Median spaces

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Espaces médians

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Résumé

Espaces médians

Le sujet de cette thèse porte sur les espaces métriques qu'on appelle espaces médians et la direction principale concerne l'étude des actions isométriques sur les espaces médians complets connexes localement compact et de rang fini. On montre d'abord une caractérisation de la compacité locale dans le contexte de cette géométrie. Puis, on donne une classification, dans cette classes, pour les espaces médians qui admettent une action transitive. On montre qu'un tel espace est nécessairement isométrique à $\mathbb{R}^n$ munie de la métrique ℓ^1. Finalement on montre que si le groupe d'isométrie d'un espace médian X vérifie certaines conditions qui sont assez naturelles, alors les orbites de n'importe quelle action isométrique sur X sont discrètes.

Mots clés : Géométrie métrique, espaces médians, algèbres médianes, algèbres universelles, dualité.

Abstract

Median spaces

The subject of this thesis is median spaces and the main direction concerns the study of isometric actions on complete connected locally compact median space of finite rank. We first give a characterization of the local compactness in this context. Then we give a classification theorem in this class for median spaces which admit a transitive action. We show that such median spaces are necessarily isometric to $\mathbb{R}^n$ endowed with the ℓ^1-metric. Finally, we prove that when the isometry group of a median space X verifies certain conditions, then the orbits of any action on X are discrete.

Key Words : Metric geometry, median spaces, median algebras, universal algebras, duality.

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Introduction

A median space (X,d) is a metric space such that for any three points $a,b,c \in X$, there exist a unique point $m(a,b,c) \in X$, called the median points between a, b and c, such that

$$[a,b] \cap [b,c] \cap [a,c] = \{m(a,b,c)\}$$

where the interval $[a,b]$ between a and b corresponds to the set of point $x \in X$ such that $d(a,b) = d(a,x) + d(x,b)$.

Motivational examples are given by simplicial (real) trees where the intervals coincide with the geodesics. First examples of median spaces go back to [BK47] and are given by metric distributive lattices, see [Bir67] Ch. V, §9 for a definition of metric lattices.

Any median space comes naturally with a ternary operation m, called the median operation, which associates to each triple, their median point. This ternary operation encapsulates a great deal of the geometry of the median space as it detects intervals as follows

$$[a,b] = \{c \in X \mid m(a,b,c) = c\}.$$

The median operation verifies the following set of equations :

$$\begin{aligned} m(x,x,y) &= x \\ m(x,y,z) &= m(y,x,z) = m(x,z,y) \\ m(m(x,y,z),u,v) &= m(x,m(y,u,v),m(z,u,v)) \end{aligned}$$

Conversely, any set (A,m) endowed with a ternary operation $m : A^3 \to A$ which verifies the above set of equations will enjoy the same property as the median operation obtained from a median space. We call (A,m) a median algebra and a generic example is given by the set of subsets of a set X, where the ternary operation is defined as follows :

$$\begin{aligned} m_{\mathcal{P}}(A,B,C) &= (A \cup B) \cap (B \cup C) \cap (A \cup C) \\ &= (A \cap B) \cup (B \cap C) \cup (A \cap C), \end{aligned}$$

for any $A,B,C \in \mathcal{P}(X)$.

The structure of median algebras first appears as a way to characterize boolean algebra in [Gra47] and distributive lattices in [BK47] by the mean of a ternary operation verifying some set of postulates.

We note that the set of axioms cited above are not canonical and median algebras were studied under different set of postulates (and different names also). In fact, median algebras can be approached from different angles : median ternary operation, intervals (or segments) seen as a map from $X^2 \to \mathcal{P}(X)$ and a ternary relation called betweenness. The equivalence between the three approaches was shown in [Sho54].

Median spaces gained interest from the field of graph theory with the notion median graph which gives a natural generalization to simplicial trees and hypercubes. The link between graphs and median algebras appeared implicitly in [BK47] and [Ava61], where the graph is the one associated with the distributive lattice. In the same vein, this association was studied in [Neb70] and [Neb71] were the median graphs are the principal objects of study. For an overview on median graphs and their applications see [Mul11].

Convexity in median algebras is defined with respect to intervals. A halfspace is a convex subset with a convex complement. Assuming Zorn's lemma, such halfspaces exist and any pair of points are separated by a halfspace, see Theorem 1 [Nie79] where ideal and prime ideal stand for convex subset and halfspace respectively. The set of halfspace of a median algebra comes with a natural structure of a partially ordered set with a complementary operation (poc set) $(P, \leqslant, *, 0)$ where 0 is a minimal element and the complementary operation is an involution $* : P \to P$ such that :

$$\begin{aligned} \forall\, p, q \in P, \quad p \leqslant q \quad &\Longrightarrow \quad q^* \leqslant p^* \\ \forall\, p \in P, \quad p \leqslant p^* \quad &\Longrightarrow \quad p = 0 \end{aligned}$$

A duality result a la Stone was shown between the category of median algebras and the category of poc sets in [Isb80] and [Wer81].

Another important class of examples of median spaces is given by CAT(0) cube complexes. These objects played a key role in understanding groups which arise as the fundamental group of three dimensional manifolds. A CAT(0) cube complex is a cube complex which is a CAT(0) space when endowed with the length metric, where each cell is endowed with the euclidean metric. If we consider the length metric where each cell is endowed with the ℓ^1-metric, we obtain a median space (see Theorem 6.8 [Che00]).

In the same spirit of Stalling's theorem about ends of group, It was shown in [Sag95] that a group has more then one end with respect to a subgroup if and only if it acts essentially on a CAT(0) cube complex.

In [NR98] and [Rol16], the authors linked Sageev construction of the CAT(0) cube complex with the duality between median algebra and poc set, where the 0-skeleton of the CAT(0) cube complex comes with a median algebra structure. The link with median algebra was implicit in [NR98] and more explicit and detailed in [Rol16].

In fact, the graph obtained from the 1-skeleton of any CAT(0) cube complex is a median graph and any median graph arises as such, as it was shown in [Che00].

In [Nic04] and [CN05], the relation between group actions on space with walls (see [HP98]) and CAT(0) cube complexes was made. In [CDH10], it was shown that a group acts isometrically on a space with measured walls, which is the non discrete version of space with walls (see [CMV04]), if and only if it acts isometrically on a median space.

Moreover, the following characterization of Kazhdan's property (T) and Haagerup property was made :

Theorem ([CDH10] Theorem 1.2)**.** Let G be a locally compact second countable group.
- The group G has Kazhdan's property (T) if and only if any continuous isometric G-action on a median space has bounded orbits.
- The group has Haagerup property if and only if there exist a continuous proper isometric G-action on some median space.

There is a suitable and practical way of speaking of dimension in the case of median spaces, given by the notion of the rank. Loosely speaking, this detects the highest dimension of discrete cubes, endowed with the ℓ^1-metric, that can be isometrically embedded into the space, see Definition 1.1.9.

In two directions of this thesis, we will be working in the realm of finite rank median spaces. In the other direction, we will be investigating the duality in a certain class of complete median spaces of infinite rank. The latter class encompasses the class of locally convex median spaces (see Definition 2.1.1).

Isometric actions on locally compact median space

Median spaces of finite rank generalize finite dimensional CAT(0) cube complexes the same way $\mathbb{R}$-trees generalize simplicial trees. Any CAT(0) cube complex has a canonical metric which makes it a median space. The converse holds for complete connected median spaces of finite rank as it was shown in [Bow16] that these spaces admit a bilipschitz equivalent metric which is CAT(0). Any geometric action, i.e. properly discontinuous cocompact action, on an $\mathbb{R}$-tree gives rise to a geometric action on a simplicial tree. It is unknown whether any geometric action on a finite rank median space gives rise to a geometric action on a finite dimensional CAT(0) cube complex, see [CD17] subsection 1.b. It is false in the infinite rank case and examples are given by irreducible lattices in a product of $\mathrm{SO}(n,1)$. It was shown in [CD17] that $\mathrm{SO}(n,1)$ acts properly cocompactly on an infinite rank median space and we know by results given in [Fio19] that any action of an irreducible lattice in a product of $\mathrm{SO}(n,1)$ on a median space of finite rank has finite orbit.

Many evidence for an affirmative answer to the latter question in the finite rank case are given by works of E. Fioravanti which extended many results concerning action on CAT(0) cube complexes to the case of finite rank median spaces.

In [Sha00], the author obtains as a consequence of his superrigidity result that irreducible lattices in higher rank lie groups, except few cases, have the fixed point property for their isometric actions on trees. In the case of CAT(0) cube complexes, it was done in the Appendix [CFI16]. A similar result follows in the case of complete median spaces of finite rank as it was shown in [Fio19].

A version of Tits alternative for groups acting on CAT(0) cube complexes was shown in [SW05], [CS11]. The same results hold in the case of median spaces of finite rank and it was done in [Fio18] by extending the machinery used in [CS11] to the latter case.

It is a well known fact that isometry group of real trees are semi simple. The same hold for CAT(0) cube complexes as it was shown in [Hag21]. The result was extended to the case of connected finite rank median spaces in [Fio21].

In one direction of this thesis, we will be investigating isometric action on complete locally compact median space of finite rank. We first show, assuming certain conditions on the isometry group of the median space, that orbits are discrete :

Theorem A. (see Theorem A) Let X be an irreducible complete connected locally compact median space of finite rank. Let us assume that the action of $G := Isom(X)$ on X is Roller non elementary, Roller minimal and minimal. Then any G-orbit is discrete.

The argument of the proof relies in an essential way on the machinery developed in [Fio19].

Assuming that the median space is locally compact imposes a certain configuration on the halfspaces which are transverse to a ball. We give the following characterization of compact subsets by the combinatoric of the halfspaces which are transverse to the subset :

Theorem B. (see subsection 4.2.2) Let X be a complete connected median space of rank n. Let C be a closed bounded subset of X. Then the following are equivalent :

1. The subset C is compact.
2. For any $x_0 \in C$ and $\epsilon > 0$, there exist $x_1, ..., x_{k_\epsilon} \in C$ such that for any $x \in C$ we have $d(x, [x_0, x_i]) \leqslant \epsilon$ for some $i \in \{1, .., k_\epsilon\}$.
3. For any $\epsilon > 0$, if $\mathcal{H}_\epsilon$ is a family of pairwise disjoint halfspaces transverse to C and of depth bigger than ϵ in the convex hull of C, then it is finite.

Where the depth of a subset $H \subset X$ inside another subset C is defined as $depth_C(H) := sup(\{d(x, H^c)/x \in C\}$.

Theorem B falls within the framework of the duality between the category of median spaces and the category of pointed measured partially ordered sets with inverse operation. It characterizes the subcategory of the latter category which is dual to the subcategory of complete compact connected median space of finite rank.

Having Theorem B in hand, we prove the following classification of complete connected locally compact median space of finite rank which admits a transitive action :

Theorem C. Let X be a connected locally compact median space of finite rank which admits a transitive action, then X is isometric to $(\mathbb{R}^n, l^1)$.

The argument consists of considering the set of halfspaces which are "branched" at an arbitrary point. A halfspace is a convex subset such that its complementary is also convex. The set $\mathcal{H}_x(X)$ of halfspaces branched at a point $x \in X$ in a complete median space of finite rank is the set of halfspaces $\mathfrak{h} \subset X$ such that $x \in \bar{\mathfrak{h}} \cap \bar{\mathfrak{h}^c}$. The set $\mathcal{H}_x(X)$ can be seen as the extension of the notion of the valency at a point x in an $\mathbb{R}$-tree to the case of complete median space of finite rank. Theorem C is obtained then as a consequence of the following results :

Theorem D. (see subsection 4.3.2) Let X be a complete connected median space of rank n which admits a transitive action. If for some or equivalently any $x \in X$:

1. The set $\mathcal{H}_x$ contains no triple of pairwise disjoint halfspaces then the space X is isomorphic to $(\mathbb{R}^n, l^1)$.
2. The set $\mathcal{H}_x$ contains three halfspaces which are pairwise disjoint then the space X is not locally compact.

We note that the result of Theorem C still holds when we assume that the action of the group of isometry is topologically transitive, although we do not give a proof.

Duality

In [Sto36], M. H. Stone showed a duality between the category of boolean algebra and the category of Stone space (Totally separated compact space). The contravariant functor from the latter category to the former one associates to each Stone space X, the set of continuous maps from X to $\{0,1\}$, where its boolean algebra structure is induced from $\mathcal{P}(X) \cong \{0,1\}^X$.

The other contravariant functor associates to each boolean algebra B the set of morphisms of boolean algebras from B to the trivial boolean algebra $\{0,1\}$, endowed with the topology of pointwise convergence. It identifies naturally with the set $\{0,1\}^B$ endowed with the product topology.

This duality extends to the framework of median algebras as it was proved in [Isb80] (see Theorem 6.13 therein), where it was shown in particular that the category of Stone median algebra and the category of poc sets are dually equivalent. A Stone median algebra is a median algebra endowed with a Stone topology such that the ternary operation is continuous.

Theorem ([Isb80] Theorem 6.13, [Rol16] Theorem 5.3)**.** A Stone median algebra is isomorphic to its double dual.

An analogue of the above duality holds in the case of median space as it was first shown in [CDH10]. The additional metric structure on the median algebra is encoded in a structure of measured space on the set of halfspaces. There is a canonical way to endow the set of halfspaces $\mathcal{H}(X)$ of a median space X with a structure of measured space such that the measure of the set of halfspaces separating two points coincides with the distance between them. To each set of halfspaces with such structure of measured space, there is a canonical median space $\mathcal{M}(X)$, the double dual of X, associated to it and there is a natural $Isom(X)$-equivariant isometric embedding of X into $\mathcal{M}(X)$.

The duality in the metric case was investigated in more details in [Fio20] and the analogy with Isbell duality is more explicit where the notion of measured poc sets is introduced. In particular, E. Fioravanti showed that a complete locally convex median space is isometric to its double dual (Theorem A [Fio20]).

We say that a median space X verifies the strong separation if for any $x, y \in X$ there exists a halfspace $\mathfrak{h} \subset X$ which contains x in its interior and contains y in the interior of its complement $\mathfrak{h}^c$. This class of median spaces strictly contains the class of locally convex median spaces. We extend the latter theorem using another proof to the case of complete median space which admits a *strong separation property* (see Definition 2.1.1).

Theorem E. (see Theorem 2.2.1) Let X be a complete median space which satisfies the strong separation property. Then X is isometric to its double dual $\mathcal{M}(X)$.

The proof of the above Theorem relies uniquely on the structure of measured space on the set of halfspaces introduced in [CDH10] which is "coarser" then the one introduced in [Fio20]. An essential ingredient in the proof of Theorem 2.2.1 is the following remark :

Proposition F. (see Lemma 2.2.6) Let X be a complete median space. Then any interval is compact with respect to the topology where the closed subsets are generated by bounded gate convex subsets of X.

Actions of S-arithmetic lattices on median spaces

By the results of Theorem 1.2 [CDH10], the action of any lattice in a locally compact group satisfying the Kazhdan's property (T) has bounded orbits. Examples involves lattices in simple algebraic group of rank greater than or equal two over a local field, lattices in $\mathrm{Sp}(n,1)$, the subgroup of special linear transformations which preserve a quaternionic hermitian form of signature $(n,1)$.

In the other side of the spectrum, groups having the Haagerup property always admit a proper isometric on a median space. Hence, lattices arising in a product of $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ or in a product of their universal cover, acts properly on a median space. In fact, lattices in a product of $\mathrm{SO}(n,1)$ acts geometrically on a locally compact median space of infinite rank (see [CD17]).

However, when the median space is of finite rank irreducible lattices in a product of $\mathrm{SO}(n,1)$ cannot even act properly on a complete median space of finite rank. This is a consequence of a result by E. Fioravanti which is much more general.

Theorem ([Fio19] Corollary D and Theorem C)**.** Let X be a complete median space of finite rank and let Γ be an irreducible lattice in a connected semisimple Lie group of higher rank. Then any isometric action of Γ on X has finite orbit.

In the other hand, there is no proper action of a discrete solvable group which is not virtually abelian on a complete median space of finite rank (see Theorem A [Fio18]). Hence, non uniform lattices in $\mathrm{SU}(n,1)$, for $n \geqslant 2$, does not act properly on a complete median space of finite rank as they contains Heisenberg subgroup obtained from the intersection of the lattices with a horospheric subgroup of $\mathrm{SU}(n,1)$. The existence of proper action of uniform lattices in $\mathrm{SU}(n,1)$ on a complete median space of finite rank is unknown, but evidences for a negative answer are shown in [DP19].

The group $\mathrm{PSL}(2,\mathbb{Q}_p)$ acts geometrically on a homogeneous simplicial tree of valency $p+1$, the Bruhat-Tits tree associated to $\mathrm{SL}(2,\mathbb{Q}_p)$. Hence, lattices in a product of $\mathrm{PSL}(2,\mathbb{Q}_p)$ acts geometrically on a median space of finite rank, given by the ℓ^1-product of the Bruhat-Tits tree. The groups $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ being isomorphic to $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3,1)$ respectively, lattices in a product of $\prod_{i\in S}\mathrm{PSL}(2,k_i)$ acts geometrically on a locally compact median space of infinite rank, where k_i is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$.

With regards to the finite rank case, we show the following :

Theorem G. (see Theorem 3.3.1) Let $\Gamma \leqslant \prod_{i\in S}\mathrm{PSL}(2,k_i)$ be a lattice such that its projection into each factor $\mathrm{PSL}(2,k_i)$ is dense, where k_i is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$. Then there is no proper action of Γ on a complete median space of finite rank.

The proof of the above theorem relies heavily on Fioravanti's machinery, especially on its superrigidity result with regards to actions of lattices in a product of locally compact groups on complete median spaces of finite rank ([Fio19] Theorem B).

Document structure

In Chapter 1, we describe the median geometry first from the general algebraic point of view of median algebras, then from the particular metric point of view of metric spaces. We describe the duality in both cases and make the analogy between the algebraic and metric one.

In Chapter 2, we introduce the class of strongly separated median space and prove all the needed results, for instance Proposition F, to prove the duality Theorem E.

Chapter 3 is devoted to the proof of Theorem G. In the first section we describe briefly the structure of the p-adic group $\mathrm{PSL}(2,\mathbb{Q}_p)$ and the needed properties satisfied by lattices in a product of $\mathrm{PSL}(2,k_i)$. In the second section, we recall Fiovaranti's machinery with regards to isometric actions on complete finite rank median space.

In Chapter 4, we investigate isometric action on complete locally compact median space of finite rank. In the first section, we prove an embedding lemma of the convex hull that we will be using in the proof of the classification Theorem C. In the second section, we prove the characterization of compactness Theorem B. The third and fourth sections are devoted to the proof of Theorems D and A.

Chapitre 1

Median geometry

1.1 Algebraic structure of the median geometry

1.1.1 Median algebra

Definition 1.1.1. (Median algebra) A ***median algebra*** is a set M with a ternary operation $m : M \times M \times M \to M$ which verifies the following set of equations :

$$\begin{aligned} m(x,x,y) &= x. \\ m(x,y,z) &= m(y,x,z) = m(x,z,y). \\ m(m(x,y,z),u,v) &= m(x,m(y,u,v),m(z,u,v)). \end{aligned}$$

A ***median morphism*** $\Phi : M \to N$ between two median algebras is a map which commutes with the ternary operation m, i.e. $\Phi(m_M(x,y,z)) = m_N(\Phi(x),\Phi(y),\Phi(z))$. Let M be a median algebra, the ***interval*** between any two point $a,b \in M$ denoted by $[a,b]$ is the set of fixed points of the ternary operation $m(a,b,*)$. The following properties ensure us that the intervals of median algebra can support a strong notion of convexity (see Section 2 Intervals [Rol16]) :

Remark 1.1.2. Let M be a median algebra and let $a,b,c,d \in M$. We have then :

— If $c,d \in [a,b]$ then $[c,d] \subseteq [a,b]$.

— $[a,b] \cap [b,d] = [b,m(a,b,c)]$.

— $[a,b] \cap [b,c] \cap [a,c] = \{m(a,b,c)\}$.

We note that by the third property can serve as an axiom to define median algebras. More precisely, we have the following alternative definition by mean of segments :

Definition 1.1.3. Let M be a set endowed with a map $I : M \times M \to \mathcal{P}(M)$ which verifies the following :

1. $I(x,x) = \{x\}$.

2. If $z \in I(x,y)$ then $I(x,z) \subseteq I(x,y)$ (Convexity).

3. For any triple $x, y, z \in M$, the three subsets $I(x,y), I(y,z)$ and $I(x,z)$ intersect in a unique point.

The equivalence between the two definitions was shown in [Sho54], see Theorem 4.11 therein. Sholander does not assume the interval map to be symmetric in his axioms, but it is easily deduced from the postulate (γ) therein and remarking from the postulate (β) that $b \in [a,b]$, by letting $a = b$ in the postulate (α) (see [Sho54] Postulates 4.10 pp.806).

Examples 1.1.4. 1. Let $(O,<)$ be a totally ordered set. The ternary operation which associates to any three points $a, b, c \in O$ such that $a < b < c$ the point b endows the set O with a structure of a median algebra. For any $a, b \in O$ such that $a < b$, the interval between them is given by $[a,b] = \{c \in O \mid a < c < b\}$. In particular, the real line has a natural structure of median algebra.

2. The arbitrary product of a family of median algebras $(M_i, m_i)_{i \in I}$ is naturally endowed with the product structure where the median point of the triple $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in (M_i)_{i\in I}$ is $(m_i(a_i, b_i, c_i))_{i\in I}$.

3. More generally, let $(L, \wedge, \vee)$ be a distributive lattice. A median structure on L is given by the following ternary operation

$$m(a,b,c) := (a \wedge b) \vee (b \wedge c) \vee (a \wedge c) = (a \vee b) \wedge (b \vee c) \wedge (a \vee c).$$

The interval between $a, b \in L$ is the set of elements which contain the meet of a, b and is contained in the join of a, b, i.e. $[a,b] = \{c \in L \mid (a \wedge b) \vee c = c \text{ and } (a \vee b) \wedge c = c\}$. In particular boolean algebras admit a natural structure of median algebra. A very particular example is the trivial boolean algebra $\{0,1\}$. We will see later that any median algebra embeds into a product of the median algebra $\{0,1\}$.

A subset $C \subset M$ is ***convex*** if for any $a, b \in M$ the interval $[a,b]$ lies in C.

A well known fact, due to Eduard Helly, is that the intersection of a finite family of convex subsets in the euclidean space $\mathbb{R}^n$ is empty if and only if the intersection of some subfamily of cardinal less than or equal $n+1$ is empty. In the particular case of the real line, the intersection of a finite family of convex subsets is empty if and only if there exist two convex subsets of the family which are disjoint. The same holds for median algebra (see Theorem 2.2 [Rol16]) :

Theorem 1.1.5 (Helly's Theorem)**.** *Let X be a median space and let $C_1, .., C_n \in X$ be a family of pairwise intersecting convex subsets. Then their intersection is not empty.*

Halfspaces and convex walls :

Definition 1.1.6. (Halfspace) Let M be a median algebra. A convex subset $\mathfrak{h} \subset M$ is a ***halfspace*** if its complementary $\mathfrak{h}^c$ is also convex. A ***convex wall*** is a couple $(\mathfrak{h}, \mathfrak{h}^c)$ where

$\mathfrak{h} \subset M$ is a halfspace. We denote by $\mathcal{H}(M)$ and $\mathcal{W}_c(M)$ the sets of halfspaces and convex walls of M respectively.

For any $A, B \subseteq M$, we denote by $\mathcal{H}(A,B)$ the set of halfspaces which separate B from A, *i.e.*$\mathcal{H}(A,B) := \{\mathfrak{h} \in \mathcal{H}(X)/B \subseteq \mathfrak{h}, A \subseteq \mathfrak{h}^c\}$. We will be using the following notation $\tilde{\mathcal{H}}(A,B) = \mathcal{H}(A,B) \cup \mathcal{H}(B,A)$ when we do not need to keep track on the "orientation". In the same vein, we define the ***convex walls interval*** between A and B as $\mathcal{W}(A,B) := \{(\mathfrak{h}, \mathfrak{h}^c)/\mathfrak{h} \in \mathcal{H}(A,B)\}$. When A and B are singletons, we simply write $\mathcal{H}(x,y)$ and $\mathcal{W}(x,y)$. When there is no confusion, we will just say walls instead of convex walls.

We say that a halfspace is ***transverse*** to a subset $A \subset M$ if both $\mathfrak{h} \cap A$ and $\mathfrak{h} \cap A^c$ are not empty. We denote by $\mathcal{H}(A)$ the set of halfspaces which are transverse to A and by $\mathcal{H}_A$ the set of halfspaces which contain A.

Assuming Zorn's lemma, not only halfspaces exist but they separate any two convex subsets in a median algebra (see Theorem 2.8 in [Rol16]) :

Theorem 1.1.7. *(Separation theorem) Let M be a median algebra and let $A, B \subseteq M$ be two disjoint convex subsets. There exist a halfspace $\mathfrak{h} \subseteq M$ such that $A \subseteq \mathfrak{h}$ and $B \subseteq \mathfrak{h}^c$.*

The above separation property is of fundamental importance, especially in the duality between the median algebra and its set of halfspaces (See Subsection 1.1.3).

We have a correspondence between the set of halfspaces of a median algebra M and the set of median morphisms between M and the median algebra $\{0,1\}$. The correspondence is given by considering the characteristic function over a halfspace $\mathfrak{h}$ and the inverse image 0 or 1 under such median morphisms. Hence, any median algebra M embeds into the product $\prod_{\mathfrak{h} \in \mathcal{H}(M)} \{0,1\}$ by the product of the characteristic functions $\mathbb{1}_h$'s. Theorem 1.1.7 ensures that the later median morphism is injective.

In the language of universal algebra, the above paragraph translates into saying that the category of median algebras with median morphisms is the same as the variety generated by median algebra $\{0,1\}$, that is, the smallest category containing $\{0,1\}$ and stable under considering submedian algebras, median quotient and direct product.

Remark 1.1.8. Let $f : M \to N$ be a morphism of median algebras. Then the inverse image of any halfspace of N by f is a halfspace of M. This is due to the fact that the inverse image of a convex subset is convex and that $f^{-1}(A^c) = (f^{-1}(A))^c$.

Halfspaces give a natural way of speaking about "dimension" in median algebras :

Definition 1.1.9. (Rank) Let M be a median algebra.

— Let $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}(M)$ be two halfspaces. We say that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are ***transverse*** if the following intersections are not empty :

$$\mathfrak{h}_1 \cap \mathfrak{h}_2 \quad \mathfrak{h}_1^c \cap \mathfrak{h}_2 \quad \mathfrak{h}_1 \cap \mathfrak{h}_2^c \quad \mathfrak{h}_1^c \cap \mathfrak{h}_2^c.$$

— We say that the median algebra is of ***rank*** n if there exist a family of pairwise transverse halfspaces $\mathfrak{h}_1, .., \mathfrak{h}_n \in \mathcal{H}(M)$ where n is maximal.

Convex hull and Gate convex subsets

Definition 1.1.10. (Convex hull) Let M be a median algebra and let $A \subset M$. The ***convex hull*** of A, that we denote by $Conv(A)$, is the intersection of all convex subsets containing A.

