\gamma(x,y) \\
&= \inf_{\gamma\in\Gamma(\mu,\nu)} \int d(fx,fy)d\gamma(x,y) \\
&\leq \inf_{\gamma\in\Gamma(\mu,\nu)} \int C\cdot d(x,y)d\gamma(x,y) = CW_1(\mu,\nu).
\end{aligned}$$

□

7.2.1 Some examples of couplings

Example 7.12. *Let* $X = [0,1]$ *with the usual Euclidean metric. Let* $x,y \in [0,1]$, *then* $W^1(\delta_x,\delta_y) = d(x,y)$. *More generally if* $\mu = \sum_{k=1}^n a_k\delta_{x_k}$ *and* $\nu = \sum_{k=1}^n b_k\delta_{y_k}$, *then we have*

$$W_1(\mu,\nu) \leq \left(2 + \sum_{k=1}^n |a_k - b_k|\right) \max d(x_k,y_k).$$

Example 7.13. *Consider the Lebesgue measure λ on $[0,1]$, and let $\mu_n = \frac{1}{2^n}\sum_{k=1}^{2^n} \delta_{\frac{k}{2^n}}$, then*

$$W_1(\lambda, \mu_n) \leq \frac{1}{2^n}.$$

In fact, the set of finite combination of Dirac measures is dense in the space of probability measures on X.

Example 7.14. *If μ and ν are two probability measures on a Polish space X, then there exist a measurable bijection $T : X \to X$ such that $T_*\mu = \nu$ and $T_*^{-1}\nu = \mu$. In that case, the measure $\gamma = (Id, T)_*\mu$ is a coupling of μ and ν which is supported on the graph of T. This coupling is called deterministic coupling.*

7.2.2 Convexity of optimal cost

Theorem 7.15. *Let $(\Omega, \mathbb{P})$ be a probability space, and $\theta \mapsto \mu_\theta, \theta \mapsto \nu_\theta$ be two measurable functions defined on Ω with values on probability measures of M. Then we have*

$$W_1\Big(\int_\Omega \mu_\theta \ d\mathbb{P}(\theta)), \int_\Omega \nu_\theta \ d\mathbb{P}(\theta)\Big) \leq \int W_1(\mu_\theta, \nu_\theta) \ d\mathbb{P}(\theta). \tag{110}$$

Corollary 7.16. *If two probability measures μ and ν have a common mass $\tau \in (0,1)$ i.e there are probability measures λ, μ_0 and ν_0 such that*

$$\mu = \tau\lambda + (1-\tau)\mu_0$$
$$\nu = \tau\lambda + (1-\tau)\nu_0,$$

then $W_1(\mu, \nu) \leq (1-\tau)W_1(\mu_0, \nu_0)$.

7.2.3 Total variation

Let μ and ν be two probability measures, the total variation between μ and ν is given by

$$\|\mu - \nu\|_{TV} := \sup_{\|\varphi\|_\infty} |\mu(\varphi) - \nu(\varphi)| = \sup_{B\in\mathfrak{B}} |\mu(B) - \nu(B)|. \tag{111}$$

Using Kantorovich duality, it also can be seen as the Wasserstein distance with the cost $1_{x\neq y}$, in other words

$$\|\mu - \nu\|_{TV} = \inf_{\gamma\in\Gamma(\mu,\nu)} \int 1_{x\neq y} \ d\gamma(x,y).$$

When X is bounded W_1 is controlled by the total variation.

Proposition 7.17. *If X is bounded, then for all probability measures μ and ν we have*

$$W_1(\mu, \nu) \leq diam(X)\|\mu - \nu\|_{TV}.$$

Proof. We have for all $\gamma \in \Gamma(\mu, \nu)$

$$\begin{aligned} W_1(\mu, \nu) \leq \int d(x, y)\ d\gamma(x, y) &\leq \int diam(X) 1_{x \neq y} d\gamma(x, y) \\ &\leq diam(X) \int 1_{x \neq y}\ d\gamma(x, y). \end{aligned}$$

□

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