The convex hull of the union of subsets which form a directed set is the union of the convex hull of each set. Therefore, we get the following proposition (Corollary 2.5 [Rol16]) :

Proposition 1.1.11. *Let X be a median algebra, then for any $A \subseteq X$ we have :*

$$Conv(A) = \bigcup_{x_1,..,x_n \in A} Conv(\{x_1,..,x_n\}).$$

For any $a, b, x \in M$ and $y \in [a,b]$, we have :

$$\begin{aligned} m(x, m(x,a,b), y) &= m(x, m(x,a,b), m(y,a,b)) \quad \text{As } y \in [a,b] \\ &= m(m(x,x,y),a,b) \\ &= m(x,a,b) \end{aligned}$$

The median point $m(x,a,b)$ lies in the interval $[x,y]$ for any $y \in [a,b]$. Hence, if we fix the two first variables of the ternary operation $m(*,*,*)$, the morphism obtained can be seen as the nearest point projection into the interval $[a,b]$. This motivates the following definition :

Definition 1.1.12. (Gate convex) A convex subset $C \subset M$ is ***gate convex*** if for any $x \in M$ there exist a point $\pi_C(x) \in C$, called the gate projection of x into C, such that for any $a \in C$ the point $\pi_C(x)$ lies in the interval $[x,a]$.
We denote by $\pi_C : M \to C$ the retraction which associates to each point x its gate projection in C.

We note that in the literature regarding median algebra, it is the term ***retract*** that it is used to denote gate convex subset, whereas the latter is used in the case of median space.

Remark 1.1.13.
1. Let $C \subseteq M$ be a gate convex subset and let $a \in M$. Any halfspace which separates a from $\pi_C(a)$ must separate a from C as for any point $c \in C$, we have $\pi_C(a) \in [a,c]$.
2. The gate projection is a median morphism from M to C where the latter is endowed with the median algebra structure induced from M.

Lemma 1.1.14. *Let M be a median algebra and let $C \subseteq M$ be a gate convex subset and $A \subset M$ a convex subset such that $A \cap C \neq \varnothing$. Then the projection of A into C lies in $A \cap C$.*

Proof. Let us consider a point $a \in A$. The intersection $C \cap A$ being assumed to be not empty, we choose a point $c \in C \cap A$. We conclude then $\pi_C(a) \in [a,c] \subseteq C$. $\square$

Remark 1.1.15. Let M be a median algebra and let $C \subseteq M$ be a gate convex subset. Any halfspace $\tilde{\mathfrak{h}}$ of C, when the latter is seen as a median algebra, lifts to a halfspace of M under the inverse of image of the gate projection onto C (see Remark 1.1.8). In the other hand, any halfspace $\mathfrak{h} \in \mathcal{H}(M)$ gives rise to a halfspace $\tilde{\mathfrak{h}} := \mathfrak{h} \cap C$ in $\mathcal{H}(C)$. By Lemma 1.1.14, we have $\pi_C^{-1}(\mathfrak{h} \cap C) = \mathfrak{h}$. Hence, we have a correspondance between the halfspace of C and the halfspace of M which are transverse to C.

Definition 1.1.16. (Join) Let M be a median algebra and let $A, B \subseteq M$.

— The ***join*** $[A, B]$ between is the union of all intervals having endpoints in A and B.

— The n***-iterated join*** A^n of a subset A is defined recursively by $A^n = [A^{n-1}, A^{n-1}]$ and $A^0 = A$.

By the paragraph preceding Definition 1.1.12, we know that intervals are gate convex. Moreover, we have the following proposition :

Proposition 1.1.17. *Let M be a median algebra and let $C_1, C_2 \subseteq M$ be two gate convex subset. Then the join $[C_1, C_2]$ is also gate convex.*

Proof. Let us consider $x \in M$ and set $\tilde{x} := m(x, \pi_{C_1}(x), \pi_{C_2}(x))$. Let $y \in [C_1, C_2]$ and let us show that $\tilde{x} \in [x, y]$. Before doing so, we show that $y \in [\pi_{C_1}(y), \pi_{C_2}(y)]$. Note that any halfspace which separates y from $\pi_{C_i}(y)$ must separate y from C_i. Hence, if y lies outside the interval $[\pi_{C_1}(y), \pi_{C_2}(y)]$, it must lie outside the convex hull of $C_1 \cup C_2$. It last to show that $\tilde{x} \in [x, y]$. By Proposition 1.1.19, there exist $y_1 \in C_1$ and $y_2 \in C_2$ such that $y \in [y_1, y_2]$. Hence, we have $m(\pi_{C_1}(x), x, y) = \pi_{C_1}(x)$ and $m(\pi_{C_2}(x), x, y) = \pi_{C_2}(x)$ as $\pi_{C_1}(x), \pi_{C_2}(x) \in [x, y]$. Therefore, we conclude by :

$$\begin{aligned} m(\tilde{x}, x, y)) &= m(m(x, \pi_{C_1}(x), \pi_{C_2}(x)), x, y) \\ &= m(x, m(\pi_{C_1}(x), x, y), m(\pi_{C_2}(x), x, y)) \\ &= m(x, \pi_{C_1}(x), \pi_{C_2}(x)) \end{aligned}$$

□

As a consequence of the above proposition, we remark the following :

Proposition 1.1.18. *Let M be a median algebra. Then the convex hull of any finite subsets is a gate convex.*

We deduce from Propositions 1.1.11 and 1.1.18, the following description of the convex hull between two convex subsets :

Proposition 1.1.19. *Let M be a median algebra and let $C_1, C_2 \subset M$ be two convex subsets. The convex hull of the union of C_1 and C_2 is equal to their join $[C_1, C_2]$.*

When the median algebra is of finite rank, it was shown in [Bow13] (Lemma 6.4) that the convex hull of any subset is obtained by iterating a finite number of time the join operation :

Proposition 1.1.20. *Let M be a median algebra of rank n and let $A \subseteq M$. We have then :*

$$Conv(A) = A^n.$$

The same result holds when the finiteness restriction is on the cardinal of the set rather than the rank of the space :

Proposition 1.1.21. *Let M be a median algebra and let $F \subset M$ be a subset of cardinal n. we have then :*

$$Conv(F) = F^n.$$

Proof. We proceed by induction on the cardinal of the set F. If F is a singleton there is nothing to show. Let us assume that the proposition holds for subset of cardinal $n-1$. By Proposition 1.1.19, we conclude that :

$$Conv(F) = [Conv(F \backslash \{a\}), \{a\}].$$

□

Let us look more closely at the convex hull between two gate convex subsets.

Proposition 1.1.22. *Let M be a median algebra and let $C_1, C_2 \subseteq M$ be two gate convex subsets. Then $Conv(\pi_{C_1}(C_2), \pi_{C_2}(C_1))$ is isomorphic to $\pi_{C_1}(C_2) \times [x, \pi_{C_2}(x)]$ where x is any point in $\pi_{C_1}(C_2)$.*

We call the convex subset $Conv(\pi_{C_1}(C_2), \pi_{C_2}(C_1))$ the ***bridge*** between C_1 and C_2, let us denote it by $\mathcal{B}(C_1, C_2)$.

We note that when the median algebra is a median graph, the above notion of bridge corresponds with the notion of bridge between halfspaces of CAT(0) cube complexes that arises in [CFI16] Section 2.G.

Before proving Proposition 1.1.22, we will be needing some lemmas.

Lemma 1.1.23. *Let M be a median algebra and let $C_1, C_2 \subset M$ be two gate convex subsets. Then the image of C_1 under the projection onto C_2 is convex.*

Proof. Let us consider $x_1, y_1 \in C_1$ and set $x_2 := \pi_{C_2}(x_1), y_2 := \pi_{C_2}(y_1) \in C_2$. After choosing a point $x \in [x_2, y_2]$ and setting $\tilde{x} = \pi_{C_2}(\pi_{C_1}(x))$, let us show that there is no halfspace that separates x and $\tilde{x}$, which implies that $x = \tilde{x}$. Let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace containing x. If $\mathfrak{h}$ does not separate x_2, y_2, then it will contain the point $\tilde{x}$ as the latter belongs to the interval $[x_2, y_2]$. Let us assume now that $\mathfrak{h}$ does separate x_2 from y_2. By Lemma 1.1.14, the halfspace $\mathfrak{h}$ would also separate x_1 from x_2. Again by Lemma 1.1.14, we deduce that $\pi_{C_1}(x)$ and $\tilde{x} = \pi_{C_2}(\pi_{C_1}(x))$ also belongs to $\mathfrak{h}$. □

Lemma 1.1.24. *Let C_1 and C_2 be two gate convex subset. Then we have*

$$(\pi_{C_1} \circ \pi_{C_2})_{|\pi_{C_1}(C_2)} = \mathbb{1}_{\pi_{C_1}(C_2)}.$$

Proof. Let us consider two gate convex subsets $C_1, C_2 \subseteq M$ and a point $x \in C_2$. We have $\pi_{C_2}(\pi_{C_1}(x)) \in [x, \pi_{C_1}(x)]$. The gate projection onto gate convex subset being a morphism of median algebra, we compose the latter inclusion by the gate projection onto C_1 to obtain

$$\pi_{C_1}(\pi_{C_2}(\pi_{C_1}(x))) \in \pi_{C_1}([x, \pi_{C_1}(x)]) = [\pi_{C_1}(x), \pi_{C_1}(\pi_{C_1}(x))] = \{\pi_{C_1}(x)\}.$$

The equality between $\pi_{C_1}([x, \pi_{C_1}(x)])$ and $[\pi_{C_1}(x), \pi_{C_1}(\pi_{C_1}(x))]$ is a consequence of Lemma 1.1.23. □

Proof of Proposition 1.1.22. Let us denote by C the convex hull between $\pi_{C_1}(C_2)$ and $\pi_{C_2}(C_1)$. By Proposition 1.1.19 and Lemma 1.1.24, we have

$$C = \bigcup_{\substack{x \in \pi_{C_1}(C_2) \\ x' \in \pi_{C_2}(C_1)}} [x, x'] = \bigcup_{x \in \pi_{C_1}(C_2)} [x, \pi_{C_2}(x)]..$$

Note that we have $[x, \pi_{C_2}(x)] \cap C_1 = \{x\}$ and $[x, \pi_{C_2}(x)] \cap C_2 = \{\pi_{C_2}(x)\}$. It last to show that for any $y \in C_1$, the intervals $[x, \pi_{C_2}(x)]$ and $[y, \pi_{C_2}(y)]$ are isomorphic. By Proposition 1.1.14, we have $m(y, x, \pi_{C_2}(x)) = x$ and $m(\pi_{C_2}(y), x, \pi_{C_2}(x)) = \pi_{C_2}(x)$. Hence, by Lemma 1.1.23, we have $\pi_{[x,\pi_{C_2}(x)]}([y, \pi_{C_2}(y)]) = [x, \pi_{C_2}(x)]$ and $\pi_{[y,\pi_{C_2}(y)]}([x, \pi_{C_2}(x)]) = [y, \pi_{C_2}(y)]$. By Lemma 1.1.24, we conclude that the intervals $[x, \pi_{C_2}(x)]$ and $[y, \pi_{C_2}(y)]$ are isomorphic. □

Remark 1.1.25. By remark 1.1.13 and Proposition 1.1.22, we deduce that for any two gate convex subset $C_1, C_2 \subset M$, there exist $c_1 \in C_1$ and $c_2 \in C_2$ such that $\mathcal{H}(C_1, C_2) = \mathcal{H}(c_1, c_2)$. It is enough to take $c_1 \in \pi_{C_1}(C_2)$ and $c_2 := \pi_{C_2}(c_1)$.

The following is a remark regarding the equivalence between the existence of a half-space which is transverse to two gate convex subsets and the bridge relating them being a singleton.

Proposition 1.1.26. *Let M be a median algebra and let $C_1, C_2 \subset X$ be two gate convex subsets. We have then $\mathcal{W}_c(C_1) \cap \mathcal{W}_c(C_2) = \mathcal{W}_c(\pi_{C_1}(C_2))$. In particular, there is no halfspace which is transverse to both C_1 and C_2 if and only if $\pi_{C_1}(C_2)$ and $\pi_{C_2}(C_1)$ are singletons.*

Proof. Note that by Proposition 1.1.22, the convex subsets $\pi_{C_1}(C_2)$ and $\pi_{C_2}(C_1)$ are isomorphic. Let $\mathfrak{h} \in \mathcal{H}(M)$ be a halfspace which is transverse to both C_1 and C_2. By Lemma 1.1.14, the images of $C_1 \cap \mathfrak{h}$ and $C_1 \cap \mathfrak{h}^c$ under the projection π_{C_2} lie in $C_2 \cap \mathfrak{h}$ and $C_2 \cap \mathfrak{h}^c$ respectively. For the other inclusion, any halfspace which separates two points of $\pi_{C_1}(C_2)$ is a halfspace which is transverse to both $\pi_{C_1}(C_2)$ and $\pi_{C_2}(C_1)$ by Lemma 1.1.14. □

Gluing median algebras along gate convex subsets As median algebras are particular case of universal algebras, they benefit from the properties of the latter. In particular, the category of median algebra with median morphisms is stable under products, projective limits and inductive limits. Thus, this give a way to construct new examples of median algebra. One may also construct new examples by gluing median algebra along gate convex subsets by the mean of a median isomorphism, that what we will describe in the following paragraph

Proposition 1.1.27. *Let (M_1, m_1) and (M_2, m_2) be two median algebras. Let $C_1 \subseteq M_1$ and $C_2 \subseteq M_2$ be two gate convex subsets such that there exist a median isomorphism $\Phi : C_1 \to C_2$. Then the set $\tilde{M} := M_1 \bigsqcup_{C_1 \sim_\Phi C_2} M2$ is naturally endowed with a median algebra structure given by the following ternary operation :*

$$m(x, y, z) := \begin{cases} m_i(x, y, z) \text{ if } x, y, z \in M_i \\ m_1(x, y, \Phi^{-1}(\pi_{C_2}(z))) \text{ if } x, y \in M_1 \text{ and } z \in M_2 \\ m_2(x, y, \Phi(\pi_{C_1}(z))) \text{ if } x, y \in M_2 \text{ and } z \in M_1 \end{cases}$$

We define the interval $[a, b]$ as it was defined for median algebra, that is the set of fixed points of $m(a, b, *)$. Note that for any $x, y \in M_i$, we have $[x, y] = [x, y]_{M_i}$, where the latter is the interval corresponding to the median algebra structure of M_i. Before proving the above proposition, let us first prove some lemmas about the properties verified by the intervals of M :

Lemma 1.1.28. *Under the same notation of Proposition 1.1.27, fr any $x, y \in \tilde{M}$ and $z \in [x, y]$, we have $[x, z] \subseteq [x, y]$.*

Proof. It is enough to consider the case where $x, z \in M_1$ and $y \in M_2$. Let us consider a point $t \in [x, z]$ and show that $t \in [x, y]$. We have :

$$\begin{aligned} m(t, x, y) &= m_1(t, x, \Phi^{-1}(\pi_{C_2}(y))) \\ &= m_1(m_1(t, x, z), x, \Phi^{-1}(\pi_{C_2}(y))) \\ &= m_1(t, m_1(x, x, \Phi^{-1}(\pi_{C_2}(y)), m_1(z, x, \Phi^{-1}(\pi_{C_2}(y))) \\ &= m_1(t, x, z) \quad (\text{As } z \in [x, y]) \\ &= t \end{aligned}$$

□

Lemma 1.1.29. *Under the same notation of Proposition 1.1.27, for any $x \in M_1$ and $y \in M_2$ we have $[x, y] \cap M_1 = [\Phi^{-1}(\pi_{C_2}(y)), x]$.*

Proof. Remark that for any $z \in [x, y] \cap M_1$ we have :

$$z = m(x, y, z) = m(x, z, y) = m_1(x, z, \Phi^{-1}(\pi_{C_2}(y))).$$

Hence, we have the inclusion $[x, y] \cap M_1 \subseteq [\Phi^{-1}(\pi_{C_2}(y)), x]$.
For the other inclusion, we have $m(x, \Phi^{-1}(\pi_{C_2}(y)), y) = m(x, \Phi^{-1}(\pi_{C_2}(y)), \Phi^{-1}(\pi_{C_2}(y)) = \Phi^{-1}(\pi_{C_2}(y))$. Hence, we conclude by Lemma 1.1.28 that $[x, \Phi^{-1}(\pi_{C_2}(y))] \subseteq [x, y]$ □

Proof of Proposition 1.1.27. It is enough to show that the intervals of M verify the axioms of Definition 1.1.3. Note that the first axiom is direct. The second one is given by Lemma 1.1.28. It last to show that it verifies the last axiom. Let us consider a triple of points $x, y, z \in M$. Without loss of generality, we assume that $x, y \in M_1$ and $z \in M_2$. We have :

$$[x, y] \cap [x, z] \cap [y, z] = [x, y] \cap M_1 \cap [x, z] \cap [y, z].$$

Hence By Lemma 1.1.29, we deduce that :

$$\begin{aligned}
[x,y]\cap[x,z]\cap[y,z] &= [x,y]\cap[x,\Phi^{-1}(\pi_{C_2}(z))]\cap[y,\Phi^{-1}(\pi_{C_2}(z))] \\
&= [x,y]_{M_1}\cap[x,\Phi^{-1}(\pi_{C_2}(z))]_{M_1}\cap[y,\Phi^{-1}(\pi_{C_2}(z))]_{M_1} \\
&= \{m_1(x,y,\Phi^{-1}(\pi_{C_2}(z)))\} \\
&= \{m(x,y,z)\}
\end{aligned}$$

□

Using the above construction, one can show that $\mathbb{R}$-trees have a natural structure of median algebra obtained from the inductive limit of $\mathbb{R}$-trees which have finitely many branching points.

1.1.2 Poc Set

The set of halfspaces $\mathcal{H}(X)$ of a median algebra X is naturally endowed with a partial order relation given by the inclusion and a complementary operation which associates to each halfspaces its complement in X, thus inverting the partial order relation. This provides $\mathcal{H}(X)$ with a natural structure of a poc set.

Definition 1.1.30. (Poc set) A ***poc set*** $(P, \leqslant, *, 0)$ is a partially ordered set $(P, \leqslant)$ with a minimal element 0 and involution $* : P \to P$ such that it inverses the order in P and the minimal element 0 is the unique element in P which is smaller than its complement 0^*.
A ***morphism of poc sets*** is a morphism which respect the order and commutes with the complementary operations.

Examples 1.1.31.
1. The trivial poc set $\{0, 1\}$ consisting a minimal and a maximal element.
2. Any boolean algebra $(B, 0, 1, \vee, \wedge, ')$ has a natural structure of a poc set. The partial order relation is given defined by $a \leqslant b$ if and only if $a \wedge b = b$. The complementary operation is given by the negation $'$.
3. Let P_1 and P_2 be two poc sets. The disjoint union poc set $P_1 \coprod P_2$ between P_1 and P_2 is defined by taking the disjoint union of P_1 and P_2, where we identify their respective minimal elements and maximal elements. The partial order relation is the same on each component P_i and the elements of P_1 are not comparable with the elements of P_2, besides the maximum and the minimum. This operation corresponds to the coproduct or the categorical sum (in [Rol16] the author used the notation $\oplus$) in the category of Poc sets and morphism of poc sets.

For any subset $A \subset P$, we define $A^* := \{p^* \mid p \in A\}$.

A filter in a set of subsets $\mathcal{P}(X)$ is a non empty subset $F \subset \mathcal{P}(X) \backslash \{\varnothing\}$ which is stable under intersection and is upward stable, that is, for any $A \in F$ and $B \in \mathcal{P}(X)$ such that $A \subseteq B$ we have $B \in F$. Its definition extends to the case of poc set :

Definition 1.1.32. (Filter) Let $(P, \leqslant, 0, *)$ be a poc set. A non empty subset $F \subset P$ is a ***filter*** if it verifies the following :
1. The minimal element 0 does not lie in F.
2. For any $p, q \in F$ the inequality $p \leqslant q*$ does not hold. (Stability under intersection)
3. For any $p \in F$ and $q \in P$ such that $p \leqslant q$, then $q \in F$. (Upward stability)

For any filter $F \subset P$, the ***transverse*** to F is defined by $\mathcal{T}(F) := P \backslash (F \cup F^*)$.

Proposition 1.1.33. *(Extension of a filter) Let P be a poc set and let $F \subset P$ be a filter. Let $a \in P \backslash \{0\}$ such that neither $a \in F$ nor $a^* \in F$. Then the set $\tilde{F} := \{p \in P \mid p \in F \text{ or } a \leqslant p\}$ is a filter which contains F and a.*

Proof. By construction the set $\tilde{F}$ is upward stable and it does not contain the minimal element 0. Last to show the stability under "intersection". Let us consider $p, q \in \tilde{F}$. We have three case :

— Case 1 : Both p and q lie in F. In this case, there is nothing to show as F is assumed to be a filter.
— Case 2 : Both p and q are greater than a. Therefore, we can not have $p \leqslant q^*$ as it would imply that $a \leqslant a^*$.
— Case 3 : We have $p \in F$ and $a \leqslant q$. Again, we can not have $p \leqslant q^*$ as it would imply that $p \leqslant a^*$ which contradicts the assumption that a^* does not belong to F.

□

An ***ultrafilter*** is a filter which is maximal with respect to the inclusion. Proposition 1.1.33, gives us the following characterization of ultrafilter :

Definition 1.1.34. (Ultrafilter) Let $(P, \leqslant, 0, *)$ be a poc set. A subset $\mathfrak{u} \subseteq P$ is an ***ultrafiler*** if it verifies the following :

1. For any $p, q \in F$ we do not have $p \leqslant q*$.
2. For any $p \in P$, we have either $p \in \mathfrak{u}$ or $p* \in \mathfrak{u}$.

When the poc set P corresponds to the set of halfspaces of a median algebra M, the subset of a halfspaces containing a convex subset C constitutes a filter of $\mathcal{H}(M)$ and its transverse $\mathcal{T}(\mathcal{H}(M))$ is $\mathcal{H}(C)$, the set of halfspaces which are transverse to C. The subset of halfspaces containing a fixed point constitutes a maximal filter in P and they are called ***principal ultrafilters***.

Remark 1.1.35. Let $f : P \to Q$ be a morphism of poc sets. Then the inverse image of any ultrafilter of Q by f is an ultrafilter of P. This is due to the fact that the inverse image of any filter is a filter and that $f^{-1}(A^c) = (f^{-1}(A))^c$.

As for the halfspaces in median algebras, by assuming Zorn's lemma ultrafilters exist and verify the following "separation" property :

Theorem 1.1.36. *Let P be a poc set and let $F_1, F_2 \subset P$ be two filters. There exist then an ultrafilter $\mathfrak{u} \subset P$ such that $F_1 \subset \mathfrak{u}$ and $F_2 \cap \mathcal{T}(F_1) \subset \mathfrak{u}$.*
Moreover, if $\mathcal{T}(F_1) \cap \mathcal{T}((F_2) = \varnothing$, then such ultrafilter $\mathfrak{u}$ is unique.

Before proving the above Theorem, we make the following remarks to give a geometric picture of it.

Remark 1.1.37. — In the case where the filters F_1 and F_2 of Theorem 1.1.36 arise as the set of halfspaces which contain a gate convex subset of a median algebra M, i.e. $F_i = \mathcal{H}_{C_i}$ where $C_1, C_2 \subseteq M$ are gate convex subsets. The ultrafilter $\mathfrak{u}$ is given by a principal ultrafilter over a point lying in the projection $\pi_{C_1}(C_2)$ of C_2 into C_1.
The condition $\mathcal{T}(F_1) \cap \mathcal{T}((F_2) = \varnothing$ translates into the non existence of a halfspace which is transverse to both C_1 and C_2. In this case, the projection of C_2 into C_1 is a singleton $\{x\}$, and the ultrafilter $\mathfrak{u}$ corresponds to the principal ultrafilter over the point x (compare with Proposition 1.1.26).

— We note that the ultrafilter $\mathfrak{u}$ obtained in Theorem 1.1.36 verifies $F_2 \cap \mathfrak{u}^* = F_2 \cap F_1^*$. The filter F_1 being included in $\mathfrak{u}$, we deduce that $F_1^* \subseteq \mathfrak{u}^*$, thus $F_2 \cap F_1^* \subseteq F_2 \cap \mathfrak{u}^*$. In the other hand, we have $\mathfrak{u}^* \subseteq F_1^* \cup \mathcal{T}(F_1)$. As $F_2 \cap \mathcal{T}(F_1) \subseteq \mathfrak{u}$ we deduce that $F_2 \cap \mathfrak{u}^* \subseteq F_1^*$.

  Again when F_1 and F_2 arise as the set of halfspaces which contain gate convex subsets C_1 and C_2 respectively, the remark above translates into saying that any halfspace which separates C_2 from the point x must separate it also from C_1.

Proof of Theorem 1.1.36. Let $\mathcal{A}$ be the set of filters F which contains F_1 and such that $\mathcal{T}(F_1) \cap F_2^* \cap F = \varnothing$. The set $\mathcal{A}$ is an inductive set and is not empty as it contains F_1. Assuming Zorn's lemma, let us consider a maximal element $\mathfrak{u} \in \mathcal{A}$. We claim that $\mathfrak{u}$ is an ultrafilter which contains $\mathcal{T}(F_1) \cap F_2$.

Let $p \in P$ such that neither $p \in F_1$ nor $p^* \in F_1$, that is, the element p lie in $\mathcal{T}(F_1)$. Without loss of generality, we assume that p does not lie in F_2^*. The subset F_2 being a filter F_2^* is downward closed, i.e. for any $a \in P$ and $b \in F_2^*$ such that $a \leqslant b$ then $a \in F_2^*$. Hence any element which is greater than p lie outside F_2^*. By proposition 1.1.33, the set $\tilde{\mathfrak{u}}$ consisting of the union of all the elements which are greater than p and the elements of $\mathfrak{u}$ is a filter which verifies $\mathcal{T}(F_1) \cap F_2^* \cap \tilde{\mathfrak{u}} = \varnothing$. By the maximality of $\mathfrak{u}$, we conclude that $p \in \mathfrak{u}$. In particular, we have shown that $F_2 \cap \mathcal{T}(F_1) \subset \mathfrak{u}$.

Let $\tilde{\mathfrak{u}}$ be another maximal element in $\mathcal{A}$. There exist then $p \in P$ such that $p \in \mathfrak{u}$ and $p^* \in \tilde{\mathfrak{u}}$. The element p necessarily lie in $\mathcal{T}(F_1)$. As we have $\mathcal{T}(F_1) \cap F_2^* \cap \tilde{\mathfrak{u}} = \varnothing$ and $\mathcal{T}(F_1) \cap F_2^* \cap \mathfrak{u} = \varnothing$, we conclude that $p \in \mathcal{T}(F_2)$. Therefore, if the intersection $\mathcal{T}(F_1) \cap \mathcal{T}((F_2)$ is empty, then the ultrafilter is unique. □

In particular we get the following corollary :

Corollary 1.1.38. *(Separation Theorem) Let P be a poc set and let $p, q \in P$. There exist then an ultrafilter which separates p and q.*

Proof. Without loss of generality we can assume that $p \nleqslant q$ and $p \nleqslant q^*$. Let us set $F_1 := \{x \in P/p \leqslant x\}$ and $F_2 := \{x \in P/q^* \leqslant x\}$. By Theorem 1.1.36, there exist an ultrafilter $\mathfrak{u} \subset P$ such that $F_1 \subseteq \mathfrak{u}$ and $F_2 \cap \mathcal{T}(F_1) \subseteq \mathfrak{u}$. As $p \nleqslant q$ and $p \nleqslant q^*$, the element q and its complementary q^* lie in the transverse set $\mathcal{T}(F_1)$ of the filter F_1. Therefore, the ultrafilter $\mathfrak{u}$ contains p and q^*. □

Let us denote by $\mathcal{U}(P)$ the sets of ultrafilters of the poc set P. With analogy to the case of median algebra, the sets of ultrafilters of a poc set P is in correspondence with the set of poc sets morphisms from P to the trivial poc set $\{0, 1\}$ by considering the characteristic map on the ultrafilters for the first direction and considering the inverse image of $\{1\}$ under the latter poc set morphisms for the other direction. We obtain then that any poc set P embeds into the product $\prod_{\mathfrak{u} \in \mathcal{U}(P)} \{0, 1\}$ where the injectivity is ensured by the Separation Theorem 1.1.38. We deduce then that the category of poc sets and poc set morphisms is the same as the variety generated by the trivial poc set $\{0, 1\}$.

The set $\mathcal{U}(P)$ is a subset of the boolean algebra $\mathcal{P}(P)$ which is not closed under intersection nor under the union. However, it is closed under the median operation defined in 1.1.4. Hence it is naturally endowed with a structure of a median algebra. Therefore, there is a canonical way to associate to each poc set P a median algebra $\mathcal{U}(P)$. This association verifies a nice functorial properties. It is the subject of the next subsection.

1.1.3 Duality

We have a duality between, due to Marshal Stone, between the category of boolean algebras with boolean morphisms and the category of Stone spaces with homeomorphisms. A ***Stone space*** is a totally separated compact topological space.

In [Isb80], the author extended the latter duality to the case of median algebras and poc sets in two ways (see Theorem 6.13 therein, J. R. Isbell refers to median alebras and poc sets by symmetric media and binary messages respectively). The first way is to see poc sets as a generalization of boolean algebras. In this case, it will be dual to the category of Stone median algebras with continuous median morphisms. A ***Stone median algebra*** is a median algebra endowed with a topology which makes it a Stone space and such that the median operation is continuous.

The second way is to see median algebras as a generalization of boolean algebras. One obtain then a duality between the category of median algebras with median morphisms and the category of Stone poc sets with continuous morphisms of poc set. A ***Stone poc set*** is a poc set endowed with a Stone topology which is compatible with the complementary operation $*$ ([Rol16], section 6).

Let us denote by $\mathcal{H}$ the contravariant functor, which associates to each median algebra M the poc set $\mathcal{H}(M)$ consisting of the halfspaces of M, and to each median morphism $f : M \to N$ the poc set morphism $\mathcal{H}(f) := f^{-1} : \mathcal{H}(N) \to \mathcal{H}(M)$, the latter is well defined (see Remark 1.1.8).

Let $\mathcal{U}$ be the functor which associates to each poc set P the median algebra consisting of ultrafilters of P and to each poc set morphism $f : P \to Q$ the median morphism $\mathcal{U}(f) := f^{-1} : \mathcal{U}(Q) \to \mathcal{U}(P)$, the latter is well defined (see Remark 1.1.35).

Let us feature how the concepts seen before translate into the dual of their category :

Convex subsets and filters Let M be a median algebra and let us consider a convex subset C. We recall that $\mathcal{H}_C$ corresponds to the set of halfspaces containing C. We have seen that it is the canonical example of a filter in the poc set $\mathcal{H}(M)$ and that if C is a singleton then $\mathcal{H}_C$ is a principal ultrafilter. Note that not all filter of $\mathcal{H}(M)$ arise as such.

Conversly we have the following :

Proposition 1.1.39. *Let P be a poc set and let $F \subset P$ be a filter. Then the set $\mathcal{U}_F \subset \mathcal{U}(P)$ consisting of ultrafilter containing F is a convex subset of the median algebra $\mathcal{U}(P)$.*

Moreover, if F is generated by an element $p \in P$, that is $F = \{q \in P / p \leqslant q\}$, then $\mathcal{U}_F$ is a halfspace.

Proof. We recall that for any two ultrafilter $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{U}(P)$, the interval $[\mathfrak{u}_1, \mathfrak{u}_2]$ is the set of ultrafilters which contains $\mathfrak{u}_1 \cap \mathfrak{u}_2$ and is contained in $\mathfrak{u}_1 \cup \mathfrak{u}_2$. Hence, for any $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{U}_F$ and $\mathfrak{u} \in [\mathfrak{u}_1, \mathfrak{u}_2]$ we have $F \subseteq \mathfrak{u}_1 \cap \mathfrak{u}_2 \subseteq \mathfrak{u}$. Therefore, we have $[\mathfrak{u}_1, \mathfrak{u}_2] \subseteq \mathcal{U}_F$.

Let us assume now that F is generated by an element p. In this case, it remains to show that $\mathcal{U}_F^c$ is also convex. Let us consider $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{U}_F^c$. There exist then $p_1 \in \mathfrak{u}_1$ and $p_2 \in \mathfrak{u}_2$ such that $p \leqslant p_i^*$. Let us consider an ultrafilter $\mathfrak{u} \in [\mathfrak{u}_1, \mathfrak{u}_2]$. The ultrafilter being contained in the union $\mathfrak{u}_1 \cup \mathfrak{u}_2$, at least one of the p_i's lies in $\mathfrak{u}$, which implies that $\mathfrak{u}$ does not contain F, which complete the proof. □

Let us consider a gate convex subset $C \subseteq M$. Then the filter $\mathcal{H}_C$ associated to the latter verifies the additional property that for any principal ultrafilter $\mathfrak{u}_x$ where $x \in M$, there exist a principal ultrafilter $\mathfrak{u}_{\tilde{x}}$ such that $\mathfrak{u}_x \cap \mathfrak{u}_{\tilde{x}}^* = \mathfrak{u}_x \cap \mathcal{H}_C^*$ (compare with Theorem 1.1.36 and Remarks 1.1.37 by setting $F_1 = \mathcal{H}_C$, $F_2 = \mathfrak{u}_x$ and $\mathfrak{u} = \mathfrak{u}_{\tilde{x}}$). The point $\tilde{x}$ corresponds to the gate projection of x into C.

Under the above translation gate convex subsets into the category of poc sets, Theorem 1.1.36 implies that for any filter $F \subset P$ the convex subset $\mathcal{U}_F \subseteq \mathcal{U}(P)$ is gate convex.

This can be also justified through the following facts :
The first is that the category of Poc sets with poc sets morphisms and the category of Stone median algebras with continuous median morphisms are dually equivalent (Theorem 6.13 [Isb80], Theorem 5.3 [Rol16]). The first functor is the functor $\mathcal{U}$ seen above which associates to each poc set P, the Stone median algebra $\mathcal{U}(P)$ consisting of ultrafilters of P and the halfspaces $\mathcal{U}_p \subseteq \mathcal{U}(P)$, where $p \in P$, constitute the subbasis for the topology of $\mathcal{U}(P)$. The converse functor associates to each Stone median morphism, the poc set of closed-open halfspaces. Hence, for any filter $F \subseteq P$, the convex subset $\mathcal{U}_F = \bigcap_{p \in F} \mathcal{U}_p$ is closed in the Stone median algebra $\mathcal{U}(P)$.
The second fact is that a convex subset of a Stone median algebra is gate convex if and only if it is closed (Proposition 5.6 [Rol16]). We conclude then that the convex subsets $\mathcal{U}_F$ corresponding to filters $F \subset P$ are gate convex.

Direct product of median algebras and direct sum of poc sets We have seen in Examples 1.1.4 and 1.1.31 that the category of median algebra with median morphisms admits a product operation over an arbitrary family and that the category of poc sets with poc set morphisms admits a coproduct operation over an arbitrary family.

Let us take a closer look on the halfspaces of a product of median algebras. Throughout the following, let $(M_i)_{i \in I}$ denote a family of median algebras and let $M := \prod_{i \in I} M_i$ be the product median algebra. We denote by π_i the canonical projection from M to M_i. We first remark the following description of the intervals in an arbitrary product :

Remark 1.1.40. 1. The median operation of the product of median algebras being defined pointwise, we have for any $a, b \in M := \prod_{i \in I} M_i$:

$$[a, b]_M = \prod_{i \in I} [a_i, b_i]_{M_i}.$$

Where a_i and b_i are the projection of a and b respectively into the factors M_i under the canonical projections π_i.

2. For any subsets $(A_i)_{i\in I}, (B_i)_{i\in I} \subseteq (M_i)_{i\in I}$, we have the following description of the join between two products :

$$[\prod_{i\in I} A_i, \prod_{i\in I} B_i]_M = \prod_{i\in I}[A_i, B_i]_{M_i}.$$

More generally, we have the following :

Proposition 1.1.41. *For any finite subset $A \subset M$, we have :*

$$Conv_M(A) = \prod_{i\in I} Conv_{M_i}(\pi_i(A)).$$

Proof. We proceed by induction on the cardinal of the subset A. The equality is trivially verified when A is a singleton. Let us assume that the claim is true for all subsets of cardinal $n-1$ and consider a subset $A \subseteq M$ of cardinal n. After fixing a point $a \in A$ and setting $\tilde{A} := A\backslash\{a\}$, we get then :

$$\begin{aligned} Conv_M(A) &= Conv_M(\tilde{A} \cup \{a\}) \\ &= [\tilde{A}, \{a\}] \quad \text{By Proposition 1.1.19.} \end{aligned}$$

By assumption we have the following splitting $Conv(\tilde{A}) = \prod_{i\in I} Conv_{M_i}(\pi_i(\tilde{A}))$. We deduce then the following :

$$\begin{aligned} Conv_M(A) &= [\prod_{i\in I} Conv_{M_i}(\pi_i\tilde{A})), \{a\}] \\ &= \prod_{i\in I}[\pi_i(\tilde{A}), \pi_i(a)] \quad \text{By Remark 1.1.40 (2)} \\ &= \prod_{i\in I}[\pi_i(\tilde{A}), \pi_i(a)] \\ &= \prod_{i\in I} Conv_{M_i}(\pi_i(A)). \end{aligned}$$

□

We conclude from the above Proposition and Proposition 1.1.11, that the convex hull of any subset $A \subseteq M$ is the union of the product of the projections of its finite subsets :

Remark 1.1.42. For any $A \subseteq M$, we have by Propositions 1.1.41 and 1.1.11 the following :

$$Conv(A) = \bigcup_{x_1,..,x_n\in A} (\prod_{i\in I} Conv_{M_i}(\{x_1, .., x_n\})).$$

In particular, if the index set I is finite we get that :

$$Conv(A) = Conv_{M_{i_1}}(\pi_{i_1}(A)) \times ... \times Conv_{M_{i_n}}(\pi_{i_n}(A)).$$

In the following proposition, we give a description of halfspaces in a finite product of median algebras :

Proposition 1.1.43. *Let $M_1, .., M_n$ be median algebras and let $\mathfrak{h} \subset M := M_1 \times ... \times M_n$ be a proper halfspace. Then there exist $k \in \{1, .., n\}$ such that $\pi_i(\mathfrak{h})$ is a proper halfspace of M_k and for all $i \neq k$ we have $\pi_i(\mathfrak{h}) = M_i$. Moreover, we have :*

$$\mathfrak{h} = \pi_k(\mathfrak{h}) \times (\prod_{\substack{i=1 \\ i\neq k}}^{n} M_i).$$

Proof. By Remark 1.1.42 (2), we have $\mathfrak{h} = \prod_{i=1}^{n} \pi_i(\mathfrak{h})$. The halfspace being assumed to be proper, there exist $k \in \{1, .., n\}$ such that $\pi_k(\mathfrak{h}) \neq M_k$. In the other hand, $\mathfrak{h}^c = \prod_{i=1}^{n} \pi_i(\mathfrak{h}^c) = (\prod_{i=1}^{n} \pi_i(\mathfrak{h}))^c$. We conclude that for the rest of the indices $i \in \{1, .., n\} \backslash \{k\}$ we have $\pi_i(\mathfrak{h}) = M_i$. □

Remark 1.1.44. Proposition 1.1.43 is no longer true when we consider a product of an infinite family of median algebras. The halfspaces described in Proposition 1.1.43 obviously constitutes halfspaces of the product but one obtain infinitely many halfspaces of other type.
Take for instance the product $M := \prod_{i\in\mathbb{N}} \mathbb{R}$ and consider the convex subsets

$$\begin{aligned} C_1 &:= \{(x_i)_{i\in\mathbb{N}} \mid x_i \geqslant 0\} \\ C_2 &:= \{(x_i)_{i\in\mathbb{N}} \mid x_i \leqslant 0 \text{ except for finitely many } i \in I\} \end{aligned}$$

Any halfspace of the form $\mathfrak{h}_k \times (\prod_{\substack{i\in\mathbb{N} \\ i\neq k}} M_i)$ where $\mathfrak{h}_k$ is a halfspace of M_k, must intersects C_2.
In the other hand, as C_1 and C_2 does not intersect, there exist a halfspace which separates them by Theorem 1.1.7.

Conversely, we have the following description of the ultrafilters in a coproduct of poc sets :

Proposition 1.1.45. *Let $(P_i)_{i\in I}$ be a family of poc sets and let $P := \coprod_{i\in I} P_i$. Then the set of ultrafilters of P is in bijection with the product of the sets of utrafilters of each P_i.*

Proof. Each $P_i \backslash \{0\}$ embeds as a filter of P. Hence, for any ultrafilter $\mathfrak{u} \subset P$ the intersection $\mathfrak{u} \cap P_i$ identifies with a filter of P_i and it is maximal as for any $p \in P$, we have either $p \in \mathfrak{u}$ or $p^* \in \mathfrak{u}^c$. For the converse direction, any pair of elements $p_i \in P_i$ and $p_j \in P_j$, where $i \neq j$, are transverse. Hence, for any ultrafilters $\mathfrak{u}_i \subset P_i$ and $\mathfrak{u}_j \subset P_j$, their union is a filter of P. Therefore, we conclude that the union $\bigcup_{i\in I} \mathfrak{u}_i$ of a family of ultrafilters $\mathfrak{u}_i \subset P_i$ is a maximal filter of P. □

We sum up the above discussion into the following proposition :

Proposition 1.1.46. *Let $M_1, .., M_n$ be a family of median algebras and let $(P_j)_{j\in J}$ be a family of poc sets. We have the following :*

- *The poc set $\mathcal{H}(M_1 \times .. \times M_n)$ is isomorphic to $\mathcal{H}(M_1) \coprod ... \coprod \mathcal{H}(M_n)$.*
- *The median algebra $\mathcal{U}(\coprod_{j\in J} P_j)$ is isomorphic to $\prod_{j\in J} \mathcal{U}(P_j)$.*

Remark 1.1.47. It is perhaps counter intuitive that the functor $\mathcal{U}$ commutes with an arbitrary co-product of poc sets and the functor $\mathcal{H}$ only commutes with finites product of median algebras. This is due to the fact that the category of poc sets is dual to the category of Stone median algebras, and the functor which goes from the latter category to the former one associates to each Stone median algebra its set of clopen halfspaces and not the set of all of its halfspaces.
Hence, if $(P_i)_{i\in I}$ is a family of poc set and $M := \prod_{i\in I} \mathcal{U}(P_i)$ the Stone median algebra associated to the coproduct of the family $(P_i)_{i\in I}$. The median algebra is endowed with the product topology where each median algebra $\mathcal{U}(P_i)$ is endowed with the topology generated by the subbasis $\mathcal{U}_p$ for $p \in P_i$. In this case, the halfspaces of M which are closed and open at the same time are those of the form $\mathcal{H}_k \times (\prod_{\substack{i\in I \\ i\neq k}} M_i)$.

1.2 Metric structure of the median geometry

1.2.1 Median spaces

Definition 1.2.1 (Median spaces)**.** A ***median space*** is a metric space (X, d) such that for any three points $a, b, c \in X$ there exist a unique point $m \in X$, called the median point between a, b and c, such that the three intervals relating each two points of the former triple intersect uniquely in $\{m\}$, i.e. :

$$[a, b] \cap [b, c] \cap [a, c] = \{m\}$$

where the ***interval*** between two points is defined as follows :

$$[a, b] := \{x \in X \mid \ d(a, b) = d(a, x) + d(x, b)\}.$$

We say that a point $x \in X$ is ***between*** a and b if it lies in the $[a, b]$.

Examples 1.2.2.
1. The real line with the usual metric is a median space.
2. The ℓ^1-product $(X_1 \times X_2, d_{\ell^1})$ of two median spaces (X_1, d_1) and (X_2, d_2), i.e. $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d(y_1, y_2)$ is again a median space. Its median algebra structure corresponds to the cartesian product of the median algebra structure of both X_1 and X_2. For instance $\mathbb{R}^n$ endowed with the ℓ^1-metric is a median space. Note that the interval between a and b corresponds to the product of the interval between the canonical projection of a and b on each factors.
3. More generally, if (A, μ) is a measured space. The space $L^1(A, \mu)$ of ℓ^1-integrable real functions over (A, μ) is a median space. The median structure comes from the median structure of the target space which is $\mathbb{R}$. A representative of the median class between any three classe of functions $[f], [g]$ and $[h]$ is defined point wise, i.e. $[m(f, g, h)(x)] = [m_{\mathbb{R}}(f(x), g(x), h(x))]$. The median algebra structure of $L^1(A, \mu)$ corresponds to the ultraproduct $(\prod_A \mathbb{R})/U$, where

$$U := \{B \subseteq A \mid B \text{ are measurable and } \mu(A \Delta B) = 0\},$$

that is the quotient of $\prod_A \mathbb{R}$ by the equivalence relation $\sim_U$ defined by $f \sim_U g$ if and only if $ker(f, g) \in U$ where $ker(f, g) := \{x \in A \mid \ f(x) = g(x)\}$. As the set of subsets U is stable under finite intersection, the equivalence relation $\sim$ is compatible with median operation of the product $\prod_A \mathbb{R}$, that is, for any $f_1, g_1, f_2, g_2, f_3, g_3 \in \prod_A \mathbb{R}$ such that $f_i \sim_U g_i$, then $m(f_1, f_2, f_3) \sim m(g_1, g_2, g_3)$. Hence, the quotient $(\prod_A \mathbb{R})/U$ is naturally endowed with a median algebra structure. See Definition 6.3 in [BS81] for a concrete overview on ultraproducts.

4. We say that a graph is median if its set of vertices, when endowed with the combinatorial metric, is a median space. Simplicial trees are examples of median graphs. More generally, the 0-skeleton of a CAT(0) cube complex, seen as a graph and endowed with the combinatorial metric is a median graph. In fact, any median graph arise as the 0-skeleton of a $CAT(0)$ cube complex (see [Che00] Theorem 6.1).
5. $\mathbb{R}$-tree are median spaces of rank 1. In fact, a metric space is a $\mathbb{R}$-tree if and only if it is a connected median space of rank 1 (see [Bow13] Lemma 9.5).
6. An ***isotone valuation*** on a lattice $(L, \vee, \wedge)$ is a real valued function $v : L \to \mathbb{R}$ such that
   - For any $a, b \in L$ we have $v(a) + v(b) = v(a \vee b) + v(a \wedge b)$.
   - For any $a, b \in L$ such that $a \leqslant b$, i.e. $a \wedge b = a$, then $v(a) \leqslant v(b)$

   Any isotone valuation gives rise to a pseudo metric on the lattice
   $$d(a, b) = v(a \vee b) - v(a \wedge b).$$
   A lattice endowed with an isotone valuation is called a ***quasi metric lattice***. When the isotone valuation gives rise to a metric, we say then that (L, d) is a metric lattice (see [Bir67] Ch V, §6 and §7 for definitions and properties).
   The relation $a \sim b$ whenever $d(a, b) = 0$ is a congruence relation, that is, if $a_1 \sim a_2$ and $b_1 \sim b_2$ then $(a_1 \wedge b_1) \sim (a_2 \wedge b_2)$ and $(a_1 \vee b_1) \sim (a_2 \vee b_2)$ (see [Bir67] Ch V, §7 Theorem 9 p77). Hence the quotient $\tilde{L} := L/\sim$ is metric distributive lattice. By Theorem 1 [BK47], we have $c \in [a, b]$ if and only if $d(a, b) = d(a, c) + d(c, b)$ for any $a, b, c \in \tilde{L}$ and where $[a, b]$ is the interval with respect to the median algebra structure associated to the distributive lattice $\tilde{L}$ (see (3) in Examples 1.1.4). A σ-algebra $\mathcal{B}$ over a set X with a measure μ is an example of a distributive metric lattice. The valuation is given by the measure μ and the pseudo metric between two measurable subsets of X corresponds to the measure of their symmetric difference.

The intervals in a median space X, seen as a map $X^2 \to \mathcal{P}(X)$, verify the axioms of median algebra by mean of intervals described in Proposition 1.1.3. Hence median space admits a natural structure of median algebra. Let us describe how the notions seen in the previous section translate into the metric framework.

As for median algebra, convexity is defined with respect to intervals. Note that median spaces need not to be geodesic spaces, nor to be even connected. However, we have the following :

Proposition 1.2.3. *Any complete connected median space is geodesic.*

Proof. Let X be a complete connected median space. The space being complete, it is enough to show that for any two points, there exist a point in the middle, i.e. for any $a, b \in X$ there exist $x \in [a, b]$ such that $d(a, x) = d(b, x)$. One may then construct a geodesic between any two points $a, b \in X$ by extending the isometric embedding of the dyadic rational of the interval $[0, d(a, b)] \in \mathbb{R}$, using the completeness of X.

As we assumed the space X to be connected, for any two points $a, b \in X$ the subset $M_{a,b} := \{x \in X \mid d(a,x) = d(b,x)\}$ is not empty as it separates the two disjoint open subsets $\{x \mid d(a,x) < d(b,x)\}$ and $\{x \mid d(a,x) > d(b,x)\}$. Then the median point between a, b and any point in $M_{a,b}$ lies in the middle of a and b. □

Obviously, if a median space X is geodesic, geodesics joining two points belongs to the interval between the latter, but it does not coincide with it. In fact geodesics and intervals in a median space coincide if and only if the space is of rank 1, and it is the case when the median space is a $\mathbb{R}$-tree.

In the following, we show that in a median space X, gate convex subsets arise as the closed convex subsets. We remark by the definition of the intervals in term of the metric of X that the projections onto gate convex subsets coincide with the nearest point projections. A median space is in general not a $CAT(0)$ space, in fact a connected complete median space is $CAT(0)$ if and only if it is a $\mathbb{R}$-tree (see Lemma 4.3.8). However, the convexity being defined with regard to intervals instead of geodesic, convex subset features some rigidity and we still have nice properties appearing in $CAT(0)$ spaces. In particular, we have the following :

Proposition 1.2.4. *Let X be a median space and let $C \subset X$ be a gate convex subset. Then the projection π_C onto C is 1-lipschitz.*

Proof. Let us consider $x_1, x_2 \in X$ and assume, without loss of generality, that $d(x_2, \pi_C(x_2)) \leqslant d(x_1, \pi_C(x_1))$. The point $\pi_C(x_1)$ lies in the interval $[x_1, \pi_C(x_2)]$, that is

$$d(x_1, \pi_C(x_2)) = d(x_1, \pi_C(x_1)) + d(\pi_C(x_1), \pi_C(x_2)).$$

We deduce then

$$\begin{aligned} d(\pi_C(x_1), \pi_C(x_2)) &= d(x_1, \pi_C(x_2)) - d(x_1, \pi_C(x_1)) \\ &\leqslant d(x_1, x_2) + d(x_2, \pi_C(x_2)) - d(x_2, \pi_C(x_2)) \\ &\leqslant d(x_1, x_2). \end{aligned}$$

□

In particular, the gate projections onto gate convex subsets in median spaces are continuous morphisms. We deduce the following :

Corollary 1.2.5. *Let X be a median space. Then gate convex subset are closed.*

Proof. Let C be a gate convex subset. Note that C corresponds to the set of points which are stabilized by projection map π_C. The latter map being continuous by Proposition 1.2.4 and the space X being Hausdorff, we deduce that C is closed. □

Proposition 1.2.6. *In a complete median space, a convex subset if gate convex if and only if it is closed.*

Proof. By Corollary 1.2.5, it is left to show that a closed convex subset $C \subset X$ in a complete median space is a gate convex subset. Let $x \in X$ and let $(x_i)_{i\in\mathbb{N}} \subseteq C$ be a sequence of point such that $lim_{i\to+\infty} d(x, x_n) = d(x, C)$. As C is convex, the median point $m(x, x_i, x_j)$ lies in C for any $i, j \in \mathbb{N}$. Hence, the sequence $(x_i)_{i\in\mathbb{N}}$ is a cauchy sequence. Its limit is the nearest point projection of the point x onto C. It is left to show that the limits does not depends on the sequence $(x_i)_{i\in\mathbb{N}}$. Let $a, b \in C$ such that $d(x, a) = d(x, b) = d(x, C)$. Then the median point $m(a, b, c)$ is a point of C which is closed to x and lies in the interval $[a, b]$. We deduce that $a = m(a, b, c) = b$ which finishes the proof. □

The intervals being a gate convex subset in median algebras, we deduce that in a median space X, for any point $a, b \in X$ the interval $[a, b]$ is closed and its projection map given by $m(*, a, b)$ is 1-lipschitz. Moreover, we have the following :

Proposition 1.2.7 (Corollary 2.15 [CDH10]). *In a median space X the median operation $m : (X^3, d_{\ell^1}) \to X$ is a 1-lipschitz morphism of median algebra.*

It is very convenient to assume the metric space to be complete and almost all of the results concerning median spaces are stated under that assumption. It is a mild condition as the median structure is preserved under metric completion.

Proposition 1.2.8. *The metric completion of a median space is a median space.*

We first prove the following Lemma :

Lemma 1.2.9. *Let (X, d) be a metric space endowed with a symmetric ternary operation m such that for any $a, b, c \in X$ we have $m(a, b, c) \in [a, b]$. We have then :*

$$[a, b] = \{c \in X \mid m(a, b, c) = c\}.$$

In particular, if (X, m) is a median algebra then (X, d) is a median space.

Proof. For any $c \in [a, b]$, we have :

$$\begin{aligned} d(a, m(a, b, c)) + d(m(a, b, c), c) &= d(a, c). \\ d(b, m(a, b, c)) + d(m(a, b, c), c) &= d(b, c). \end{aligned}$$

After we sum the equations above, we get :

$$d(a, b) + 2d(m(a, b, c), c) = d(a, b).$$

We conclude that $d(m(a, b, c), c) = 0$, that is $c = m(a, b, c)$. □

Proof of Proposition 1.2.8. Let X be a median space and let $\hat{X}$ be its metric completion. The median ternary operation m extends to a continuous ternary operation on $\tilde{X}$. We first show that the interval between two points $a, b \in \tilde{X}$ corresponds to the stabilizer of the map

$m(a, b, *)$. By Lemma 1.2.9, it is enough to show that $m(a, b, x) \in [a, b]$ for any $x \in X$. Note that it is the same as saying that the map $f : \tilde{X}^3 \to \tilde{X}$ defined by

$$f(x, y, z) := d(x, y) - d(x, m(x, y, z)) - d(y, m(x, y, z))$$

is null. The map f is continuous and it vanishes in the dense subset X^3, hence it vanishes in the whole domain $\tilde{X}$.

By the equivalence of the two Definitions 1.1.1 and 1.1.3, it last to show that the ternary operation m verifies the axiom of median algebras stated in Definition 1.1.1. The solution of the equations arising in Definition 1.1.1 corresponds to the zero of real continuous functions defined over a power of $\tilde{X}$, where the power depends on the number of variable. The median space X being dense in $\tilde{X}$ and being a median algebra, we deduce that the latter functions vanishes at a dense subset. Hence by continuity of those functions, the equations defining a median algebra are verified by all points of $\tilde{X}$, which finishes the proof. □

Median spaces of finite rank The ***rank*** of a median space X is the rank of the median algebra associated to it. There is a dichotomy between the finite rank case and the infinite one. For instance, it was shown in [Fio21] that isometries without wall inversion in a finite rank median space X are semi simple ([Fio21] Corollary D), where an isometry g is said to be ***without wall inversion*** if for any halfspace $\mathfrak{h} \in \mathcal{H}(X)$ we have $g.\mathfrak{h} \neq \mathfrak{h}^c$. This is no longer the case when the median space is of infinite rank. Examples are given by proper action of group which have a distorted abelian subgroup, like some lie groups which verifies Haagerup property. It was shown in [CDH10] (Theorem 2.2) that a locally compact second countable group has the Haagerup property if and only if it admits a proper continuous isometric action on a median space. Thus such actions of $\tilde{\mathrm{PSL}}(2, \mathbb{R})$, the universal cover of $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{SO}(n, 1)$ on median spaces cannot be semisimple, as there exist distorted real line which is not relatively compact inside these groups (one may consider the one parameter group of a parabolic element in $\mathrm{SO}(n, 1)$).

Another major difference between the finite and the infinite rank case is local convexity. Let X be a median space, then for any $x \in X$ and $\epsilon > 0$ the join $[B(x, \epsilon), B(x, \epsilon)]$ of the ball centred at x and of radius x is, by triangular inequality, inside the ball $B(x, 2\epsilon)$. If X is of rank n then by Proposition 1.1.20 we have $Conv(B(x, \epsilon)) \subseteq B(x, 2^n\epsilon)$. Hence median spaces of finite rank are locally convex (see [Bow13] Lemma 7.1). This not necessarily the case in infinite rank median space. One may consider for example $L^1(\mathbb{R})$ where the convex hull of any ball is the whole space. The iterated join between the elements $f_n = \mathbb{1}_{[n,n+1]}$ yields elements in the convex hull of the unit ball around the null function which get arbitrary far from the center (see [Fio20] example 2.24). This affects the shape of halfspaces as in the latter cases, they become dense in the space.

Even if we assume the space to be locally convex, the space may contain halfspaces which are dense. Consider for instance the space

$$X = (\prod_{i \in \mathbb{N}} [-\frac{1}{2^n}, \frac{1}{2^n}], \ell^1) \cong \{f \in L^1(\mathbb{N}) \mid f(i) \in [-\frac{1}{2^n}, \frac{1}{2^n}]\}.$$

The median space X is compact and locally convex. The following two convex subsets

$$\begin{aligned} \mathfrak{h}_{i,<0} &:= \{f \in X / \ f(i) < 0 \ \text{ except for finitely many } i \in \mathbb{N}\}, \\ \mathfrak{h}_{i,>0} &:= \{f \in X / \ f(i) > 0 \ \text{ except for finitely many } i \in \mathbb{N}\} \end{aligned}$$

are disjoint and dense in X. Thus so is the halfspace which separates them and such halfspaces exist by Theorem 1.1.7.

In the finite rank case, halfspaces are more manageable geometrically :

Proposition 1.2.10 ([Fio20] Corollary 2.23)**.** *Let X be a complete median space of finite rank. Then any halfspace are either open, closed or possibly both.*

When the median space is of finite rank, all of its halfspaces are bounded by a hyperplane which is formally defined as follows :

Definition 1.2.11 (Hyperplanes)**.** Let X be a complete finite rank median space and let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace. The closed convex subset $\hat{\mathfrak{h}} := \bar{\mathfrak{h}} \cap \bar{\mathfrak{h}}^c$ is the ***hyperplane*** associated to $\mathfrak{h}$.

If X is a complete median space of rank n, then any hyperplane is of rank less or equal $n-1$ ([Fio20] Proposition 2.22). Note that if the median space is not connected, then the intersection $\bar{\mathfrak{h}} \cap \bar{\mathfrak{h}}^c$ may be empty. It occurs exactly when $\mathfrak{h}$ and $\mathfrak{h}^c$ are both closed. Hyperplanes in finite rank median spaces appear to be very useful, for instance, they are practical to prove properties of median spaces of finite rank using an argument by induction on the rank of the spaces.

For a complete connected median space of finite rank, we set $\mathcal{H}_x := \{\mathfrak{h} \in \mathcal{H}(X) \mid x \in \bar{\mathfrak{h}} \cap \overline{\mathfrak{h}^c}\}$. It consists of the set of halfspaces which are "branched at the point x. It is the natural generalization of the valency from $\mathbb{R}$-trees to the higher rank case.

Remark 1.2.12. By Proposition 4.3.7, any halfspace $\mathfrak{h} \in \mathcal{H}_x$ which contains x, in a complete median space of finite rank, is necessarily closed.

Before closing this part, we take a brief look at the convex hull of balls in median spaces of finite rank. In a median space, the balls are convex if and only if the median space is of rank 1, that is, it is tree like.

One may remark from Proposition 1.1.20 that the convex hull of the ball of radius r lies in a ball of radius $2^{n-1}r$. In the following proposition we show that in a median space of rank n, the convex hull of the ball of radius $r > 0$ is contained in a ball of radius nr :

Proposition 1.2.13. *Let X be a median space of rank n. Then for any $a \in X$ and $r > 0$, we have :*

$$Conv(B(a,r)) \subseteq B(a,nr).$$

Before proving the statement, we will be needing some lemmas. The following lemma is a strengthening of the separation Theorem 1.1.7 in the case of complete median space of finite rank :

Lemma 1.2.14. *Let X be a complete connected median space of finite rank. Then for any $a, b \in X$, there exists a halfspace $\mathfrak{h} \in \mathcal{H}(a, b)$ such that $d(a, \mathfrak{h}) = 0$.*
In particular, we have $\bigcap_{\substack{\mathfrak{h} \in \mathcal{H}_a(X) \\ a \in \mathfrak{h}}} \mathfrak{h} = \{a\}$

Proof. Let us consider two distinct points $a, b \in X$. The median space X being complete and connected, there exists a midpoint $b_1 \in [a, b]$, i.e. $d(a, b_1) = d(b_1, b) = \frac{d(a,b)}{2}$. Let $\mathfrak{h}_1 \in \mathcal{H}(a, b_1)$ be a halfspace separating b_1 from a. Let b_2 be a midpoint of $[a, \pi_{\hat{\mathfrak{h}}_1}(a)]$ and let $\mathfrak{h}_2 \in \mathcal{H}(a, b_2)$ be a halfspace separating b_2 from a. The halfspace $\mathfrak{h}_2$ separates $\hat{\mathfrak{h}}_1$ from a, hence it contains the halfspace $\mathfrak{h}_1$. Proceeding by iteration, we obtain an ascending sequence of halfspaces $(\mathfrak{h}_n)_{n \in \mathbb{N}*}$ separating b from a and such that $\lim_{n \to \infty} d(\mathfrak{h}_n, a) = 0$. The subset $\mathfrak{h} := \bigcup_{n \in \mathbb{N}*} \mathfrak{h}_n$ is our desired halfspace. □

Remark 1.2.15. Lemma 1.2.14 above remains true in the case of a complete connected locally convex median space. One may adapt the argument given in Theorem 2.8 [Rol16] by looking at a maximal element in the family of convex subsets which contains b in their interior and separates it from a, and show that such a maximal element is a halfspace which contains a on its closure.

We deduce the following lemma :

Lemma 1.2.16. *Let X be a median space of rank n, and $a, b \in X$. Then there exists a halfspace $\mathfrak{h} \in \mathcal{H}(a, b)$ such that $d(a, \mathfrak{h}) = 0$ and $d(b, \mathfrak{h}^c) \geqslant \frac{d(a,b)}{n}$.*

Proof. We proceed by induction on the rank of the space X. The lemma is trivial for a connected $\mathbb{R}$-tree. Let us assume that the lemma is true for median spaces of rank $n-1$. Let X be a median space of rank n, and let $a, b \in X$. Let us take a halfspace $\mathfrak{h} \in \mathcal{H}(a, b)$ such that $d(a, \mathfrak{h}) = 0$, Lemma 1.2.14 ensures the existence of such halfspace. Let us assume that $\mathfrak{h}$ is not our desired halfspace, that is $d(b, \mathfrak{h}^c) < \frac{d(a,b)}{n}$. Let us consider then the projections $\tilde{b} := \pi_{\hat{\mathfrak{h}}}(b)$, $\pi_{\hat{\mathfrak{h}}}(a) = a$. We have then :

$$d(a, b) = d(a, \tilde{b}) + d(\tilde{b}, b).$$

We deduce the following :

$$\begin{aligned} d(a, \tilde{b}) &= d(a, b) - d(b, \tilde{b}) \\ &\geqslant d(a, b) - \frac{d(a, b)}{n} \\ &\geqslant \frac{(n-1)d(a, b)}{n}. \end{aligned}$$

The interval $[a, \tilde{b}]$ lies in the hyperplane $\hat{\mathfrak{h}}$, which is a median space of rank $n-1$. Hence, there exists a halfspace $\tilde{\mathfrak{h}} \in \mathcal{H}(a, \tilde{b})$ such that $d(a, \tilde{\mathfrak{h}}) = 0$ and $d(\tilde{b}, \tilde{\mathfrak{h}}^c) \geqslant \frac{d(a,\tilde{b})}{n-1}$. The projection into a convex subset being 1-lipschitz, we get $d(b, \tilde{\mathfrak{h}}^c) \geqslant d(\tilde{b}, \tilde{\mathfrak{h}}^c)$. We conclude then :

$$\begin{aligned} d(b, \tilde{\mathfrak{h}}^c) &\geqslant d(\tilde{b}, \tilde{\mathfrak{h}}^c) \geqslant \frac{d(\tilde{a}, b)}{n-1} \\ &\geqslant \frac{\frac{(n-1)d(a,b)}{n}}{n-1} \geqslant \frac{d(a,b)}{n}. \end{aligned}$$

Therefore $\tilde{\mathfrak{h}}$ is our desired halfspace. □

Proof of Proposition 1.2.13. Note that there is no loss of generality if we assume X to be complete. Let $b \in X$ such that $d(a, b) > nr$. By Lemma 1.2.16, there exists a halfspace $\mathfrak{h} \in \mathcal{H}(a, b)$ such that $d(a, \mathfrak{h}) = 0$ and $d(b, \mathfrak{h}^c) \geqslant \frac{d(a,b)}{n}$. Thus, the ball $B(b, r)$ is contained in the halfspace $\mathfrak{h}$. Hence, the convex hull of $B(b, r)$ also lies in $\mathfrak{h}$. We conclude that any point which is at distance bigger then nr from the point b lies outside the convex hull of $B(b, r)$. □

Remark 1.2.17. As it was pointed out to us by M. Hagen, Proposition 1.2.13 remains true when we replace the point a by a closed convex subset C and consider the tubular neighbourhood of it. More precisely, for any $r > 0$ we have :

$$Conv(\mathcal{N}_r(C)) \subseteq \mathcal{N}_{nr}(C).$$

To see this, let us consider a point $x \in X$ which is at distance greater than $n.r$ and show that there exist a halfpsace $\mathfrak{h} \in \mathcal{H}(C, x)$ such that $d(C, \mathfrak{h}) > r$. By Lemma 1.2.16, there exist a halfspace $\mathfrak{h} \in \mathcal{H}(x_C, x)$, where $x_C := \pi_C(x)$ and such that $d(x_C, \mathfrak{h}) > r$. By the bridge Lemma, we have $d(C, \mathfrak{h}) = d(x_C, \pi_{\bar{\mathfrak{h}}}(x_C))$. Hence the r-tubular neighbourhood $\mathcal{N}_r(C)$ of C is contained in $\mathfrak{h}^c$. Therefore we conclude that $Conv(\mathcal{N}_r(C)) \subseteq \mathfrak{h}^c$ which implies that the point x is not contained in $Conv(\mathcal{N}_r(C))$.

1.2.2 Measured poc sets

As it was displayed in Subsection 1.1.3, a median algebra is determined by the poc set of its halfspaces and there is a duality between the category of median algebra and the category of poc set. The duality extends in a sense to the case of median spaces where the additional metric structure on the median algebra gives rise to a structure of measured space on its set of halfspaces. This was first shown in [CDH10] using the language of space with measured walls and the study was extended in [Fio20]. The idea used in [CDH10] to construct the structure of measured space on the set of halfspaces is to consider the ring generated by walls intervals (see Definitions 1.1.6) and define on it a premeasure in order to apply the Carathéodory extension theorem. Let us describe briefly the construction.

Let X be a complete median space and let $\mathcal{W}_c(X)$ be its set of convex walls. Let us denote by $\mathcal{R} \subseteq \mathcal{P}(\mathcal{W}_c(X))$ the ***ring*** generated by wall intervals which separates two points (with the convention $\mathcal{W}_c(x, x) = \varnothing$), that is, the smallest subset of $\mathcal{P}(\mathcal{W}_c(X))$ containing the wall intervals, closed under finite union and closed under relative complements. Note that by Remark 1.1.25, any walls interval between two disjoint gate convex subsets corresponds to the walls interval between two points. We have the following :

Proposition 1.2.18 (Lemma 3.3 [CDH10]). *Let X be a median space and let $\mathcal{R}$ be the ring generated by wall intervals of the form $\mathcal{W}_c(x,y)$. Then any for any $A \in \mathcal{R}$, there exist $x_1, y_1, .., x_n, y_n \in X$ such that $A = W(x_1,y_1) \sqcup ... \sqcup W(x_n,y_n)$.*

A ***premeasure*** over a ring $\mathcal{R}$ is a function $\mu : \mathcal{R} \to [0,+\infty]$ such that $\mu(\varnothing) = 0$ and for any countable sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}$ of pairwise disjoint sets, we have :

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

Proposition 1.2.19 (Corollary 5.16 [CDH10]). *Let $\mathcal{R}$ be the ring as defined above. Then*

$$\begin{array}{rcl} \mu : \mathcal{R} & \to & [0,+\infty] \\ I & \mapsto & \mu(I) := \sum_{i=0}^{n_I} d(x_i,y_i) \quad \textit{Where} \;\; I = \bigsqcup_{i=0}^{n_I} \mathcal{W}_c(x_i,y_i) \end{array}$$

is a well defined premeasure over $\mathcal{R}$.

By Carathéodory extension theorem (see [Bog07] Ch 1, § 11 Theorem 1.11.8), the premeasure μ extends to a measure over the σ-algebra generated by walls intervals.

Theorem 1.2.20 (Theorem 5.1 [CDH10]). *Let X be a median space. Then the set of halfspaces $\mathcal{H}(X)$ admits a natural structure of a measured space $(\mathcal{H}(X), \mathcal{B}, \mu)$, where $\mathcal{B}$ is the σ algebra generated by the sets $\tilde{\mathcal{H}}(x,y) = \mathcal{H}(x,y) \cup \mathcal{H}(y,x)$ and $\mu(\tilde{\mathcal{H}}(x,y)) = d(x,y)$.*

In [CDH10], the structure of measured space is constructed on the set of convex walls $\mathcal{W}_c(X)$ and the structure considered on $\mathcal{H}(X)$ in Theorem 1.2.20 is the one induced from the latter under the natural identification of elements in $\mathcal{P}(\mathcal{W}_c(X))$ with the elements of $\mathcal{P}(\mathcal{H}(X))$ which are stable under the complementary operation of $\mathcal{H}(X)$. Hence, the σ-algebra considered in $\mathcal{H}(X)$ is the one generated by the subsets of the form $\tilde{\mathcal{H}}(x,y)$.

Remark 1.2.21. Let X_1, X_2 be two median spaces and let $f : X_1 \to X_2$ be an isometry. Then the map $f^{-1} : (\mathcal{H}(X_2), \mu_2) \to (\mathcal{H}(X_1), \mu_1)$ is a measurable poc set morphism which preserves the measure.

In [Fio20], a finer σ-algebra is considered in order to obtain a semi finite measure on $\mathcal{W}_c(X)$. A measure μ on X is semi finite if for any measurable subset $A \subset X$ such that $\mu(A) = +\infty$, there exist another measurable subset $B \subset A$ such that $\mu(B) < +\infty$.

The measure arising in Theorem 1.2.20 is not necessarily semi finite. For instance let us consider a $\mathbb{R}$-tree X which is not separable. Not that in this case, the hyperplane of a halfspace is a point. A uncountable family consisting of halfspaces which give rise to a family of hyperplanes which are discrete is a measurable subset with an infinite measure.

In [CDH10], a duality result was shown between the category of median spaces and the category of spaces with measured walls.

Definition 1.2.22. [Space with measured walls] A ***Wall*** in a set X is a couple (A, A^c) where $A \subseteq X$. We define the wall interval $\mathcal{W}(x,y)$ between two points $x, y \in X$ to be the set of walls (A, A^c) such that $x \in A$ and $y \in A^c$.

A ***space with measured wall*** is the data $(X, \mathcal{W}, \mathcal{B}, \mu)$ where $\mathcal{W}$ is a set of walls in X, $\mathcal{B}$ is a σ-algebra on $\mathcal{W}$ and μ a measure on the latter such that for any $x, y \in X$, the wall interval $\mathcal{W}(x,y)$ is measurable and is of finite measure.

The structure of a space with measured wall on a set X defines a natural pseudo metric on the latter called the wall pseudo metric $d_{\mathcal{W}}$ where $d_{\mathcal{W}}(x,y) = \mu(\mathcal{W}(x,y))$. If it is a metric, we say that $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a faithful space with measured walls.

To each median space, there is a canonical space with measured walls associated to it which is faithful and the wall pseudo metric coincides with the median metric (Theorem 1.1 [CDH10]). To make the analogy with the duality in the context of median algebra and poc set, we will be using the language of pointed measured poc set introduced in [Fio20]. We recall the definition :

Definition 1.2.23 (Pointed measured poc set)**.** A ***pointed measured Poc set*** is a quadruple $(P, \mathcal{B}, \mathfrak{u}_0, \mu)$ where P is a poc set, $\mathcal{B}$ is a σ-algebra over P, μ a measure defined on $\mathcal{B}$ and $\mathfrak{u}_0$ an ultrafilter in $\mathcal{U}(P)$.

We will assume that the σ- algebra $\mathcal{B}$ consists of subset which are stable under the complementary operation $*$ of P, i.e. $A^* = A$ for all $A \in \mathcal{B}$.

The latter assumption was not made in the original definition (Definition 2.13 [Fio20]) and is restrictive as it gives no chance for ultrafilters to be measurable subset. However, throughout this section, we will be only using the structure of measured poc set given by Theorem 1.2.20. Hence, the assumption is made for the convenience of switching between the language of measured halfspaces and measured convex walls of the median space.

Note that any space with measured wall $(X, \mathcal{W}, \mathcal{B}, \mu)$ gives rise to a pointed measured poc set $(P, \mathcal{B}', \mathfrak{u}_0, \mu')$ such that $P := \pi^{-1}(\mathcal{W})$, $\mathcal{B}' := \pi^{-1}(\mathcal{B})$, $\mathfrak{u}_0$ a principal utrafilter over an arbitrary point $x_0 \in X$ i.e $\mathfrak{u}_0 := \{A \in P \mid x_0 \in A\}$ and $\mu'(B) = \mu(\pi(B))$, where π is the canonical projection which associates to each subset of X the wall corresponding to it, i.e. $A \mapsto \pi(A) = (A, A^c)$.

1.2.3 Duality

Let $(P, \mathcal{B}, \mathfrak{u}_0, \mu)$ be a pointed measured poc set. Let X be a median space. In the end of Subsection 1.1.2, we have seen that the set of ultrafilters over P carries a natural structure of median algebra. Let us set $\mathcal{U}_{\mathfrak{u}_0}(P) := \{\mathfrak{u} \in \mathcal{U}(P) \mid \mathfrak{u}\Delta\mathfrak{u}_0 \in \mathcal{B} \text{ and } \mu(\mathfrak{u}\Delta\mathfrak{u}_0) < +\infty\}$ and show that it is stable under the median operation of $\mathcal{U}(P)$:

Lemma 1.2.24. *The set $\mathcal{U}_{\mathfrak{u}_0}(P)$ is a median subalgebra of $\mathcal{U}(P)$.*

Proof. Let us denote by m the median operation of $\mathcal{U}(P)$ induced from the median algebra structure of the set of subsets of $\mathcal{P}(P)$ (see Example 1.1.4 (3)). For any $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}_{\mathfrak{u}_0}(P)$ we have :

$$\mathfrak{u}_1 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) = (\mathfrak{u}_1 \Delta \mathfrak{u}_2) \cap (\mathfrak{u}_1 \Delta \mathfrak{u}_3).$$

In the other hand, we have $\mathfrak{u}_0 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) = (\mathfrak{u}_0 \Delta \mathfrak{u}_1)\Delta(\mathfrak{u}_1 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3))$. Hence, the subset $\mathfrak{u}_0 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ is measurable. It remains to show that it is of finite measure.

We remark that for any $A, B, C, D \subset P$, we have :

$$m_{\mathcal{P}(P)}(A, B, C)\Delta D = ((A \cap B) \cup (A \cap C) \cup (B \cap C))\Delta D \subset (A\Delta D) \cup (B\Delta D) \cup (C\Delta D).$$

Hence, for any $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}_{\mathfrak{u}_0}(P)$ we have :

$$\mu(m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta \mathfrak{u}_0) \leqslant \mu(\mathfrak{u}_1 \Delta \mathfrak{u}_0) + \mu(\mathfrak{u}_2 \Delta \mathfrak{u}_0) + \mu(\mathfrak{u}_3 \Delta \mathfrak{u}_0) < +\infty.$$

□

As for any $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{U}_{\mathfrak{u}_0}(P)$, we have :

$$(\mathfrak{u}_1 \Delta \mathfrak{u}_0)\Delta(\mathfrak{u}_2 \Delta \mathfrak{u}_0) = \mathfrak{u}_1 \Delta \mathfrak{u}_2,$$

the measure of the subset $\mathfrak{u}_1 \Delta \mathfrak{u}_2$ is finite. Therefore, the measure μ endows the median algebra $\mathcal{U}_{\mathfrak{u}_0}(P)$ with a pseudo metric d_μ where $d(\mathfrak{u}_1, \mathfrak{u}_2) = \mu(\mathfrak{u}_1 \Delta \mathfrak{u}_2)$. Let us denote by $\tilde{\mathcal{U}}_{\mathfrak{u}_0}$ the metric space obtained by identifying the ultrafilters which are at null pseudo distance.

Proposition 1.2.25 ([CDH10] Proposition 3.14)**.** *The space $\tilde{\mathcal{U}}_{\mathfrak{u}_0}(P)$ is a median space.*

Proof. Let us first show that the median operation of $\mathcal{U}_{\mathfrak{u}_0}$ is well defined on the quotient $\tilde{\mathcal{U}}_{\mathfrak{u}_0}$. Let us consider $\mathfrak{u}_1, \mathfrak{u}_1', \mathfrak{u}_2, \mathfrak{u}_2', \mathfrak{u}_3, \mathfrak{u}_3' \in \mathcal{U}_{\mathfrak{u}_0}(X)$ such that $\mu(\mathfrak{u}_i \Delta \mathfrak{u}_i') = 0$. We remark that

$$m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3') \subseteq \mathfrak{u}_3 \Delta \mathfrak{u}_3' \quad \text{(Compare with Proposition 1.2.4).}$$

In the same spirit of the proof of Proposition 1.2.7 given in [CDH10] Corollary 2.15, we remark that

$$\begin{aligned} m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3') &= (m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3'))\Delta(m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3')\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3')). \\ m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3')\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3') &= (m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1, \mathfrak{u}_2', \mathfrak{u}_3'))\Delta(m(\mathfrak{u}_1, \mathfrak{u}_2', \mathfrak{u}_3')\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3')). \end{aligned}$$

We deduce then that

$$m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3') \subseteq (\mathfrak{u}_1 \Delta \mathfrak{u}_1') \cup (\mathfrak{u}_2 \Delta \mathfrak{u}_2') \cup (\mathfrak{u}_3 \Delta \mathfrak{u}_3'),$$

which implies that $\mu(m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta m(\mathfrak{u}_1', \mathfrak{u}_2', \mathfrak{u}_3')) = 0$.

By Lemma 1.2.9, to conclude that $\tilde{\mathcal{U}}_{\mathfrak{u}_0}$ is a median space, it is enough to show that for any $[\mathfrak{u}_1], [\mathfrak{u}_2], [\mathfrak{u}_3] \in \tilde{\mathcal{U}}_{\mathfrak{u}_0}$ we have $[m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)] \in [[\mathfrak{u}_1], [\mathfrak{u}_2]]_{d_\mu}$, that is

$$\mu(\mathfrak{u}_1 \Delta \mathfrak{u}_2) = \mu(\mathfrak{u}_1 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)) + \mu(m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)\Delta \mathfrak{u}_2).$$

To simplify the notation, let us set $m = m((\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$. We have :

$$\mathfrak{u}_1 \Delta \mathfrak{u}_2 = (\mathfrak{u}_1 \Delta m)\Delta(\mathfrak{u}_2 \Delta m)) = ((\mathfrak{u}_1 \Delta m) \cup (\mathfrak{u}_2 \Delta m))\backslash((\mathfrak{u}_1 \Delta m) \cap (\mathfrak{u}_2 \Delta m))).$$

In the other hand, we have :

$$\mathfrak{u}_1 \Delta m(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) = (\mathfrak{u}_1 \Delta \mathfrak{u}_2) \cap (\mathfrak{u}_1 \Delta \mathfrak{u}_3).$$

Thus

$$((\mathfrak{u}_1 \Delta m) \cap (\mathfrak{u}_2 \Delta m))) = (\mathfrak{u}_1 \Delta \mathfrak{u}_2) \cap (\mathfrak{u}_1 \Delta \mathfrak{u}_3) \cap (\mathfrak{u}_2 \Delta \mathfrak{u}_3) = \varnothing.$$

Therefore, we conclude that

$$\mathfrak{u}_1 \Delta \mathfrak{u}_2 = (\mathfrak{u}_1 \Delta m) \Delta (\mathfrak{u}_2 \Delta m)) = ((\mathfrak{u}_1 \Delta m) \sqcup (\mathfrak{u}_2 \Delta m)).$$

Which completes the proof □

Remark 1.2.26. Note that one can obtain Proposition 1.2.25 from the results of Birkhoff, See Examples 1.2.2 (6). Lemma 1.2.24 is equivalent to say that the relation which identifies ultrafilter which have a symmetric difference of null measure is a congruence with respect to the median operation of $\mathcal{U}(P)$ (compare with Theorem 9 in [Bir67] Ch V, §7 pp 77). Proposition 1.2.25 shows that the metric given by the measure coincides with the median algebra structure induced from the quotient of $\mathcal{U}(P)$, which is the analogue of Theorem 1 in [BK47] in this particular case.

Each ultrafilter $\mathfrak{u} \in \mathcal{U}(P)$ defines a, possibly trivial, median space $\mathcal{U}_{\mathfrak{u}}$. Hence, when we consider the quotient of $\mathcal{U}(P)$ under the relation $\mathfrak{u}_1 \sim \mathfrak{u}_2$ if and only if $\mathfrak{u}_1 \Delta \mathfrak{u}_2 \in \mathcal{B}$ is of finite measure. The median algebra $\mathcal{U}(P)$ splits into components which are at infinite "distance" and each components is a median space. Hence, when we fix the ultrafilter $\mathfrak{u}_0$, we are fixing a component in $\mathcal{U}(P)/\sim$

Remark 1.2.27.

1. Let $(P_1, \mathcal{B}_1, \mathfrak{u}_1, \mu_1)$, $(P_2, \mathcal{B}_2, \mathfrak{u}_2 \mu_2)$ be two measured poc sets and let $f : P_1 \to P_2$ be a measurable morphism of poc set which preserves the measure and such that $\mu_1(f^{-1}(\mathfrak{u}_2) \Delta \mathfrak{u}_1) < +\infty$. Then the morphism $f^{-1} : \mathcal{U}_{\mathfrak{u}_2} \to \mathcal{U}_{\mathfrak{u}_1}$ is an isometry.
2. Let $(P_1, \mathcal{B}_1, \mathfrak{u}_0, \mu_1)$ be a measured poc set. In the same vein as proposition 1.1.39, any element p gives rise to a halfspace $\mathcal{U}_p$ in $\mathcal{U}_{\mathfrak{u}_0}(P)$ by considering the set of ultrafilters of $\mathcal{U}_{\mathfrak{u}_0}$ which contains p. In the proof of Proposition 1.2.25, we have seen that the quotient of $\mathcal{U}_{\mathfrak{u}_0}(P)$ onto $\tilde{\mathcal{U}}_{\mathfrak{u}_0}(P)$ is a surjective morphism of median algebra. Hence, the image $\tilde{\mathcal{U}}_p$ of the halfspace $\mathcal{U}_p$ under the latter quotient is a halfspace of $\tilde{\mathcal{U}}_{\mathfrak{u}_0}(P)$. The halfspace $\tilde{\mathcal{U}}_p$ is characterized as follows :

$$\tilde{\mathcal{U}}_p := \{[\mathfrak{u}] \in \tilde{\mathcal{U}}_{\mathfrak{u}_0}(P) \mid \text{there exist } \mathfrak{u}' \in [\mathfrak{u}] \text{ such that } p \in \mathfrak{u}'\}.$$

In contrast with pointed measured poc sets, the dual median space $\tilde{\mathcal{U}}_{\mathfrak{u}_{x_0}}$ ($\mathcal{M}(X)$ in [CDH10]) to a space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$, where $x_0 \in X$, comes with the additional canonical morphism $\Phi : X \to \tilde{\mathcal{U}}_{\mathfrak{u}_{x_0}}$ which associates to each point x, the principal ultrafilter over it. This morphism is injective if for any $x, y \in X$ we have $\mu(\mathcal{W}(x, y)) > 0$. If X is a metric space and its structure of a space with measured walls is faithfull, then the morphism Φ is an isometric embedding.

The hyperbolic space $\mathbb{H}^n$ has a natural structure of a space with measured walls. The set of walls $\mathcal{W}(\mathbb{H}^n)$ considered is given by the set of hyperplanes of $\mathbb{H}^n$, where to each hyperplane $\hat{\mathfrak{h}}$, we associate the couples $(\mathfrak{h}, \mathfrak{h}^c)$ such that $\mathfrak{h}$ is a halfspace bounded by $\hat{\mathfrak{h}}$. The group $\mathrm{SO}(n,1)$ acts by isometries on $\mathbb{H}^n$ and the action is transitive on the set of hyperplanes. Each hyperplane is isometric to a hyperbolic space of dimension $n-1$ and the stabilizer of a hyperplane in $\mathrm{SO}(n,1)$ is isomorphic to $\mathrm{SO}(n-1,1)$. Thus the space of hyperplanes of $\mathbb{H}^n$ corresponds to the quotient $\mathrm{SO}(n,1)/\mathrm{SO}(n-1,1)$. We endow the set of walls $\mathcal{W}(\mathbb{H}^n)$ with μ, the push-forward of a Haar measure on $\mathrm{SO}(n,1)$ to the quotient $\mathrm{SO}(n,1)/\mathrm{SO}(n-1,1)$. Up to rescaling μ, we have the following Crofton formula $x, y \in \mathbb{H}^n$ $\mu(\mathcal{W}(x,y)) = d(x,y)$ for any $x, y \in \mathbb{H}^n$, that was proved in [Rob98] Proposition 2.1 (see [CMV04] Proposition 3 for a different proof).

In [CD17], it was shown that the median space associated to $(\mathbb{H}^n, \mathcal{W}(\mathbb{H}), \mu)$ is locally compact, of infinite rank, and the image of $\mathbb{H}^n$ under the canonical isometric embedding Φ is at finite Hausdorff distance from the ambient median space. In particular, they proved the following :

Theorem 1.2.28 ([CD17] Corollary 1.2)**.** *There exist an* $\mathrm{SO}(n,1)$*-equivariant isometric embedding* $\Phi : \mathbb{H}^n \to X$*, where* X *is a locally compact median space of infinite rank and* $\Phi(\mathbb{H}^n)$ *is at bounded Hausdorff distance from* X*.*

X connected versus $(\mathcal{W}_c(X), \mu)$ "atomless" :

In [Fio20], the author shows that a median space is connected if and only if the finer algebra defined on its set of halfspaces contains no atoms (see Lemma 3.5 therein). In the following, we give a reformulation of the latter characterization when $\mathcal{W}_c(X)$ is endowed with the structure of measured set obtained from Theorem 5.1 in [CDH10] (see Theorem 1.2.20).

Definition 1.2.29. Let $(X, \mathcal{B}, \mu)$ be a measurable space. A subset $A \in \mathcal{B}$ is an ***atom*** if $\mu(A) > 0$ and for any $B \in \mathcal{B}$ such that $B \subseteq A$ we have either $\mu(A \cap B) = 0$ or $\mu(A \cap B^c) = 0$. We say that $(X, \mathcal{B}, \mu)$ is atomless if $\mathcal{B}$ contains no atoms.

Proposition 1.2.30. *Let* X *be a complete median space. Then* X *is connected if and only if* $(\mathcal{W}_c(X), \mu)$ *contains no finite atoms.*

We first remark the following characterization of connectedness in term of intervals.

Lemma 1.2.31. *Let* X *be a complete median space. Then* X *is connected if and only if for any* $x, y \in X$ *we have* $[x,y] \neq \{x,y\}$*.*

Proof. Note that for any $x, y \in X$, the map $m(x, y, *)$ is a retraction from X onto $[x,y]$. Hence, if X is connected any interval is connected.

We assume now that X is not connected and let $U \subset X$ be a proper clopen subset. We set $\mathcal{F} := \{[a,b] \mid a \in U, \quad b \in U^c\}$. We endow $\mathcal{F}$ by the reverse order $\leqslant$ obtained from the inclusion, i.e. $[a,b] \leqslant [c,d]$ iff $[c,d] \subseteq [a,b]$. By Lemma 2.2.6 $(\mathcal{F}, \leqslant)$ is an inductive set. This is also due to the fact that both U and U^c are closed, which ensures us that for any interval $[a,b]$ which is obtained from the intersection of an increasing (with respect to $\leqslant$) sequence of intervals in $\mathcal{F}$, we have $a \in U$ and $b \in U^c$. Any maximal element $[a,b] \in \mathcal{F}$ is necessarily of the form $[a,b] = \{a,b\}$ and by Zorn's lemma such maximal element exists, which finishes the proof. $\square$

Proof of Proposition 1.2.30. If X is not connected, there exist two distinct points $x, y \in X$ such that $[x,y] = \{x,y\}$. By Remark 1.1.15, the inverse image of y under the retraction $m(x,y,*)$ is the unique halfspace of X which separates y from x, that is $\mathcal{W}_c(x,y) = \{(\mathfrak{h}, \mathfrak{h}^c)\}$. Hence $\{\mathfrak{h}\}$ is measurable and $\mu(\{\mathfrak{h}\}) = d(x,y) \neq 0$, that is $\{\mathfrak{h}\}$ is an atom.

Conversely, let us assume that there exist an atom $A \subseteq \mathcal{W}_c(X)$ such that $0 < \mu(A) < +\infty$. By construction, we have :

$$\mu(A) = \inf\{\sum_{n\in\mathbb{N}} \mu(\mathcal{W}_c(x_i,y_i)) \mid A \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{W}_c(x_i,y_i)\}.$$

The measure of A being finite, there exist sequences of points $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \subseteq X$ such that $A \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{W}_c(x_i,y_i)$. Because the measure of A is positive, there exist at least one $n \in \mathbb{N}$ such that $\mu(A \cap \mathcal{W}_c(x_n,y_n)) > 0$. As A is an atom, we have $\mu(A \cap \mathcal{W}_c(x_n,y_n)) = \mu(A)$, that is $\mu(A \backslash \mathcal{W}_c(x_n,y_n)) = 0$. Hence, we deduce that

$$\mu(A) = \inf\{\mu(\mathcal{W}_c(x,y)) = d(x,y) \mid \mu(A \backslash \mathcal{W}_c(x,y)) = 0\}.$$

We claim that the limit above is attained by some $a, b \in X$ and $[a,b] = \{a,b\}$. Let us denote by I_A the set of intervals $[x,y]$ such that $\mu(A\backslash\mathcal{W}_c(x,y)) = 0$. We have already shown that I_A is not empty. Let us endow it with the reverse order given by the inclusion, i.e. $[x,y] \leqslant [z,t]$ iff $[z,t] \subseteq [x,y]$, and show that it is an inductive set. Let $([x_i,y_i])_{i\in\mathbb{N}}$ be an increasing sequence with respect to $\leqslant$. By Lemma 2.2.5, we may assyme that $x_{i+1} = m(x_i, x_{i+1}, y_{i+1})$ and $y_{i+1} = m(y_i, x_{i+1}, y_{i+1})$. By Lemma 2.2.6, the intersection $\bigcap_{i\in\mathbb{N}} [x_i,y_i]$ is not empty and equal $[\tilde{x}, \tilde{y}]$ where $\tilde{x}$ and $\tilde{y}$ are the limits of $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ respectively. By Lemma 2.2.7 we have $\mathcal{W}'_c(x_{i+1},y_{i+1}) \subseteq \mathcal{W}'_c(x_i,y_i)$ and $\mathcal{W}'_c(\tilde{x},\tilde{y}) \subseteq \bigcap_{i\in\mathbb{N}} \mathcal{W}'_c(x_i,y_i)$. As $\lim_{i\to+\infty} d(x_i,y_i) = d(\tilde{x},\tilde{y})$ we get

$$\mu(\mathcal{W}_c(\tilde{x},\tilde{y}) \backslash (\bigcap_{i\in\mathbb{N}} \mathcal{W}_c(x_i,y_i))) = 0.$$

We deduce then

$$\begin{aligned}
\mu(A\backslash\mathcal{W}_c(\tilde{x},\tilde{y})) &= \mu(A\backslash(\bigcap_{i\in\mathbb{N}} \mathcal{W}_c(x_i,y_i))) \\
&= \mu(\bigcup_{i\in\mathbb{N}} A\backslash\mathcal{W}_c(x_i,y_i)) \\
&= \lim_{i\to+\infty} \mu(A\backslash\mathcal{W}_c(x_i,y_i)) \\
&= 0.
\end{aligned}$$

We conclude that $(I_A, \leqslant)$ is an inductive set. Assuming Zorn's lemma, there exist a maximal element $[a,b] \in (I_A, \leqslant)$. Note that a is distinct from b as $\mu(A) > 0$. For any $c \in [a,b]$ we have $\mathcal{W}_c(a,b) = \mathcal{W}_c(a,c) \sqcup \mathcal{W}_c(c,b)$. Thus as A is an atom, we have either $\mu(A \backslash \mathcal{W}_c(a,c)) = 0$ or $\mu(A \backslash \mathcal{W}_c(c,b)) = 0$. Therefore, by the maximality of $[a,b]$ we have $[a,b] = \{a,b\}$ which, by Lemma 1.2.31, implies that X is not connected. □

Remark 1.2.32. It is not clear if any measurable subset $A \subseteq \mathcal{W}_c(X)$ of infinite measure contains a subset B such that none of the two subsets $A \cap B$ and $A \backslash B$ is of null measure. This is one of the points where the finer algebra with the new measure constructed in [Fio20] appears to be very helpful, being semi finite.

Product and co-product

Definition 1.2.33. We say that a median space X is ***reducible*** if X splits as the ℓ^1-product of two median spaces which are not singletons.

We say that it is ***irreducible*** if it is not reducible.

We have seen in Paragraph 1.1.3 of Subsection 1.1.3 how the functors $\mathcal{U}$ and $\mathcal{H}$ intertwines between products of median algebras and co-product of poc sets. By construction, the same hold in the metric case :

Proposition 1.2.34. *Let $(X_1, d_1), (X_2, d_2)$ be two median spaces and let $(P_1, \mathcal{B}_1, \mathfrak{u}_1, \mu_1), (P_2, \mathcal{B}_2, \mathfrak{u}_2, \mu_2)$ be two pointed measured poc sets. We have then :*

1. *The measured poc set $\mathcal{H}_{\mu_{(d_1 \times_{\ell^1} d_2)}}(X_1 \times X_2)$ is isomorphic to the coproduct $\mathcal{H}_{\mu_{d_1}}(X_1) \coprod \mathcal{H}_{\mu_{d_2}}(X_2)$.*
2. *The median space $\mathcal{U}_{d_{(\mu_1+\mu_2)}}(P_1 \coprod P_2)$ is isometric to the ℓ^1-product $\mathcal{U}_{d_{\mu_1}} \times \mathcal{U}_{d_{\mu_2}}$.*

By Theorem 2.2.1, we deduce the following :

Proposition 1.2.35. *Let X be a complete median space with the strong separation property. Then the following are equivalent :*

1. *The median space X is reducible.*
2. *The measured set of walls of X decomposes into the following*

$$(\mathcal{W}_c(X), \mu) = (\mathcal{W}'_c(X), \mu') \coprod (\mathcal{W}''_c(X), \mu''),$$

where each wall of $\mathcal{W}'_c$ is a transverse to any wall of $\mathcal{W}''_c$.

Chapitre 2

Strongly separated median space

Let us denote by $\mathcal{H}_\mu$ the contravariant functor which associates to each median spaces X the pointed measured poc set $(\mathcal{H}(X), \mathcal{B}, \mu, \mathfrak{u}_x)$ where $x \in X$. Note that the structure of measured poc set $(\mathcal{H}(X), \mathcal{B}, \mu, \mathfrak{u}_x)$ does not depend on the base point x as all points in X define the same component in $\mathcal{U}/\sim$.

Let us denote by $\mathcal{U}_d$ the contravariant functor which associates to each pointed measured poc set $(P, \mathcal{B}, \mathfrak{u}_0, \mu)$ the median space $(\tilde{\mathcal{U}}_{\mathfrak{u}_0}, d_\mu)$. If there is no risk of confusion, we will simply write $\mathcal{H}$ and $\mathcal{U}$ to denote $\mathcal{H}_\mu$ and $\mathcal{U}_d$ respectively.

To each median algebra M there is a natural embedding of median algebra into its double dual $\mathcal{U}(\mathcal{H}(M))$, through the map which associates to each point $x \in M$ the principal ultrafilter $\mathfrak{u}_x$. In [Isb80], it was shown that the embedding is a homeomorphism when M is a Stone median algebra and the poc set considered is the set of closed open halfspaces (Theorem 6.13 [Isb80]). An analogue to this in the metric case was shown in [Fio20] (Theorem A), where it shows that the isometric embedding from X to $\mathcal{U}_d(\mathcal{H}_\mu(M))$ is surjective when X is locally convex and complete.

We give another proof to the latter Theorem without considering the finer sigma algebra constructed in [Fio20] and extend it to the case where the median space X is complete and for any two points $x, y \in X$, there exist a halfspace which contains x in its interior and the interior of its complementary contains y.

Before stating the duality theorem, we fix some terminology and notation with regard to the latter class of median spaces in the following section.

2.1 Median space with strong separation property

Definition 2.1.1. Let X be a complete median space. For any $x, y \in X$, we denote by $\mathcal{H}'(x, y)$ the set of halfspaces $\mathfrak{h} \in \mathcal{H}(X)$ such $x \in \mathfrak{h}^\circ$ and $y \in (\mathfrak{h}^c)^\circ$, where $\mathfrak{h}^\circ$ is the interior of $\mathfrak{h}$. Following the same notation of Definition 1.1.6, we set $\mathcal{W}'_c(x, y) := \{(\mathfrak{h}, \mathfrak{h}^c) \in \mathcal{W}_c(x, y) \mid \mathfrak{h} \in \mathcal{H}'(x, y)\}$ and $\tilde{\mathcal{H}}'(x, y) = \mathcal{H}'(x, y) \cup \mathcal{H}'(y, x)$.

We say that a median space has the ***strong separation property*** if for any points $x, y \in X$, the subset $\mathcal{H}'(x, y)$ is not empty.

Note that by virtue of Theorem 1.1.7, any locally convex median space has the strong separation property (see also [Bow13] Lemma 7.3). In the other hand, there are median spaces which have the strong separation property and are not locally convex as shown in the following examples.

Example 2.1.2. Let $X := \ell^1(\mathbb{R}) = L^1(\mathbb{N}, \mu)$ where μ is the counting measure. The space X is not locally convex as the convex hull of any ball is unbounded.
For each $i \in \mathbb{N}$ and $r \in \mathbb{R}$, the following sets

$$\begin{aligned} \mathfrak{h}_{i,<r} &:= \{f \in X / \ f(i) < r\}, \\ \mathfrak{h}_{i,>r} &:= \{f \in X / \ f(i) > r\} \end{aligned}$$

are open halfspaces of X and $\mathfrak{h}_{i,<r} \subset (\mathfrak{h}_{i,>r})^c$. For any $f, g \in X$, there exist $i \in \mathbb{N}$ and $r \in \mathbb{R}$ such that $f \in \mathfrak{h}_{i,<r}$ and $g \in \mathfrak{h}_{i,>r}$. Hence for any $f, g \in X$, the set $\mathcal{H}'(f, g)$ is not empty. We remark that there are also infinitely many halfspaces which are dense in the space X. In fact, for any $f, g \in X$ which differ in an infinite subset of $\mathbb{N}$ there exist a halfspace $\mathfrak{h} \in \mathcal{H}(x, y)$ which is dense in X.

In the same spirit of the above example, one may generate many examples of median spaces which satisfy the strong separation property through the direct sum of locally convex median space.

Definition 2.1.3 (See also [CMV04] Definition 5)**.** Let $((X_i, a_i))_{i\in\mathbb{N}}$ be a family of pointed median space. We define ***the direct sum*** $(\bigoplus_{i\in\mathbb{N}}(X_i, a_i), d)$ of the family $((X_i, a_i))_{i\in\mathbb{N}}$ to be

$$\bigoplus_{i\in\mathbb{N}}(X_i, a_i) := \{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \mid \sum_{i\in\mathbb{N}} d(x_i, a_i) < +\infty\}.$$

And for $\tilde{x} = (x_i)_{i\in\mathbb{N}}, \tilde{y} = (y_i)_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}}(X_i, a_i)$, the distance between them is defined by $d(\tilde{x}, \tilde{y}) = \sum_{i\in\mathbb{N}} d(x_i, y_i)$.

The direct sum $\bigoplus_{i\in\mathbb{N}}(X_i, a_i)$ is a median space of infinite rank (if we assume that X_i is different from a singleton for infinitely many $i \in \mathbb{N}$).

Let us denote by $\tilde{X} := \bigoplus_{i\in\mathbb{N}}(X_i, a_i)$. If the family $((X_i, a_i))_{i\in\mathbb{N}}$ consists of complete median spaces, then their direct sum is also a complete median space. Each of the complete median space X_k embeds as a closed convex subset of $\tilde{X}$ through the isometric embedding ϕ_k defined by $\phi_k(x) = (x_i)_{i\in\mathbb{N}}$ such that $x_i = a_i$ for all $i \neq k$ and $x_k = x$. By Remark 1.1.15, any halfspace $\mathfrak{h}_k \in \mathcal{H}(X_k)$ lifts into a halfspace $\tilde{\mathfrak{h}}_k$ in $\tilde{X}$ where

$$\tilde{\mathfrak{h}}_k = \{(x_i)_{i\in\mathbb{N}} \in \tilde{X} \mid x_k \in \mathfrak{h}_k\}.$$

The projection into gate convex subset being 1-Lipschitz by Proposition 1.2.4, we deduce that for $x, y \in X_k$ and $\mathfrak{h}_k \in \mathcal{H}'_{X_k}(x, y)$, and for any $\tilde{x} = (x_i)_{i\in\mathbb{N}}, \tilde{y} = (y_i)_{i\in\mathbb{N}}$ such that $x_i = x$ and $y_i = y$, we have $\tilde{\mathfrak{h}}_k \in \mathcal{H}'(\tilde{x}, \tilde{y})$. In particular we deduce the following

Proposition 2.1.4. *The direct sum of a family of complete locally convex median space is a median space which satisfies the strong separation property.*

The direct sum of complete locally convex median spaces is in general not locally convex. When each of the median space is assumed to be connected, we have the following criterion

Proposition 2.1.5. *Let $(X_i, a_i)_{i\in\mathbb{N}}$ be a family of complete connected median space. Then $\tilde{X} := \bigoplus_{i\in\mathbb{N}}(X_i, a_i)$ is locally convex if and only if each X_i is locally convex and $\sum_{i\in\mathbb{N}} diam(X_i) < +\infty$, where $diam(X_i)$ is the diameter of X_i.*

Proof. Note That as each X_i embeds as a closed convex subset of $\tilde{X}$, if one of the X_i is not locally convex then $\tilde{X}$ will not be locally convex. We assume that each X_i is locally convex and show that $\tilde{X}$ is locally convex if and only if $\sum_{i\in\mathbb{N}} diam(X_i) < +\infty$. Let us assume first that $\sum_{i\in\mathbb{N}} diam(X_i) < +\infty$ and fix $r > 0$, $\tilde{b} = (b_i)_{i\in\mathbb{N}} \in \tilde{X}$. There exist $N \in \mathbb{N}$ such that $\sum_{i>N} diam(X_i) < r$. The spaces X_i's being locally convex, there exist for each $i \in \{1, .., N\}$ a convex subset C_i which contains b_i in its interior such that $\sum_{i\leqslant N} diam(C_i) < r - \sum_{i>N} diam(X_i)$. Then the closed convex subset $C := \{(x_i)_{i\in\mathbb{N}} \mid x_i \in C_i \text{ for each } i \leqslant N\}$ contains $\tilde{b}$ in its interior and is contained in the ball of radius r, thus we have shown that $\tilde{X}$ is locally convex.

Let us assume now that $\sum_{i\in\mathbb{N}} diam(X_i) = +\infty$ and show that the convex hull of any ball in $\tilde{X}$ is unbounded. Let us fix an integer $m \in \mathbb{N}$ and consider the sequence $\tilde{x}_n = (x_{i,n})_{i\in\mathbb{N}}$ such that $x_{i,n} = a_i$ for each $i \neq n$ and $x_{n,n} \in C_n$ be such that $d(x_{n,n}, a_n) = min(\frac{1}{n}, diam(X_i)(1 - \frac{1}{2^n}), \frac{1}{m})$. We note that such point $x_{n,n}$ exist as each X_i is assumed to be connected. The convex hull of the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ contains the points $\tilde{y}_n = (y_{i,n})_{i\in\mathbb{N}}$ defined by $y_{i,n} = x_i$ for each $i \leqslant n$ and $y_{i,n} = a_i$ for each $i > n$. The sequence of points $(\tilde{y}_n)_{n\in\mathbb{N}}$ goes arbitrarily far from the origin $(a_i)_{i\in\mathbb{N}}$, which proves that $\tilde{X}$ is not locally convex. □

2.2 Duality for median spaces with the strong separation property

The aim of this paragraph is to prove the following theorem

Theorem 2.2.1. *Let X be a complete median space with the strong separation property. Then the canonical isometric embedding of X into $\mathcal{U}_d(\mathcal{H}_\mu(X))$ is surjective.*

We obtain the above theorem as a consequence of the following proposition :

Proposition 2.2.2. *Let X be a complete median space with the strong separation property. Then the image of X under the canonical embedding is a closed convex subset of the median space $\mathcal{U}_d(\mathcal{H}_\mu(X))$.*

We already know that the space X embeds isometrically into $\mathcal{U}_d(\mathcal{H}_\mu(X))$, to show that its image is a convex of $\mathcal{U}_d(\mathcal{H}_\mu(X))$, we need to prove that any ultrafilter in $\mathcal{U}_d(\mathcal{H}_\mu(X))$ which lies between two principal ultrafilter has a null symmetric difference with some principal ultrafilter $\mathfrak{u}_c$. We will be needing some lemmas before proving the latter proposition.

Lemma 2.2.3. *Let X be a complete median space such that for any $x, y \in X$, the set $\mathcal{W}'_c(x,y)$ is not empty. Then for any $a, b \in X$ and any $\epsilon > 0$ small enough there exist $a_1 \in [a,b]$ such that $d(a,a_1) < \epsilon$ and any halfspace $\mathfrak{h} \in \mathcal{H}(b,a_1)$ contains a in its interior.*

Proof. Let $X_a \subseteq [a,b]$ be the set of points x such that any halfspace separating x from b contains a in its interior. We first show that the subset X_a is not empty. Let us consider $\mathfrak{h} \in \mathcal{H}'(b,a)$ and set $a' := \pi_{\bar{\mathfrak{h}}}(b)$. Any halfspace which separates a' from b contains $\mathfrak{h}$, hence contains a in its interior. Note that a' is distinct from b as the latter is contained in the interior of $\mathfrak{h}^c$.

The claim of the lemma translates into saying that $r := inf\{d(a,x)/\ x \in X_a\} = 0$. We note that for any $x, y \in X_a$, the median point $m(a,x,y)$ belongs to X_a as any halfspace which separates the latter point from b must either separate x from b or y from b. Let us consider a sequence of points $(x_i)_{i\in\mathbb{N}} \subseteq X_a$ such that $\lim\limits_{i\to+\infty} d(a,x_i) = r$ and show that it is a Cauchy sequence. Let $\epsilon > 0$ and $n \in \mathbb{N}$ such that $|d(a,x_i) - r| \leqslant \epsilon$ for all $i > n$. Let us fix indices i and j bigger than n and set $m_{i,j} := m(a,x_i,x_j)$. We have then :

$$d(x_i,x_j) = d(x_i,m_{i,j}) + d(m_{i,j},x_j) = d(a,x_i) - d(a,m_{i,j}) + d(a,x_j) - d(a,m_{i,j}).$$

In the other hand, any halfspace which separates $m_{i,j}$ from a must at least separate x_i or x_j from a. Hence, any such halfspace must contains a in its interior. We deduce that $m_{i,j} \in X_a$, thus $d(a,m_{i,j}) \geqslant r$. We conclude that :

$$d(x_i,x_j) = (d(a,x_i) - d(a,m_{i,j})) + (d(a,x_j) - d(a,m_{i,j})) \leqslant 2\epsilon.$$

The space X being complete and the intervals being closed, the sequence $(x_i)_{i\in\mathbb{N}}$ converges to a point $x \in [a,b]$, with $d(a,x) = r$. Let us assume, for sake of contradiction, that x is distinct from a. We consider a halfspace $\mathfrak{h} \in \mathcal{H}'(x,a)$ and set $\tilde{x}_i := \pi_{\bar{\mathfrak{h}}}(x_i)$. Any halfspace which separates $\tilde{x}_i$ from b either :

- Separates $\tilde{x}_i$ from x_i, in which case contains a in interior as it would contain $\mathfrak{h}$.
- Separates both $\tilde{x}_i$ and x_i from b. In this case, the halfspace would also contain a in its interior as the points x_i belong to X_a.

We deduce that the sequence $(\tilde{x}_i)_{i\in\mathbb{N}}$ lies in X_a. As it converge to the point $\pi_{\bar{\mathfrak{h}}}(x)$, there exist $k \in \mathbb{N}$ such that $d(a,\tilde{x}_i) < r$, which is a contradiction. □

Lemma 2.2.4. *Let X be a complete median with the strong separation property and let $(\mathcal{W}_c(X),\mu)$ be the canonical structure of a measure set defined on the set of walls of X (see Theorem 1.2.20). Then the set $\mathcal{W}_c(a,b)\backslash\mathcal{W}'_c(a,b)$ is measurable and it has null measure.*

Proof. Let us consider $a, b \in X$. For any $n \in \mathbb{N}$ there exist, by Lemma 2.2.3, $a_n \in [a, b]$ and $b_n \in [a_n, b]$ such that $\mathcal{W}_c(a_n, b_n) \subseteq \mathcal{W}'_c(a, b)$ with $d(a, a_n) \leqslant \frac{1}{n}$ and $d(b, b_n) \leqslant \frac{1}{n}$. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq [a, b]$ be sequences of such points. One may moreover assume that $[a_n, b_n] \subseteq [a_{n+1}, b_{n+1}]$ up to considering the sequences $\tilde{a}_i := m(a, a_i, \tilde{a}_{i-1})$ and $\tilde{b}_i := m(b, b_i, \tilde{b}_{i-1})$ where $\tilde{a}_0 = a_0$ and $\tilde{b}_0 = b_0$. Remark that for any $\mathfrak{h} \in \mathcal{H}'(a, b)$ there exist $n \in \mathbb{N}$ such that $\mathfrak{h} \in \mathcal{H}(a_n, b_n)$. Thus we get :

$$\mathcal{W}'_c(a, b) = \bigcup_{n\in\mathbb{N}} \mathcal{W}_c(a_n, b_n).$$

We conclude that $\mathcal{W}'_c(a, b)$ is measurable and

$$\mu(\mathcal{W}'_c(a, b)) = \lim_{n\to+\infty} \mu(\mathcal{W}_c(a_n, b_n)) = \lim_{n\to+\infty} d(a_n, b_n) = d(a, b).$$

Which finishes the proof. ∎

In the following lemma, we remark that any closed convex subset of any interval in a complete median space is also an interval

Lemma 2.2.5. *Let M be a median algebra and let $x, y \in M$. Then for any gate convex subset $C \subseteq [x, y]$, we have $C = [\pi_C(x), \pi_C(y)]$.*

Proof. Let M be a median algebra and let $x, y \in M$. Let $C \subseteq [x, y]$ be a gate convex subset. Note that any wall which separates two points in $[x, y]$ must separate x and y. Let $c \in C$ and let $\mathfrak{h} \in \mathcal{H}(x, y)$ be a halfspace which contains x. By Lemma 1.1.14, we have $\pi_C(x) \in \mathfrak{h}^c$ and $\pi_C(y) \in \mathfrak{h}$. Hence there is no halfspace which separates c from both $\pi_C(x)$ and $\pi_C(y)$. Therefore by Theorem 1.1.7, the point c belongs to the interval $[\pi_C(x), \pi_C(y)]$ which finishes the proof. ∎

In a general complete median space, intervals need not to be compact. But if we consider the topology generated by closed convex subsets, intervals become compact as shown in the following lemma. See Theorem 14 in [Mon06] for the analogue in the case of CAT(0) metric spaces.

Lemma 2.2.6. *Let X be a complete median space and let $(C_i)_{i\in I} \subset [a, b]$ be a family of pairwise intersecting closed convex subsets. Then the intersection $\bigcap_{i\in I} C_i$ is not empty.*

Proof. By Helly's Theorem, we have for any finite subset $J \subset I$, the intersection $C_J := \bigcap_{i\in J} C_i$ is a non empty closed convex subset. By Lemma 2.2.5, we have $C_J = [\pi_{C_J}(a), \pi_{C_J}(b)]$. Let us set $a_J := \pi_{C_J}(a)$ and $b_J := \pi_{C_J}(b)$. Note that for any two finite subsets $J, K \subseteq I$, we have $[a_{J\cup K}, b_{J\cup K}] \subseteq [a_J, b_J] \cap [a_K, b_K] \subseteq [a, b]$. We deduce that the nets $(a_J)_{J\in\mathcal{P}_f(I)}$ and $(b_J)_{J\in\mathcal{P}_f(I)}$ are Cauchy nets, where $\mathcal{P}_f(I)$ denote the set of finite subsets of I endowed with the partial order relation given by the inclusion. The metric space X being complete, the nets $(a_J)_{J\in\mathcal{P}_f(I)}$ and $(b_J)_{J\in\mathcal{P}_f(I)}$ converges to $\tilde{a}$ and $\tilde{b}$ respectively. Therefore we conclude that $\tilde{a},\tilde{b}\in C := \bigcap_{i\in I} C_i$.

Let us moreover show that $C = [\tilde{a},\tilde{b}]$. The points $\tilde{a}$ and $\tilde{b}$ maximize the distances $d(a,a_J)$ and $d(b,b_J)$ respectively. In the other hand, for any $J\in\mathcal{P}_f(I)$ we have $\pi_C(a)\in[a,\pi_{C_J}(a)]$ and $\pi_C(b)\in[b,\pi_{C_J}(b)]$. Thus, we get $d(a,\pi_C(a))\geqslant d(a,\tilde{a})$ and $d(b,\pi_C(b))\geqslant d(b,\tilde{b})$. As $\pi_C(a)\in[a,\tilde{a}]$ and $\pi_C(b)\in[b,\tilde{b}]$, we conclude that $\tilde{a}=\pi_C(a)$ and $\tilde{b}=\pi_C(b)$. □

Proof of Proposition 2.2.2. Let us denote by Φ the canonical isometric embedding of X into its double dual $\mathcal{U}(\mathcal{H}(X))$. Note that X being complete and Φ an isometric embedding, then $\Phi(X)$ is closed. It last to show that $\Phi(X)$ is a convex subset of $\mathcal{U}_d(\mathcal{H}_\mu(X))$. Let us fix $a,b\in X$, $\mathfrak{u}\in[\Phi(a),\Phi(b)]$ and show that there exists $c\in[a,b]$ such that $\mu(\Phi(c)\Delta\mathfrak{u})=0$. By Lemma 1.2.24 and Proposition 1.2.25, we have $\mu(m(\Phi(a),\Phi(b),\mathfrak{u})\Delta\mathfrak{u})=0$. This means that for μ almost all wall $w=(\mathfrak{h},\mathfrak{h}^c)\in(\mathcal{W}_c(a,b))^c$, that is $a,b\in\mathfrak{h}$ or $a,b\in\mathfrak{h}^c$, we have $\mathfrak{h}\in\mathfrak{u}$. The idea of the proof is to approximate the ultrafilter $\mathfrak{u}$ by the set of halfspaces which contains some convex subset of $[a,b]$, and at each "time" we consider a smaller convex subset until we reach a singleton.

By Lemma 2.2.5, each closed convex subset $F\subseteq[a,b]$ is of the form $[\pi_F(a),\pi_F(b)]$, hence the set of walls which are not transverse to F constitutes a μ measurable set as it is the complement of $\mathcal{W}_c(\pi_F(a),\pi_F(b))$. Let $\mathcal{F}_\mathfrak{u}$ be the set of closed convex subsets $F\subseteq[a,b]$ such that for μ almost any wall $w=(\mathfrak{h},\mathfrak{h}^c)\in\mathcal{W}_c(a,b)$ with $F\subset\mathfrak{h}$, we have $\mathfrak{h}\in\mathfrak{u}$. The set $\mathcal{F}_\mathfrak{u}$ is not empty as it trivially contains the interval $[a,b]$. Let us endow $\mathcal{F}_\mathfrak{u}$ with the partial order relation $\leqslant$ where $F_1\leqslant F_2$ if and only if $F_2\subseteq F_1$. Lemma 2.2.6 implies that $(\mathcal{F}_\mathfrak{u},\leqslant)$ is an inductive set, hence by Zorn's lemma, it contains a maximal element F_m. Let us show that F_m is a singleton. By Lemma 2.2.5, there exist $x,y\in[a,b]$ such that $F_m=[x,y]$. Let us assume for sake of contradiction that x and y are distinct. Let us consider then a halfspace $\mathfrak{h}\in\mathcal{H}'(x,y)$ and assume, without loss of generality, that $\mathfrak{h}\in\mathfrak{u}$. Let us set $\tilde{x}:=\pi_{\overline{\mathfrak{h}}}$ and note that any halfspace which contains both $\tilde{x}$ and y either contains both x and y or it separates $\tilde{x}$ from x, that is $\mathcal{W}_c(\tilde{x},y)^c=\mathcal{W}_c(x,y)^c\sqcup\mathcal{W}_c(x,\tilde{x})$. In the other hand, any halfspace which separates $\tilde{x}$ from x must contains the halfspace $\mathfrak{h}$. Therefore, we get $\mathcal{H}(x,\tilde{x})\subset\mathfrak{u}$. We deduce then that $[\tilde{x},y]$ belongs to $\mathcal{F}_\mu$ which contradicts the maximality of F_m.

We have shown that F_m is a singleton, let us denote it by $\{c\}$. By construction, we have $\mu(\mathfrak{u}_c\Delta\mathfrak{u})=0$, which finishes the proof. □

We will be needing more lemmas before deducing Theorem 2.2.1 from Proposition 2.2.2.

Lemma 2.2.7. *Let X be a complete median space and let us consider $a,b\in X$ and $c\in[a,b]$. Then any halfspace $\mathfrak{h}\in\mathcal{H}(a,c)$ which contains c in its interior contains b in its interior.*

Proof. Let us consider $\mathfrak{h}\in\mathcal{H}(a,c)$ such that $c\in\mathfrak{h}^\circ$. By assumption c lies outside $\overline{\mathfrak{h}^c}$. Hence c does not belong to the interval $[\pi_{\overline{\mathfrak{h}^c}}(a),\pi_{\overline{\mathfrak{h}^c}}(b)]=[a,\pi_{\overline{\mathfrak{h}^c}}(b)]$ which is inside $\overline{\mathfrak{h}^c}$. We conclude that b is distinct from $\pi_{\overline{\mathfrak{h}^c}}(b)$. □

Lemma 2.2.8. *Let X be a complete median space with the strong separation property and let $(\mathcal{H}(X), \mu)$ be the canonical structure of measurable set on the set of halfspaces of X. Let us consider a point $x \in X$ and an ultrafilter $\mathfrak{u} \in \mathcal{U}(\mathcal{H}(X))$ such that $\mathfrak{u}_x \Delta \mathfrak{u}$ is measurable with $\mu(\mathfrak{u}_x \Delta \mathfrak{u}) < +\infty$. Then $\mu(\mathfrak{u}_x \Delta \mathfrak{u}) = 0$ if and only if there is no halfspace $\mathfrak{h} \in \mathfrak{u}_x \Delta \mathfrak{u}$ which contains x in its interior.*

Proof. Let us assume that there exist $\mathfrak{h} \in \mathfrak{u}_x \Delta \mathfrak{u}$ which contains x in its interior and set $\tilde{x} = \pi_{\overline{\mathfrak{h}^c}}(x)$. The point x being in the interior of $\mathfrak{h}$, it is distinct from $\tilde{x}$. In the other hand, any halfspace which separates x from $\tilde{x}$ contains $\mathfrak{h}^c$, hence it belongs to $\mathfrak{u}$. Therefore we get $\tilde{\mathcal{H}}(x, \tilde{x}) \subseteq \mathfrak{u}_x \Delta \mathfrak{u}$ with $\mu(\tilde{\mathcal{H}}(x, \tilde{x})) = d(x, \tilde{x}) > 0$.
For the other inclusion, let us assume that $\mathfrak{u}_x \Delta \mathfrak{u}$ contains no halfspace $\mathfrak{h}$ which contains x in its interior and show that $\mu(\mathfrak{u}_x \Delta \mathfrak{u}) = 0$. The measure μ being build from the Caratheodory extension theorem, we have :

$$\mu(\mathfrak{u}_x \Delta \mathfrak{u}) = \inf\{\sum_{n\in\mathbb{N}} \mu(\tilde{\mathcal{H}}(x_i, y_i)) / \ \mathfrak{u}_x \Delta \mathfrak{u} \subseteq \bigcup_{i\in\mathbb{N}} \tilde{\mathcal{H}}(x_i, y_i)\}.$$

Its measure being finite, there exist a sequence of pairs $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \subseteq X$ such that $\mathfrak{u}_x \Delta \mathfrak{u} \subseteq \bigcup_{i\in\mathbb{N}} \tilde{\mathcal{H}}(x_i, y_i)$. In one hand, we have $\mathcal{H}(x, a) \cap \mathcal{H}(b, a) = \mathcal{H}(m(a, x, b), a)$. In the other one, Lemma 2.2.7 implies that $\mathcal{H}'(m(x, x_i, y_i), y_i) \subseteq \mathcal{H}'(x, y_i)$. Thus we get

$$(\mathcal{H}(x, y_i) \backslash \mathcal{H}'(x, y_i)) \cap \mathcal{H}(x_i, y_i) \subseteq \mathcal{H}(m(x, x_i, y_i), y_i) \backslash \mathcal{H}'(m(x, x_i, y_i), y_i).$$

The point $m(x, x_i, y_i)$ being in the interval $[x_i, y_i]$, we have

$$\tilde{\mathcal{H}}(x_i, y_i) = \tilde{\mathcal{H}}(x_i, m(x, x_i, y_i)) \sqcup \tilde{\mathcal{H}}(m(x, x_i, y_i), y_i).$$

As we assumed that $\mathfrak{u}_x \Delta \mathfrak{u}$ contains no halfspace which contains x in its interior, we deduce the following inclusion

$$(\mathfrak{u}_x \Delta \mathfrak{u}) \cap \tilde{\mathcal{H}}(x_i, y_i) \subseteq (\tilde{\mathcal{H}}(x_i, m(x, x_i, y_i)) \backslash \tilde{\mathcal{H}}'(x_i, m(x, x_i, y_i))) \cup (\tilde{\mathcal{H}}(m(x, x_i, y_i), y_i) \backslash \tilde{\mathcal{H}}'(m(x, x_i, y_i), y_i)).$$

The right hand of the above inclusion being of null measure by Lemma 2.2.4, we conclude that $\mu(\mathfrak{u}_x \Delta \mathfrak{u}) = 0$. □

Proof of Theorem 2.2.1. Let us consider an ultrafilter $\mathfrak{u} \in \mathcal{U}(\mathcal{H}(X))$ such that $\mathfrak{u}_x \Delta \mathfrak{u} < +\infty$ for some, hence for all, $x \in X$. By Proposition 2.2.2, the image of X under the canonical embedding into $\mathcal{U}_d(\mathcal{H}_\mu(X))$ is a closed convex subset. Let $a \in X$ be such that $[\mathfrak{u}_a]$ is the gate projection $\pi_{\Phi(X)}([\mathfrak{u}])$ of $[\mathfrak{u}]$ into $\Phi(X)$. We claim that $\mu(\mathfrak{u} \Delta \mathfrak{u}_a) = 0$. The class $[\mathfrak{u}_a]$ being the gate projection of $\mathfrak{u}$ into $\Phi(X)$, we have $[\mathfrak{u}_a] \in [[u], [u_x]]$ for any $x \in X$. Which translate into saying that for any $x \in X$

$$\mu(m(\mathfrak{u}_a, \mathfrak{u}_x, \mathfrak{u}) \Delta \mathfrak{u}_a) = 0.$$

Let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace separating $\mathfrak{u}$ from $\mathfrak{u}_a$, that is $\mathfrak{h} \in (\mathfrak{u}\Delta\mathfrak{u}_a) \cap \mathfrak{u}$ and set $\tilde{a} := \pi_{\bar{\mathfrak{h}}}(a)$. Any halfspace in X which separates $\tilde{a}$ from a must be contained in $\mathfrak{u}$. Hence we have the following inclusion $\tilde{\mathcal{H}}(a, \tilde{a}) \subseteq (\mathfrak{u}_a, \mathfrak{u}_{\tilde{a}}, \mathfrak{u})\Delta\mathfrak{u}_a$. We deduce then

$$d(a, \tilde{a}) = \mu(\tilde{\mathcal{H}}(a, \tilde{a}) \leqslant \mu(m(\mathfrak{u}_a, \mathfrak{u}_{\tilde{a}}, \mathfrak{u})\Delta\mathfrak{u}_a) = 0.$$

Therefore, for any $\mathfrak{h} \in (\mathfrak{u}\Delta\mathfrak{u}_a) \cap \mathfrak{u}$ we have $a = \pi_{\bar{\mathfrak{h}}}(a)$, which implies that there is no halfspace $\mathfrak{h} \in \mathfrak{u}_a\Delta\mathfrak{u}$ which contains a in its interior. We conclude by Lemma 2.2.8 that $\mu(\mathfrak{u}_a\Delta\mathfrak{u}) = 0$. □

Chapitre 3

Action of S-arithmetic lattices on median spaces

3.1 Lattices in $\prod_{i \in I} \mathrm{PSL}(2, k_i)$

3.1.1 $(p+1)$-regular tree associated to $\mathrm{PSL}(2, \mathbb{Q}_p)$

p-adic field : Let $\mathbb{Q}$ be the field of rational numbers.

Definition 3.1.1. A ***valuation*** on $\mathbb{Q}$ is a group homomorphism $v : (\mathbb{Q}^*, *) \to (\mathbb{R}, +)$ such that :

$$v(x+y) \geqslant min(v(x), v(y)).$$

Note that each valuation is determined uniquely by the values that it takes on the integer. We say that two valuation v_1 and v_2 are equivalent if there exist a constant $\lambda > 0$ such that $v_1 = \lambda v_2$.

Examples 3.1.2.

1. There is the trivial valuation which take value 0 on $\mathbb{Q}^*$.
2. The p-adic valuation v_p, where p is a prime number, associates to each integer k the maximal n such that p^n divides k, which corresponds to the first power of p in the development of k in the basis p, $k = \sum_{i \geqslant n} a_i.p^i$ where $a_n \neq 0$.

Each valuation v gives rise to an absolute value $|\ |$ on $\mathbb{Q}$ by $|r|_v := e^{-v(r)}$ which verifies the ultra-metric inequality $|r_1 + r_2| \leqslant max(|r_1|, |r_2|)$. A consequence of Ostrowski's theorem (see Ch 2, §2.1 p85), is that the valuations arising in Examples 3.1.2 are the unique valuation on the field $\mathbb{Q}$, up to equivalence.

The ultrametric absolute value $|\ |_{v_p}$ associated to a p-adic valuation v_p is called the p-adic norm. The metric space $(\mathbb{Q}, d_p)$ arising from this norm is totally disconnected and the integers accumulates around 0. The space $(\mathbb{Q}, d_p)$ is not complete, one may consider for instance a Cauchy sequence of the form $k_i = \sum_{i_0}^{k} a_i.p^i$. The metric completion of $(\mathbb{Q}, d_p)$ denoted by $\mathbb{Q}_p$ is the field of ***the*** p***-adic numbers***. Note that the valuation v_p extends to $\mathbb{Q}_p$. The unit ball around zero, which is a clopen, is called ***the rings of*** p***-adic integers*** or ***the valuation ring*** of $\mathbb{Q}_p$ and is denoted by $\mathbb{Z}_p$. It corresponds to the set of points which have positive valuation and it can be described as the set of formal series with bases p

$$\mathbb{Z}_p := \{\sum_{i=k} a_i.p^i \mid k \leqslant 0 \text{ and } a_i \in \{0, .., p-1\}\}.$$

For a reference about the theory of p-adic field, see [Rob00].

The homogeneous simplicial tree associated to PSL$(2, \mathbb{Q}_p)$ **:** Throughout this paragraph, we will follow the construction explained in [Ser80] Ch II §1.1 p.69. Let V be a vector space of dimension 2 over $\mathbb{Q}_p$. A ***lattice*** $(L, +) < (V, +)$ is a free $\mathbb{Z}_p$-module of rank 2, i.e. $L = \mathbb{Z}_p.e_1 + \mathbb{Z}_p.e_2$, where $e_1, e_2 \in V$ are linearly independent. Let us denote by $\mathcal{L}$ the set of lattices in V. We say that two lattices L_1 and L_2 are equivalent if there exist $k \in \mathbb{Q}_p$ such that $L_1 = k.L_2$, we denote this equivalence by $\sim$.

The linear action of SL$(2, \mathbb{Q}_p)$ on V induces a natural action on $\mathcal{L}$. The stabilizer of a lattice L is conjugate, in GL$(2, \mathbb{Q}_p)$, to SL$(2, \mathbb{Z}_p)$. Note that SL$(2, \mathbb{Z}_p)$ acts transitively on the set of basis of the lattice that it stabilizes.

Let us consider two lattices $L_1, L_2 \in \mathcal{L}$ and a p-adic integer $k \in \mathbb{Z}_p$ such that $k.L_2 \subseteq L_1$. The existence of such p-adic integer is due to the fact that $\mathbb{Q}_p$ is the fraction field over $\mathbb{Z}_p$. By the elementary divisors theorem (see [Lan02] Theorem 7.8 Ch III §7 p.153), there exist a basis $\mathcal{B} = \{e_1, e_2\}$ of L_1, and $n_1, n_2 \in \mathbb{N}^*$ such that $k.L_2 =< p^{n_1}.e_1, p^{n_2}.e_2 >$, with $n_1 \leqslant n_2$. Note that the difference $n_2 - n_1$ does depend only on the equivalence classes of the lattices L_1 and L_2, we denote it by $d([L_1], [L_2])$. We set $\tilde{\mathcal{L}} := \mathcal{L}/\sim$.

Proposition 3.1.3 (Theorem 1, [Ser80] Ch II §1.1)**.** *The space $(\tilde{\mathcal{L}}, d)$ is a $(p+1)$-homogeneous simplicial tree.*

The action of SL$(2, \mathbb{Q}_p)$ descends to an isometric action on $\tilde{\mathcal{L}}$. Let us consider $\tilde{L}_1, \tilde{L}_2 \in \tilde{\mathcal{L}}$ and let $e_1, e_2 \in V$, $n \in \mathbb{N}$ such that $\{e_1, e_2\}$ and $\{e_1, p^n.e_2\}$ are basis for some $L_1 \in \tilde{L}_1$ and $L_2 \in \tilde{L}_2$ respectively. Then the application corresponding to the matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right) \in$ GL$(2, \mathbb{Q}_p)$ maps $\tilde{L}_1$ to $\tilde{L}_2$. If n is even, then the element represented by the matrix $\left(\begin{smallmatrix} p^{-\frac{n}{2}} & 0 \\ 0 & p^{\frac{n}{2}} \end{smallmatrix}\right) \in$ SL$(2, \mathbb{Q}_p)$ induces the same action as A on $\tilde{\mathcal{L}}$. If n is odd, the action of A can not be induced from an element in SL$(2, \mathbb{Q}_p)$. This is due to the fact that p does not have a square root in $\mathbb{Q}_p$ (see [Rob00] Ch.1, §.6.6, p.49). Hence, the action of SL$(2, \mathbb{Q}_p)$ is transitive on the set of points which are at even distance. There exist two conjugacy class of maximal compact subgroup in SL$(2, \mathbb{Q}_p)$. Each element in the conjugacy class corresponds to the stabilizer of an element in $\tilde{\mathcal{L}}$, and each such maximal compact subgroup is a conjugate in GL$(2, \mathbb{Q}_p)$ to SL$(2, \mathbb{Z}_p)$.

3.1.2 Lattices in $\prod_{i\in I} \mathrm{PSL}(2, k_i)$

In the next sections, we will be considering isometric action of lattices $\Gamma \leqslant G = \prod_{s\in S} \mathrm{PSL}(2, k_s)$ where S is finite and each k_s is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$ and the projection of Γ into each factors is dense.

We say that Γ is ***irreducible*** if for any $J \subset S$ the projection of Γ into $\prod_{s\in J} \mathrm{PSL}(2, k_s)$ is dense.

Example 3.1.4. — The group $\mathrm{PSL}(2, \mathbb{Z}[\frac{1}{p}])$ is a non cocompact lattice in $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{Q}_p)$.

— Let $p \in \mathbb{Z}$ a prime number such that -1 is a square in $\mathbb{Q}_p$. Note that is is equivalent to require that $p \equiv 1[4]$ as the polynomial $X^p - 1$ has always a solution which generates the cyclic group of roots of unity, which is of order $p-1$ (see [Rob00] Ch.1, §.6.7, p.51). The ring $\mathbb{Z}[\frac{i}{p}]$ embeds as a lattices in $\mathbb{C} \times \mathbb{Q}_p \times \mathbb{Q}_p$ through the $\mathbb{Z}$-linear morphism ϕ which sends i into $(i, \tilde{i}, -\tilde{i})$ where $\tilde{i}$ is a square root of -1 in $\mathbb{Q}_p$.
The group $\mathrm{PSL}(2, \mathbb{Z}[\frac{i}{p}])$ embeds as a non cocompact lattice in $\mathrm{PSL}(2, \mathbb{C}) \times \mathrm{PSL}(2, \mathbb{Q}_p) \times \mathrm{PSL}(2, \mathbb{Q}_p)$, where the entries of the matrices representing elements of $\mathrm{PSL}(2, \mathbb{Z}[\frac{i}{p}])$ are mapped through the map ϕ.

In the proof of Theorem 3.3.1, we will be considering separately the case where the lattice Γ is cocompact and the case where it is not. To deal with the latter case, we will be using the following fact

Theorem 3.1.5. *Let $\Gamma \leqslant \prod_{s\in S} \mathrm{PSL}(2, k_s)$ be an irreducible lattice which is not cocompact and $|S| \geqslant 2$. There exist then a solvable subgroup of Γ which is not virtually abelian.*

The subgroup is obtained by considering the intersection of a borel subgroup of G with Γ. For example, when $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\frac{1}{p}])$ the subgroup represented by the upper triangular matrices of $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ is a solvable subgroup which is not virtually abelian.

Proof of Theorem 3.1.5. By Margulis's arithmeticity theorem, any such irreducible non-cocompact lattice is commensurable with a conjugate of $\mathrm{PSL}(2, \mathcal{O}_S)$, where S is the ring of S-integers, that is

$$\mathcal{O}_S = \{x \in \mathbb{Q} \mid |x|_{v_p} \leqslant 1 \text{ for any } p\text{ -adic norm on } \mathbb{Q} \text{ which is not in the class of the } |\ |_{k_s}\text{'s}\}.$$

Hence, it is enough to show that $\mathrm{PSL}(2, \mathcal{O}_S)$ contains a solvable subgroup which is not virtually abelian. By The Dirichlet Theorem (See [Lan94] Unit Theorem, Ch.V, §.1, p.104), there exist a non trivial invertible element $t \in \mathcal{O}$. Hence, such solvable non virtually abelian group is given by the group generated by the element $A = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and the horospherical subgroup $H_S = \{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathcal{O}_S\}$. □

Another property of Γ which plays a key role in the proof of Theorem 3.3.1 is the quasi-simplicity.

Theorem 3.1.6 ([Mar91] Ch IX, §5 Theorem 5.4 p.325). *Let* $\Gamma \leqslant \prod_{s \in S} \mathrm{PSL}(2, k_s)$ *be an irreducible lattice. Then :*

- *For any normal subgroup* $N \trianglelefteq \Gamma$*, either* N *or* Γ/N *is finite.*
- *The quotient* $\Gamma/[\Gamma, \Gamma]$ *is finite.*

3.2 Fioravanti's machinery

3.2.1 Roller boundary

Let X be a complete locally convex median space. A natural way to define points at infinity in the median case is through ultrafilters. But not all ultrafilters of $\mathcal{U}(\mathcal{H}(X))$ points toward a direction to "infinity". Consider for example a principal ultrafilter $\mathfrak{u}_x \in \mathcal{U}(\mathcal{H}(X))$ over $x \in X$, that is, the ultrafilter consisting of all halfspaces of X which contain x. Let $\mathfrak{h} \in \mathfrak{u}_x$ be a minimal element (note that such element always exist by Zorn's lemma). Note that x must be contained in the closure of $\mathfrak{h}^c$. Then $\mathfrak{u}_x \backslash \{\mathfrak{h}\} \cup \{\mathfrak{h}^c\}$ is also an ultrafilter which is "close-fitting" to $\mathfrak{u}_x$.

When we endow $\mathcal{U}(\mathcal{H}(X))$ with the pseudo metric given by the canonical measure over $\mathcal{H}(X)$, the pseudo-distance between ultrafilters which differ with an countably many halfspaces is zero. For instance, when X is of finite rank, there are only finitely many minimal element in each ultrafilter. However, when the rank of X is infinite, one can have uncountably many such minimal elements and find two ultrafilters with infinite distance and which differ only on a set of halfspaces which contain a common point of X on their hyperplane.

In [Fio20] section 3, E. Fioravanti dealt with this issue in the case of complete locally convex median space by considering a finer sigma algebra $\mathcal{B}'$ over $\mathcal{H}(X)$ and a measure $\hat{v}$ over it which coincides with the canonical measure μ on the elements with finite measure of the canonical σ-algebra $\mathcal{B}$. Let us first point out that in [Fio20], the canonical structure of measured space over $\mathcal{H}(X)$ is the one generated through the Caratheodory extension theorem on the ring of sets generated by the sets of oriented halfspace intervals $\mathcal{H}(x, y)$, instead of the set of the non oriented halfspaces intervals $\widetilde{\mathcal{H}}(x, y)$. Hence, measurable subsets include non symmetric set of halfspaces, that is, set of halfspaces which are not stable under the complementary operation ^{c}.

Roughly speaking, the construction of $\mathcal{B}'$ consist of adding all subset of $\mathcal{H}(X)$ such that their intersection with any directed halfspace interval $\mathcal{H}(x, y)$ is measurable with respect to the canonical sigma algebra. It was shown then that for any $x \in X$, the set $adj(x) := \{\mathfrak{h} \in \mathcal{H}(X) \mid x \in \mathfrak{h}^c,\ x \in \bar{\mathfrak{h}}\}$ is measurable and of zero measure with respect to $\hat{v}$ (see [Fio20] Lemma 3.6). Remark that $adj(x)$ contains complements of all minimal halfspaces of the principal ultrafilter $\mathfrak{u}_x$.

All principal ultrafilter are measurable with respect to the sigma algebra $\mathcal{B}'$ (Lemma 3.6 [Fio20]). As in Subsection 1.2.2 the measure $\hat{v}$ defines a pseudo metric $d_{\hat{v}}(\mathfrak{u}_1, \mathfrak{u}_2) := \frac{1}{2}\hat{v}(\mathfrak{u}_1 \Delta \mathfrak{u}_2)$, with possible infinite value, on $\mathcal{U}(\mathcal{H}(X))$ and the relation $\sim$ defined by $\mathfrak{u}_1 \sim \mathfrak{u}_2$ if and only if $\mathfrak{u}_1 \Delta \mathfrak{u}_2 = 0$ is a congruence. The quotient of $\mathcal{U}(\mathcal{H}(X))$ by the congruence $\sim$ is a median algebra that is denoted by $\bar{X} = \mathcal{U}(\mathcal{H}(X))/\sim$. It decomposes into components, constituted of elements which are at finite distance $d_{\hat{v}}$. The restriction of $d_{\hat{v}}$ on each component gives rise to a median space and each component is a convex in $\bar{X}$ (Proposition 4.19 [Fio20]). There is a unique component which contains an isometrically embedded copy of X, through the canonical isometric embedding which associates to each point of X the principal ultrafilter corresponding to it. This component is denoted by $\mathcal{M}(X)$ in [Fio20] and as the median space is assumed to be locally convex, the component $\mathcal{M}(X)$ is isometric to X. Hence, there is no confusion if we still denote this component by X. The ***Roller boundary*** of X is the set $\partial X := \bar{X} \backslash X$.

Every isometric action $\Gamma \curvearrowright X$ gives rise to an action $\Gamma \curvearrowright \bar{X}$ which sends each component of $\bar{X}$ into another one and preserves the component corresponding to X.

Definition 3.2.1. Let $\Gamma \curvearrowright X$ be an isometric action on a complete locally convex median space. We say that the action is :

— ***Roller Elementary*** if it has a finite orbit in $\bar{X}$

— ***Roller minimal*** it does not fix a component of ∂X and there is no proper Γ-invariant closed convex subset in X.

If the median space X is of finite rank, then for any isometric action $\Gamma \curvearrowright X$, one can always find a closed convex Γ-invariant subset where the action of Γ is Roller minimal as stated in the following proposition :

Proposition 3.2.2 (Proposition 2.9 [Fio19])**.** *Let X be a complete median space of finite rank and let Γ be a group acting non elementarily on it. Then there exist a Γ-invariant closed convex subset $C \subseteq Z$, where Z is a Γ-invariant component in $\bar{X}$ such that the restriction of the action of Γ on C is Roller minimal.*

The following proposition states that for any convex subset $C \subseteq \bar{X}$, there exist a unique component in $Z \subseteq \bar{X}$ such that $C \subseteq \bar{Z}$

Proposition 3.2.3 (Corollary 4.31 [Fio20])**.** *Let X be a complete locally convex median space with compact intervals. Then any convex subset $C \subset \bar{X}$ intersects a unique component of $\bar{X}$ of maximal rank.*

3.2.2 Barycentric subdivision

Almost all the properties obtained from group action on median spaces of finite rank require the assumption on the action being Roller minimal. The assumption is necessary to ensure that the action have a good mixing property on the Roller boundary of the median space, and can move the space in the transverse direction to almost all halfspace. But the condition is not sufficient as one may have an action which is artificially Roller minimal, and this may occur especially when the space is not connected.

Consider for example the space $X = \mathbb{R} \times \{-1, 1\}$ endowed with the ℓ^1-product metric. The action of its group of isometries is Roller minimal, but one can imagine that it fixes an imaginary axes $\mathbb{R} \times \{0\}$. This prevent the action to move points transversally to this imaginary line. This behaviour can be detected by the inversion which maps the halfspaces $\mathbb{R} \times \{-1\}$ to its complement $\mathbb{R} \times \{1\}$.

The same pathology may arise in the assumption of the action being free, where the freeness of the action can be artificial. If the action of a group on a simplicial tree is free only on the set of vertices, one can deduce nothing about the group being free or not. Consider for instance the action of the infinite dihedral group on $\mathbb{Z}$.

To avoid this particular cases, one need to assume that there is no elements which maps a halfspace to its complement.

Definition 3.2.4. Let Γ be a group acting by isometries on a complete median space X which is f finite rank. We say that $\gamma \in \Gamma$ acts ***without wall inversion*** if for any $\mathfrak{h} \in \mathcal{H}(X)$ we have $\gamma.\mathfrak{h} \neq \mathfrak{h}^c$.

We say that the action of Γ is without wall inversion if all of its elements act without wall inversion.

To get rid of the above assumption, one consider the barycentric subdivision of the space which consist of adding all the imaginary convex parts which lie between each clopen halfspace and its complement. The construction was done in [Fio18] Section 2.3. In the case of a simplicial tree, it consists of considering the new simplicial tree where we add a vertex in the middle of each edge.

More generally, let X be a complete median space of finite rank. We assume that X is not connected which is, by Lemma 1.2.31 and Remark 1.1.8, equivalent to the existence of a halfspace which is clopen. Let $\mathfrak{h} \in \mathcal{H}(X)$ be such clopen halfspace. By Lemma 1.1.24, the two convex subsets $\pi_{\mathfrak{h}}(\mathfrak{h}^c)$ and $\pi_{\mathfrak{h}^c}(\mathfrak{h})$ are isometric and the isometry is given by the projections $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{h}^c}$, see Figure 3.2.2 below.

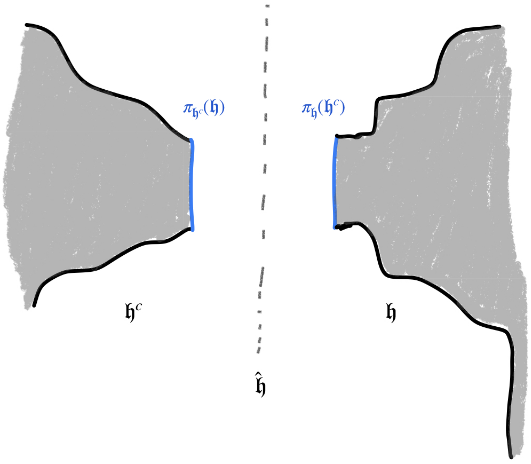

FIGURE 3.2.1 –

Roughly speaking, the barycentric subdivision of the space X consists of adding between each clopen halfspace $\mathfrak{h}$ and its complement a copy of the convex subset $\pi_{\mathfrak{h}}(\mathfrak{h}^c)$ which lies at distance $\frac{d(\mathfrak{h},\mathfrak{h}^c)}{2}$ from both $\mathfrak{h}$ and $\mathfrak{h}^c$ as shown in Figure 3.2.2 below.

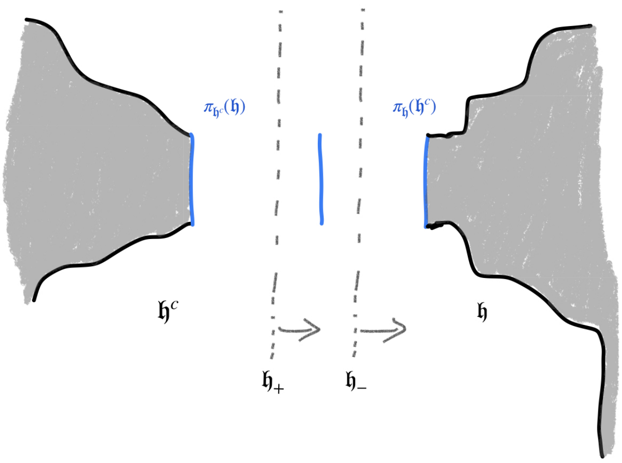

FIGURE 3.2.2 –

Note that there are other convex subsets which are added when we have many pairwise transverse clopen halfspaces.

Formally, We consider a finer poc set $\mathcal{H}'(X)$ induced from $\mathcal{H}(X)$, which contains all the halfspaces of $\mathcal{H}(X)$ which are not atoms and each atom $\mathfrak{h}$ of $\mathcal{H}(X)$ is split into two "halfpsaces" $\mathfrak{h}_-$ and $\mathfrak{h}_+$, each one of them has half the measure of $\mathfrak{h}$. The structure of poc set of $\mathcal{H}'(X)$ is constructed by keeping the same structure on the halfspaces which are not atoms and setting $\mathfrak{h}_- \leqslant \mathfrak{h}_+$, $\mathfrak{h}_-^* = (\mathfrak{h}^c)_+$, $\mathfrak{h}_+^* = (\mathfrak{h}^c)_-$ and $\mathfrak{h}' \leqslant \mathfrak{h}_-$ (respectively $\mathfrak{h}' \geqslant \mathfrak{h}_+$) if and only if $\mathfrak{h}' \leqslant \mathfrak{h}$ (respectively $\mathfrak{h}' \geqslant \mathfrak{h}$) for all $\mathfrak{h}' \in \mathcal{H}(X)$ and atoms $\mathfrak{h} \in \mathcal{H}(X)$. Note that by construction, any halfspace $\mathfrak{h}' \in \mathcal{H}(X)$ is transverse to $\mathfrak{h} \in \mathcal{H}(X)$ if and only if either one or equivalently both of $\mathfrak{h}_-$ and $\mathfrak{h}_+$ are transverse to $\mathfrak{h}'$, in the sense that $\mathfrak{h}_- \nleqslant \mathfrak{h}'$, $\mathfrak{h}_- \nleqslant \mathfrak{h}'^*$, $\mathfrak{h}_-^* \nleqslant \mathfrak{h}'$ and $\mathfrak{h}_-^* \nleqslant \mathfrak{h}'^*$.

Let us denote by $\mathcal{A}(X)$ the subset of atoms of $\mathcal{H}(X)$. We have a canonical projection $p : \mathcal{H}'(X) \to \mathcal{H}(X)$ which is the identity on the part corresponding to $\mathcal{H}(X)\backslash\mathcal{A}(X)$ and which associates to each $\mathfrak{h}_-$ and $\mathfrak{h}_+$ there corresponding clopen halfspace $\mathfrak{h}$.

Let $(\mathcal{H}(X), \mathcal{B}, \mu)$ be the measured poc set structure described in Theorem 1.2.20. Let us denote by $\mathcal{B}_{\mathcal{A}} := \{\{\mathfrak{h}, \mathfrak{h}^c\} \mid \mathfrak{h} \in \mathcal{A}\}$. The structure of measured poc set of $(\mathcal{H}(X), \mathcal{B}, \mu)$ lifts to a structure of measured poc set on $(\mathcal{H}'(X), \mathcal{B}', \mu')$ where $\mathcal{B}' := p^{-1}(\mathcal{B}) \cup (\bigcup_{\mathfrak{h}\in\mathcal{A}(X)} \{\mathfrak{h}, \mathfrak{h}^c\})$ and $\mu'(E) = \mu(p(E) \cap \mathcal{A}(X)^c) + \sum_{\substack{\mathfrak{h}\in\mathcal{A} \\ \mathfrak{h}_-\in E}} \frac{\mu(\{\mathfrak{h},\mathfrak{h}^c\})}{2} + \sum_{\substack{\mathfrak{h}\in\mathcal{A} \\ \mathfrak{h}_+\in E}} \frac{\mu(\{\mathfrak{h},\mathfrak{h}^c\})}{2}$.

Definition 3.2.5. Let X be a complete median space. We call the median space $X' := \mathcal{U}((\mathcal{H}'(X))$ associated to the pointed measured poc set $(\mathcal{H}'(X), \mathcal{B}', \mu', \mathfrak{u}_x)$, where $x \in X$ is arbitrary, the ***barycentric subdivision*** of X. Throughout the following we will abuse the notation and identify X with its image in X'.

The surjective measurable poc set morphism $p : \mathcal{H}'(X) \to \mathcal{H}(X)$ gives rise to an isometric embedding Φ_b of X into X'. Moreover, we have :

Proposition 3.2.6 (Proposition 2.15 (2) [Fio18])**.** *Let X be a complete median space of finite rank, we have then $Conv_{X'}(X) = X'$.*

Remark 3.2.7. Let X be a complete median space of finite rank. For any convex subset $C \subseteq \bar{X}$ we have a natural isometric identification between C' and $\Phi(C)$.

The following Lemma ensures us that there is no additional halfspaces that appear in the barycentric subdivision in addition to the one coming from the non atomic halfspaces of $\mathcal{H}(X)$ and the imaginary ones corresponding to the split of atoms.

Lemma 3.2.8 (Lemma 2.14 [Fio18])**.** *Let X be a complete median space of finite rank. We have then $\mathcal{H}(X') = \mathcal{H}'(X)$.*

Remark 3.2.9. We point out that we are not being totally faithful in our transcription of the definition and construction given in [Fio18], as we are using the measured poc set structure described in [CDH10] instead of the one introduced in [Fio20]. Nevertheless, this will not change the properties satisfied by the barycentric subdivision and it is the propositions concerning these properties shown by Fioravanti that we are going to appeal that matters.

Any isomorphism f of the measured poc set $(\mathcal{H}(X), \mathcal{B}, \mu)$ gives rise to an isomorphism f' where $f'(\mathfrak{h}_-) = (f(\mathfrak{h}))_-$, $f'(\mathfrak{h}_+) = (f(\mathfrak{h}))_+$ and $f'(\mathfrak{h}) = f(\mathfrak{h})$ for any $\mathfrak{h} \in \mathcal{H}(X) \backslash \mathcal{A}(X)$.

Hence any isometric action on X induces an isometric action on its barycentric subdivision X', and the isometric embedding of X into X' is equivariant whit respect to these actions.

We have the following :

Proposition 3.2.10 (Lemma 2.13 [Fio19])**.** *Let X be a complete median space of finite rank and let Γ be a group acting by isometries on X. Then the action of Γ on X is Roller non elementary if and only if its action on X' is Roller non elementary.*

Roller minimal actions on a median space does not necessarily induces a Roller minimal action on its barycentric subdivision, even if we assume that the action to be Roller non elementary. Consider for instance the ℓ^1-product of a homogeneous simplicial tree with $\{0, 1\}$ and the action of its group of isometries on it. However if we assume the space to be irreducible we get the following :

Proposition 3.2.11. *Let X be a complete irreducible median space of finite rank. Let Γ be a group acting Roller non elementarily and Roller minimally on X. Then the induced action of Γ on X' is also Roller minimal.*

The above Proposition is proven in [Fio19] Lemma 2.14 using Lemma 2.13. Since I am not sure to understand the argument in Lemma 2.13, we will proceed differently for the proof of Proposition 3.2.11. The main line of the argument is as follows. Given a Roller non elementary and Roller minimal action of Γ on a complete median space of finite rank X, we consider a closed Γ invariant convex subset $E \subseteq \bar{X}'$ for the sake of contradiction. By Proposition 3.2.3 and Remark 3.2.7 there is no loss of generality to assume that $E \subset X'$. Since the Γ-action on X is Roller minimal, the subset E does not intersect X, because the closure of $Conv(\Gamma.(E \cap X))$ is a Γ-invariant closed convex subset of X. For any convex subset $E \subset X' \backslash X$ there exist a halfspace $\mathfrak{h} \in \mathcal{A}(X)$ such that $E \subseteq \mathfrak{h}_+ \cap \mathfrak{h}_-^c$, this is shown in Lemma 3.2.12. Finally, one show that any other halfspace of X is transverse to $\mathfrak{h}$, which gives a splitting of X.

Lemma 3.2.12. *Let X be a complete median space of finite rank. Then for any closed convex subset $E \subseteq X' \backslash X$, there exist a clopen halfspace $\mathfrak{h} \in (X)$ such that $E \subseteq \mathfrak{h}_+ \cap \mathfrak{h}_-^c$.*

Proof. Let n be the rank of the space X and let $x \in E$. By Lemma 2.13 [Fio18], there exist an embedding of median algebras $i_x : \{-1, 1\}^k \to X$ which extends to an embedding $\tilde{i}_x : \{-1, 0, 1\} \to X'$, where $1 \leqslant k \leqslant n$ and such that $C(x) := i_x(\{-1, 1\}^k)$ and $C'(x) := \tilde{i}_x(\{-1, 1\}^k)$ are gate convex subset of X and X' respectively.

As the gate convex subset E lies entirely inside $X' \backslash X$, its intersection with $C'(x)$ belongs to the image under $\tilde{i}_x$ of a hyperplane of the form $\{(x_1, .., x_k) \in \{-1, 0, 1\}^k \mid x_i = 0$ for some fixed $i \in \{1, .., k\}\}$. By Lemma 1.1.14 the lift under $\pi^{-1}_{C'(x)}$ of the halfspaces of $C'(x)$ given by the image under $\tilde{i}(x)$ of the halfspaces $\{(x_1, ..., x_k) \in \{-1, 0, 1\}^k \mid x_i \leqslant 1\}$ and $\{(x_1, ..., x_k) \in \{-1, 0, 1\}^k \mid x_i \geqslant -1\}$ are two halfspaces such that their intersection contains E and lies in $X'\backslash X$. By Lemma 3.2.8, the latter halfspaces necessarily arise as $\mathfrak{h}_-$ and $\mathfrak{h}_+$ for some $\mathfrak{h} \in \mathcal{A}(X)$, which completes the proof. □

Lemma 3.2.13. *Let X be a complete median space of finite rank and let $\mathfrak{h} \in A(X)$ be a clopen halfspace of X. Then any halfspace $\mathfrak{h}' \in \mathcal{H}'(X)$ which separates two points of $\mathfrak{h}_+ \cap \mathfrak{h}_-^c$ is a halfspace which is transverse to both $\mathfrak{h}_+$ and $\mathfrak{h}_-$.*

Proof. Despite the natural identification between $\mathcal{H}'(X)$ and $\mathcal{H}(X')$, given by Lemma 3.2.8, to avoid confusion between their elements, we will denote the element of the former poc set by $\mathfrak{h}$ and their corresponding one in the latter poc set by $\mathfrak{h}'$. Let $x, y \in \mathfrak{h}'_+ \cap (\mathfrak{h}'_-)^c$ and let $\mathfrak{h} \in \mathcal{H}'(x, y)$ be a halfspace which separates y from x with $x \in \mathring{\mathfrak{h}}$ and $y \in (\mathring{\mathfrak{h}^c})$. By Lemma 1.2.10 there is no loss of generality to assume that $\mathfrak{h}$ is closed. By construction x and y are classes of ultrafilters which contains $\mathfrak{h}_+$ and $\mathfrak{h}_-$. Note that despite the measure of a non clopen halfspace is zero, each ultrafilter in the class x (respectively y) contains $\mathfrak{h}^*$ (respectively $\mathfrak{h}$). This is due to the fact that any ultrafilter which contains $\mathfrak{h}$ (respectively $\mathfrak{h}^c$) will contains the chains $\mathcal{H}(x, \pi_{\mathfrak{h}}(x))$ (respectively $\mathcal{H}(y, \pi_{\overline{\mathfrak{h}^c}}(y))$) which is of positive measure.

Let $\tilde{x}$ be an ultrafilter in the class of x and let $\tilde{y}$ be an ultrafilter in the class of y. We claim that $\tilde{x}_+ := (\tilde{x}\backslash\{\mathfrak{h}_+\}) \cup \{\mathfrak{h}_+\}$ and $\tilde{y}_+ := (\tilde{y}\backslash\{\mathfrak{h}_+\}) \cup \{\mathfrak{h}_+\}$ are ultrafilters, it is enough to show it for $\tilde{x}_+$. Note that by construction, we have $\tilde{x}_+ \cup \tilde{x}_+^* = \mathcal{H}'(X)$. It last to show that there is no $\mathfrak{t} \in \tilde{x}_+$ such that $\mathfrak{h}_+ \leqslant \mathfrak{t}^*$. By construction of $(\mathcal{H}'(X), \leqslant)$, any element $\mathfrak{t} \in \tilde{x}_+$ which verifies the latter inequality, must verify $\mathfrak{h}_- \leqslant \mathfrak{t}^*$. As both $\mathfrak{t}$ and $\mathfrak{h}_-$ lies in $\tilde{x}$ which is an ultrafilter, such inequality can not hold. Therefore, both $\tilde{x}_+$ and $\tilde{y}_+$ are ultrafilters which contain $\mathfrak{h}_+^*$ and $d([\tilde{x}_+], x) = d([\tilde{y}_+], y) = \mu(\mathfrak{h}_+)$. In the other hand we have $d([\tilde{x}_+], [\tilde{y}_+]) = d(x, y)$. Thus we conclude that the intersections $\mathfrak{h}'_+ \cap \mathfrak{h}'$, $\mathfrak{h}'^*_+ \cap \mathfrak{h}'$, $\mathfrak{h}'_+ \cap \mathfrak{h}'^*$ and $\mathfrak{h}'^*_+ \cap \mathfrak{h}'^*$ are not empty, which finishes the proof. □

Proof of Proposition 3.2.11. Let $E \subseteq \bar{X}'$ be a proper Γ-invariant closed convex subset. By Proposition 3.2.3 E intersects a unique component $Z' \subset \bar{X}$ of $\bar{X}$ of maximal rank. By Remark 3.2.7, the component Z' is the barycentric subdivision of a component $Z \subseteq \bar{X}$ of X. If Z' is distinct from X' then we consider the restriction of the action of Γ on Z', the latter is Γ-invariant as it is the unique component of maximal rank which intersects E. If $E = Z'$, then $Z' \cap \bar{X}$ is a proper Γ invariant closed subset of $\bar{X}$, which contradicts the assumption on the action $\Gamma \curvearrowright X$ being Roller minimal. Hence up to considering the restriction of the action Γ on Z' and the intersection of E with Z', there is no loss of generality to assume that E is a proper subset of X'. By Lemma 3.2.12 there exist $\mathfrak{h} \in A(X)$ such that $E \subseteq \mathfrak{h}_+ \cap \mathfrak{h}_-^c$. We claim that all other halfspace of $\mathfrak{t} \in \mathcal{H}'(X)\backslash\{\mathfrak{h}_-, \mathfrak{h}_-^c, \mathfrak{h}_+, \mathfrak{h}_+^c\}$ are transverse to both $\mathfrak{h}_-$ and $\mathfrak{h}_+$. By lemma 3.2.13, any halfspace of X' which is transverse to E is transverse to both $\mathfrak{h}_+$ and $\mathfrak{h}_-$. In the other hand, as there is no halfspace $\mathfrak{h}' \in \mathcal{H}'(X)$ such that $\mathfrak{h}_- \leqslant \mathfrak{h}' \leqslant \mathfrak{h}_+$, any halfspace which contains E either contains $\mathfrak{h}_-^c$ or $\mathfrak{h}_+$. Let us consider then a halfspace $\mathfrak{h}' \in \mathcal{H}'(X)\backslash\{\mathfrak{h}_-, \mathfrak{h}_-^c, \mathfrak{h}_+, \mathfrak{h}_+^c\}$. If $\mathfrak{h}'$ is transverse to E, there is nothing to show. If it contains it, up to considering $\mathfrak{h}_-^c$ instead of $\mathfrak{h}_+$ or the complements of $\mathfrak{h}'$, we can assume that $\mathfrak{h}_+ \subset \mathfrak{h}'$. Then for any $x \in X \cap \mathfrak{h}' \cap \mathfrak{h}_+^c$ and any $g \in \Gamma$ we have

$d(E,x) = d(E,g(x))$. Hence the orbit $\Gamma.x$ lies inside $\mathfrak{h}'$. Therefore the closure of the convex hull is a proper Γ invariant subset of X, which contradicts the minimality of X. □

3.2.3 Mixing on the set of halfspaces

The argument for the proof of Theorem A relies heavily on the existence of a strongly separated facing triple which are determined by one point. In this subsection, we recall the machinery needed to prove their existence.

Throughout this section X will be a complete finite rank median space and Γ a group acting without wall inversion on it. We say that a halfspace $\mathfrak{h} \in \mathcal{H}(X)$ is ***flipped*** by $g \in Isom(X)$ if $d(g.\mathfrak{h}, \mathfrak{h}) > 0$ and $g.\mathfrak{h} \neq \mathfrak{h}^c$ (g does not inverse the wall $(\mathfrak{h}, \mathfrak{h}^c)$).

We say that a halfspace $\mathfrak{h} \in \mathcal{H}(X)$ is ***thick*** if both $\mathfrak{h}$ and $\mathfrak{h}^c$ are of non empty interior. If the action $\Gamma \curvearrowright X$ is Roller minimal, then for any thick halfspace $\mathfrak{h} \in \mathcal{H}(X)$, the intersection $\bigcap_{g \in \Gamma} \overline{g.\mathfrak{h}}$ must be empty in $\bar{X}$. By the compactness of $\bar{X}$ ([Fio20] Theorem 4.14), one deduce that there exist $g \in \Gamma$ such that $g.\mathfrak{h} \cap \mathfrak{h} = \varnothing$. Note that the assumption on $\mathfrak{h}$ being thick is to ensure that $\bar{\mathfrak{h}} \neq X$. We summarize the above discussion into the following proposition :

Proposition 3.2.14 ([Fio18] Theorem 5.1)**.** *Let X be a complete median space of finite rank and let $\Gamma \curvearrowright X$ be a Roller minimal action without wall inversion. Then any thick halfspace is flipped by some element in Γ.*

Through flipping operation, one generate hyperbolic type isometries g in the sense that there exist a halfspace $\mathfrak{h}_g \in \mathcal{H}(X)$ such that $g.\mathfrak{h}_g \subset \mathfrak{h}_g$. Under the assumption that any thick halfspace is flipped by some isometry, there exist for any thick halfspace a hyperbolic element which translates transversally to it. The latter isometry is obtained by composing two elements $\gamma_1, \gamma_2 \in \Gamma$ such that $d(\gamma_1.\mathfrak{h}, \mathfrak{h}) > 0$ and $d(\gamma_2.\mathfrak{h}^c, \mathfrak{h}^c) > 0$. Using the same construction, we get the double skewering lemma (see [CS11] Double skewering Lemma and [Fio18] Corollary 5.4) :

Proposition 3.2.15. *Let X be a complete finite rank median space and let $\Gamma \curvearrowright X$ be a Roller minimal action without wall inversion. Then for any thick halfspaces $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}(X)$ with $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$, there exist $g \in \Gamma$ such that $g.\mathfrak{h}_2 \subset \mathfrak{h}_1 \subseteq \mathfrak{h}_2$.*

If the median space X is irreducible, Proposition 3.2.15 can be strengthened into requiring $\mathfrak{h}_1^c$ and $g.\mathfrak{h}_2$ to be strongly separated. The irreducibility of X ensure the existence of two strongly separated halfspace through the following criterion :

Theorem 3.2.16 ([Fio18] Theorem 5.9)**.** *Let X be a median space which admits a Roller minimal action. Then X is irreducible if and only if for any thick halfspace $\mathfrak{h}_1 \in \mathcal{H}(X)$ there exist two thick halfspaces $\mathfrak{h}, \mathfrak{t} \in \mathcal{H}(X)$ such that $\mathfrak{h} \subseteq \mathfrak{h}_1 \subseteq \mathfrak{t}$ where $\mathfrak{h}$ and $\mathfrak{t}^c$ are strongly separated.*

We deduce the following lemma :

Lemma 3.2.17. *Let X be a complete irreducible finite rank median space and let $\Gamma \curvearrowright X$ be a Roller minimal action without wall inversion. Then for any halfspaces $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}(X)$ such that $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$ there exist two halfspace $\mathfrak{h}, \mathfrak{t} \in \mathcal{H}(X)$ such that $\mathfrak{t} \subseteq \mathfrak{h}_1 \subseteq \mathfrak{h}_2 \subseteq \mathfrak{h}$ where $\mathfrak{h}^c$ and $\mathfrak{t}$ are strongly separated.*

Proof. By Theorem 3.2.16, there exist $\mathfrak{h}, \mathfrak{h}' \in \mathcal{H}(X)$ such that $\mathfrak{h}' \subseteq \mathfrak{h}_2 \subseteq \mathfrak{h}$ where $\mathfrak{h}^c$ and $\mathfrak{h}'$ are strongly separated. If $\mathfrak{h}' \subseteq \mathfrak{h}_1$ then we are done. If $\mathfrak{h}_1 \subseteq \mathfrak{h}'$, we set $\mathfrak{t} = \mathfrak{h}_1$. Hence, let us assume that $\mathfrak{h}_1$ and $\mathfrak{h}'$ are transverse. Let $\gamma_1 \in \Gamma$ be such that $d(\gamma_1.\mathfrak{h}'^c, \mathfrak{h}'^c) > 0$ and remark that $\mathfrak{h}'^c \subseteq \gamma_1.\mathfrak{h}' \subseteq \gamma_1.\mathfrak{h}$. Hence, the halfspace $\mathfrak{h}'^c$ and $\gamma_1.\mathfrak{h}^c$ are strongly separated. As $\mathfrak{h}_1$ is transverse to $\mathfrak{h}'$, it cannot be transverse to $\gamma_1.\mathfrak{h}^c$, which yields $\gamma_1.\mathfrak{h} \subseteq \mathfrak{h}_1$. We complete then the proof by setting $\mathfrak{t} := \gamma_1.\mathfrak{h}$. ☐

Composing through elements which flip around strongly separated halfspaces, one generate contracting hyperbolic isometries.

Lemma 3.2.18. *Let X be a complete irreducible finite rank median space and let $\Gamma \curvearrowright X$ be a Roller minimal action without wall inversion. Then for any $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}(X)$ such that $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$, there exits $g \in \Gamma$ such that $\mathfrak{h}_1 \subseteq \mathfrak{h}_2 \subset g.\mathfrak{h}_1$ where $g.\mathfrak{h}_1^c$ and $\mathfrak{h}_2$ are strongly separated.*

Proof. By Lemma 3.2.17, there exist $\mathfrak{h}, \mathfrak{t} \in \mathcal{H}(X)$ such that $\mathfrak{t} \subseteq \mathfrak{h}_1 \subseteq \mathfrak{h}_2 \subseteq \mathfrak{h}$ where $\mathfrak{h}^c$ and $\mathfrak{t}$ are strongly separated. The isometry $g \in \Gamma$ is given by setting $g = \gamma_1 \circ \gamma_2$ where $\gamma_1, \gamma_2 \in \Gamma$ such that $d(\gamma_1.\mathfrak{t}^c, \mathfrak{t}^c) > 0$ and $d(\gamma_2.\mathfrak{h}, \mathfrak{h}) > 0$. ☐

It is left to prove the existence of a facing triple of halfspaces, that is, a triple of halfspaces which are pairwise disjoint. If we assume in addition that the action of Γ is Roller non elementary, then such facing triple exists.

Proposition 3.2.19 ([Fio18] Proposition 6.2)**.** *Let X be a complete irreducible median space of finite rank and let $\Gamma \curvearrowright X$ be a Roller non elementary and Roller minimal action without wall inversion. Then there exist a triple of thick halfspaces which are pairwise strongly separated.*

The idea of the proof is to start with a thick halfspace $\mathfrak{h}$ and consider a hyperbolic isometry g such that $g.\mathfrak{h} \subset \mathfrak{h}$ where $\mathfrak{h}^c$ and $g.\mathfrak{h}$ are strongly separated. Then one look at the interval $[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in \partial X$ along which the element g translates. the two point ξ_1 and ξ_2 are chosen in $\bigcap_{i \in \mathbb{N}} g^n.\mathfrak{h}$ and $\bigcap_{i \in \mathbb{N}} g^{-n}.\mathfrak{h}^c$ respectively (see Figure 3.2.3. The action being Roller non elementary, the median space X is not contained in $[\xi_1, \xi_2]$. Let $x \in X \backslash [\xi_1, \xi_2]$ and consider $\mathfrak{t} \in \mathcal{H}([\xi_1, \xi_2], \{x\}) = \mathcal{H}(m(\xi_1, \xi_2, x), x)$, that is a halfspace which separates x from $[\xi_1, \xi_2]$. The projection of x into $[\xi_1, \xi_2]$ lies between two halfspace $g^n.\mathfrak{h}$ and $g^{n+1}.\mathfrak{h}^c$. For each n, the couple $g^n.\mathfrak{h}^c$ and $g^{n+1}.\mathfrak{h}$ being strongly separated, the halfspace $\mathfrak{t}$ is contained in $g^{n-1}.\mathfrak{h}$ and $g^{n+2}.\mathfrak{h}^c$. Therefore, the sequence $(g^3.\mathfrak{t})_{n \in \mathbb{N}}$ constitutes a family of pairwise strongly separated halfspaces.

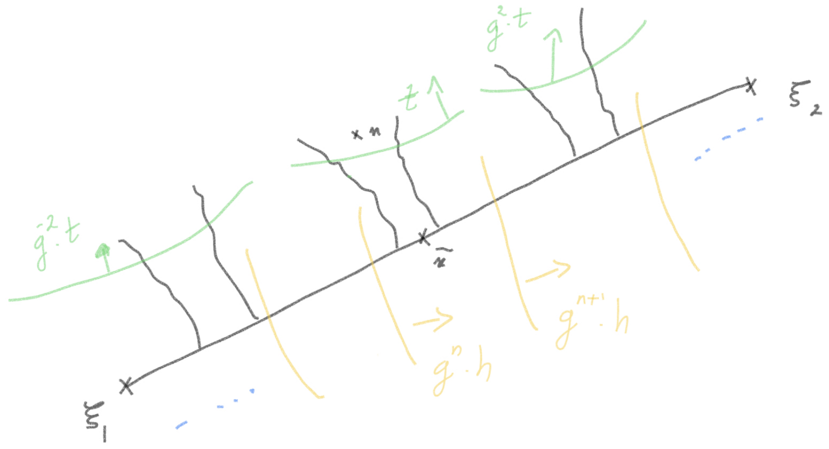

FIGURE 3.2.3 – The configuration arising in the proof of the Proposition 3.2.19

Combining all the results stated above, we deduce the following

Proposition 3.2.20. *[See also [Fio19] Lemma 4.1] Let X be an irreducible median space of finite rank. Let us assume that the action of $Isom(X)$ is Roller non elementary and Roller minimal without walls inversion. Then there exists a strongly separated facing triple of thick halfspaces $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{H}(X)$, and a point $c \in X$ such that for any $x_i \in \mathfrak{h}_i$, we have $m(x_1, x_2, x_3) = c$. Note that the point c lies in $\mathfrak{h}_1^c \cap \mathfrak{h}_2^c \cap \mathfrak{h}_3^c$.*

Proof. Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{H}(X)$ be a triple of halfspaces which are pairwise strongly separated. By Proposition 3.2.14, there exist $g_1, g_2, g_3 \in Isom(X)$ such that $d(g_i.\mathfrak{h}_i^c, \mathfrak{h}_i^c) > 0$ for all $i \in \{1,2,3\}$. Then the set of halfspaces $\mathfrak{h}_1' := g_1.\mathfrak{h}_2$, $\mathfrak{h}_2' := g_2.\mathfrak{h}_1$ and $\mathfrak{h}_3' := g_3.\mathfrak{h}_1$ constitute a strongly separated facing triple. Remark that for each $i \in \{1,2,3\}$ the halfspaces $\mathfrak{h}_i^c$ and $\mathfrak{h}_i'$ are strongly separated. By Proposition 1.1.26, the halfspace $\mathfrak{h}_i'$ projects into a point $a_i := \pi_{\bar{\mathfrak{h}}_i^c}(\mathfrak{h}_i)$ in $\bar{\mathfrak{h}}_i$. Hence, for any $a, b \in \mathfrak{h}_i^c$ and any $x \in \mathfrak{h}_i'$ we have $m(a,b,x) = m(a,b,a_i)$. We deduce that for any triple $(x_1, x_2, x_3) \in \mathfrak{h}_1' \times \mathfrak{h}_2' \times \mathfrak{h}_3'$ we have $m(x_1, x_2, x_3) = m(a_1, a_2, a_3)$, which completes the proof. □

3.2.4 Stabilizer of points

Pointwise convergence topology on $Isom(X)$: In a $\mathbb{R}$-tree, the stabilizer of any point is open in the isometry group when the latter is endowed with the topology of pointwise convergence. The same hold for any the stabilizer of any point $x \in T$ which has valency greater than 2 in a $\mathbb{R}$-tree T. This is due to the fact for any sequence $(g_n)_{n \in \mathbb{N}} \subseteq Isom(X)$ such that $g_n(x) \neq x$ with $g_n(x)$ converges to x, and for any neighbourhood of x, there exist points whose orbits accumulate nowhere.

The same holds for the median space of finite rank which admits a Roller non elementary and Roller minimal action as for space, there exist points which exhibits the same characteristic as the points with valency greater than 3 in a $\mathbb{R}$-tree. This was remarked in [Fio19] (Remark 4.5 therein).

Proposition 3.2.21. *Let X be a complete irreducible median space of finite rank. Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{H}(X)$ be a facing triple of thick halfspaces and $a \in X$ such that for any $x_1 \in \mathfrak{h}_1, x_2 \in \mathfrak{h}_2$ and $x_3 \in \mathfrak{h}_3$ we have $m(x_1, x_2, x_3) = a$. Then the stabilizer of a is open $Isom(X)$, where the latter is endowed with the topology of pointwise convergence.*

If the median space X admits a Roller minimal and Roller non elementary action then such configuration exists (see Proposition 3.2.20).

Proof. Let $(g_n)_{n\in\mathbb{N}} \subseteq Isom(X)$ be a sequence of isometries of X. If the sequence $g_n.(a)$ is infinite, we apply the same argument as the one given in the proof of Theorem A to find a infinite family of pairwise disjoint halfspaces $g_{i_n}.h_k$. Then for any point $x \in \mathfrak{h}_k$, the sequence $g_{i_n}(x)$ accumulates nowhere. This implies that if a sequence $(g_n)_{n\in\mathbb{N}} \subseteq Isom(X)$ converges to an isometry in $Stab(a)$, then it necessarily stabilizes the point a from some $n \geqslant N$. ☐

Stabilizer of points at infinity Let T be a $\mathbb{R}$-tree. Each element $g \in Isom(X)$ which does not inverse a wall either stabilizes a subset of T or leaves invariant a geodesic line inside T, the minimum of $d(gx, x)$ is verified in this axe. The former isometry is called elliptic and the latter is called hyperbolic.

Let $\eta \in \partial T$ be a point in the Roller boundary of T. Note that the Roller boundary of T coincides with the visual boundary. The stabilizer $Stab(\eta)$ of η in the isometry group consist either of hyperbolic elements which their axe of translation have η as one of their extremities, or of elements which fix a geodesic ray pointing to η.

Let us denote the length translation of an isometry of g by $(g) := inf_{x\in X}\{d(x, gx)\}$. The application $\chi : Stab(\eta) \to \mathbb{R}$ such that

$$\chi(g) = \begin{cases} l(g) \text{ if } \eta \text{ is an attractive point of } g \\ -l(g) \text{ if } \eta \text{ is a repelling point of } g \end{cases}$$

is a homomorphism. Hence, the group $Stab(\eta)$ fits into the following exact sequence :

$$0 \longrightarrow N_\eta \longrightarrow Stab(\eta) \longrightarrow \mathbb{R},$$

where the kernel N_η corresponds to the subgroup of isometries where each isometry fixes some geodesic ray pointing toward η.

In the higher rank case, an analogue holds for a subgroup of finite index of $Stab(\eta)$ and this was shown in [Fio18] Section 4. The argument is much more complicated as the points at infinity can lie in the corner of many transverse "directions" and one need a formal description of the latter. The argument is an extension of a result of P.E Caprace in the case of CAT(0) cube complexes (see the Appendix of [CFI16]) to the case of median spaces of finite rank. The canonical "directions" towards a point at infinity are described by the notion of unidirectional boundary sets (UBS) that was first introduced in the case of CAT(0) cube complexes by M.F. Hagen in [Hag13] and adapted to the case of median space of finite rank by E. Fioravanti in [Fio18].

Definition 3.2.22. Let X be a complete median space of finite rank and let $\eta \in \partial$ be a point in the Roller boundary of X. An ***inseparable subset*** $\mathcal{U} \subseteq \mathcal{H}(X)$ is a subset such that for any $\mathfrak{h}_1, \mathfrak{h}_2$ and $\mathfrak{h}_1 \subseteq \mathfrak{h} \subseteq \mathfrak{h}_2$ then $\mathfrak{h} \in \mathcal{H}(X)$.

A ***unitary boundary set*** of η is an inseparable subset $\Omega \subset \mathcal{H}(X)$ such that it contains a diverging sequence of halfspaces to η, that is a decreasing sequence $(\mathfrak{h}_i)_{i\in\mathbb{N}} \subseteq \mathfrak{u}_\eta \backslash \mathfrak{u}_x$ for some $x \in X$, such that $\lim\limits_{i\to+\infty} d(x, \mathfrak{h}_i) = +\infty$.

A partial order relation $\leqslant$ is defined on the set of UBS of η by $\Omega_1 \leqslant \Omega_2$ if and only if there exist, or equivalently, for any $x \in X$ we have $\sup\limits_{\mathfrak{h}\in\Omega_1\backslash\Omega_2} \{d(x, \mathfrak{h})\} < +\infty$.

We say that two UBS Ω_1 and Ω_2 of η are equivalent if $\Omega_1 \leqslant \Omega_2$ and $\Omega_2 \leqslant \Omega_1$.

We denote by $\mathcal{U}(\eta)$ the set of equivalence classes of UBS of η. The partial order relation defined on the set of UBS of η descends naturally to a partial order relation on $\mathcal{U}(\eta)$. We say that an $[\Omega_1] \in \mathcal{U}(\eta)$ is minimal if there is no $[\Omega_2] \in \mathcal{U}(\eta)$ such that $[\Omega_2] \leqslant [\Omega_1]$.

The minimal equivalence classes of UBS of η will constitutes the canonical directions pointing toward η. Any UBS of η is equivalent to the inseparable closure of the union of representatives of finitely many minimal classes of UBS of η.

Proposition 3.2.23 (Proposition 4.7 [Fio18])**.** *Let X be a complete median space of rank n and let $\eta \in \partial X$.*

1. *The cardinal of the set of minimal classes of UBS in $\mathcal{U}(\eta)$ is bounded by the rank of X.*
2. *For any $[\Omega] \in \mathcal{U}(\eta)$ there exist minimal classes of UBS $[\Omega_1], ..., [\Omega_k] \in \mathcal{U}(\eta)$, where $k \in \{1, ..., n\}$ and such that for any $x \in X$ we have $\sup\limits_{\mathfrak{h}\in\Omega\Delta(\Omega_1\cup...\cup\Omega_k)} \{d(x, \mathfrak{h})\} < +\infty$.*

   Moreover, this decomposition of $[\Omega]$ is unique.

The action of $Stab(\eta)$ on X extends naturally to an action on $\mathcal{U}(\eta)$. Let $[\Omega_1], ..., [\Omega_k] \in \mathcal{U}(\eta)$ be all the minimal equivalence classes of UBS of η and let $K_\eta \leqslant Stab(\eta)$ be the subgroup of isometries which fix all the classes $[\Omega_i]$'s. We have then :

Theorem 3.2.24 (Lemma 4.8, Proposition 4.9 in [Fio18] and Proposition 2.19 in [Fio19])**.** *The map*

$$\begin{aligned} \chi_\eta : K_\eta &\to \mathbb{R}^k \\ g &\to (\hat{v}(g^{-1}\Omega_i \backslash \Omega_i) - \hat{v}(\Omega_i \backslash g^{-1}\Omega_i))_i \end{aligned}$$

where $\hat{v}$ is the measure constructed in [Fio20], is a homomorphism. Moreover

1. *When $Stab(\eta)$ is endowed with the topology of pointwise convergence, the homomorphism χ_η is continuous.*

2. *The orbit of any point $x \in X$ by any finitely generated subgroup of $ker(\chi_\eta)$ has at most 2^n elements, where n is the rank of the space X.*

The map χ_η is called the ***full transfer homomorphism***. Note that when X is a median space of rank 1, the equivalence class of UBS of η is a singleton and the full transfer homomorphism coincide with the map described in 3.2.4.

3.2.5 Tits alternative

We say that a group G verifies the Tits alternative if for any finitely generated subgroup $\Gamma \leqslant G$, either Γ is virtually solvable or it contains a free non abelian subgroup. This property was first shown by J. Tits in [Tit72] for linear subgroup over a field of characteristic zero answering a conjecture by J.P. Serre.

This property holds for a large classes of groups : hyperbolic groups, mapping class group, outer automorphism group of a free group... It is still open whether groups acting geometrically on a CAT(0) spaces verify the Tits alternative. For groups acting on CAT(0) cube complexes, the following version of Tits alternative was shown by M. Sageev and D. Wise :

Theorem 3.2.25 (Theorem 1.1 [SW05])**.** *Let G be a group acting properly on a finite dimensional CAT(0) cube complex. We assume that there is a uniform bound on the order of finite subgroups of G. Then either G is virtually finitely generated abelian group or it contains a non abelian free subgroup.*

The argument goes by induction on the rank of the CAT(0) cube complex X by considering a multi-ended subgroup H of G. This subgroup corresponds to the stabilizer of a hyperplane of X and its existence is ensured by Theorem 5.1 [Sag95]. If H contains a non abelian free subgroup so does G and if it is virtually finitely generated abelian group, then the algebraic torus theorem ([DS00]) gives three possibilities for the structure of the group G, where each case is treated separately.

Two other version of the Tits alternative in the case of CAT(0) cube complexes were shown by P.E. Caprace and M. Sageev in [CS11] (Theorem F and G therein) using different argument. They showed that under the assumption on the action being non elementary (with respect to the visual boundary), it has a nice mixing property on the set of halfspaces of the CAT(0) cube complex. One may then find two hyperbolic isometries and use a ping pong argument in order to show that they generate a non abelian free subgroup.

Both versions of the Tits alternative shown in [CS11] were extended to the case of median space of finite rank by E. Fioravanti in [Fio18].

Theorem 3.2.26 (Theorem E [Fio18])**.** *Let Γ be a group acting by isometries on a complete median space of finite rank. Then either the action is Roller elementary or Γ contains a non abelian free group.*

The idea of the proof is to go first by induction on the rank of X to reduce the study to the case when X is irreducible. Up to restricting the action to an invariant subset in an invariant component $\bar{X}$, one can assume the action to be Roller minimal (Proposition 3.2.2). By Proposition 3.2.11, there is no loss of generality to assume that the action of Γ is without wall inversion. Under these assumptions, for any $n \in \mathbb{N}^*$ there exist a family of pairwise disjoint halfspaces $\mathfrak{h}_1, \mathfrak{k}_1.., \mathfrak{h}_n, \mathfrak{k}_n \in \mathcal{H}(X)$ of positive depth in X (Lemma 6.3 [Fio18]) and a family of elements $g_1, ..., g_n \in \Gamma$ such that $g_n.\mathfrak{h}_n^* \subset \mathfrak{k}_n$ (Corollary 5.4 [Fio18]). Remark that we also have $g_n^{-1}.\mathfrak{k}_n^* \subset \mathfrak{h}_n$. One conclude then with the ping-pong lemma that the subgroup generated by the elements $g_1, ..., g_n$ is free.

Using Theorem 3.2.26 and the structure of the stabilizer of points lying in the Roller boundary (Theorem 3.2.24), one deduce the following other formulation of the Tits alternative in the median case :

Theorem 3.2.27 (see Theorem A [Fio18])**.** *Let Γ be a group, with no non-abelian free subgroup, which acts on a complete finite rank median space X. If the action $\Gamma \curvearrowright X$ is proper, then Γ is virtually (locally finite)-by-abelian.*

3.2.6 Superrigidity

Let $G_1, ..., G_n$ be a finite family of locally compact compactly generated group. We say that a lattice $\Gamma \leqslant G_1 \times .. \times G_n$ is irreducible if its projection into each factor is dense. We will be needing the following version in the irreducible case of Fioravanti's superrigidity results :

Theorem 3.2.28 (Theorem 4.4 [Fio19])**.** *Let $\Gamma \leqslant G = G_1 \times .. \times G_n$ be an irreducible uniform lattice which acts Roller non elementarily on a complete irreducible median space X of finite rank. Then there exist a Γ-invariant component $Z \subseteq \bar{X}$ with a Γ-invariant median subalgebra $Y \subseteq Z$ such that the action of Γ on Y extends continuously to G. Moreover, the action of G on Y factors through the canonical projection onto a factor G_i for some $i \in \{1, ..., , n\}$.*

Remark 3.2.29. If we assume in addition that the action arising in Theorem 3.2.28 is Roller minimal, the Γ-invariant median subalgebra Y_i lies in X.

The above result does not restrict to the case of lattice which are uniform. The original statement is stated for lattices filling the square integrability condition, which is verified when they are uniform. This condition allows to extend unitary representation of Γ into another unitary representation of the whole group G.

Let us explicit the main lines of the proof of the above theorem. Let X be a complete median space of finite rank and let Γ be a group acting isometrically on X. After fixing a point $x \in X$, the action $\Gamma \curvearrowright X$ induces an affine isometric action ρ_x of Γ on the Hilbert space $L^2(\mathcal{H}(X), \hat{v})$ defined as follows :

$$(\rho_x(g).F)(h) := F(g.\mathfrak{h}) + (\mathbb{1}_{\mathfrak{u}_{g(x)}} - \mathbb{1}_{\mathfrak{u}_x}), \quad \forall F \in L^2(\mathcal{H}(X), \hat{v}).$$

If we change the base point x by $y \in X$, the action with respect to the base point y is obtained from the former one by the following cocycle $2(\mathbb{1}_{g.\mathcal{H}(x,y)} - \mathbb{1}_{\mathcal{H}(x,y)})$. We say that an isometric action of a topological group G on a metric space $\mathcal{H}$ have almost fixed points if for for any $\epsilon > 0$, and any compact subset $K \subseteq G$, there exists a point $v \in \mathcal{H}$ such that $d(g(v), v) < \epsilon$.

Remark that for any $x, y \in X$, the action ρ_x has almost fixed points if and only if ρ_y has it.

Theorem 3.2.30 (Theorem A [Fio19]). *Let X be a complete finite rank median space with an isometric action $\Gamma \curvearrowright X$. Then the action $\Gamma \curvearrowright X$ is Roller elementary if and only if $\Gamma \curvearrowright L^2(\mathcal{H}(X), \hat{v})$, through ρ_x, has almost fixed point for a given or, equivalently, for any $x \in X$.*

Let $\Gamma \curvearrowright X$ be a Roller non elementary action on a complete median space of finite rank. Up to restrict the action on a Γ-invariant component of $\bar{X}$ and up to consider its barycentric subdivision, we may assume that the action is Roller minimal and without wall inversion. By Theorem 3.2.30, the isometric affine action ρ_x is without almost invariant fixed point. Shalom's superrigidity results on affine isometric action of such lattices on Hilbert space ensures the existence of a non trivial Γ-invariant subspace $\mathcal{H}_i \subseteq L^2(\mathcal{H}(X), \hat{v})$ such that the restriction of the unitary part of ρ_x on it extends to a continuous unitary representation of G_i. One consider then a vector $f \in \mathcal{H}_i$ and remark that for any $\gamma \in \Gamma$ and any sequence $(\gamma_k)_{k \in \mathbb{N}} \in \Gamma$, such that $\pi_i(\gamma_k)$ tends to identity in G_i, the vector $\gamma.f$ is an almost invariant vector for the latter sequence. The second step is to show from the latter data the following claim :

Claim 3.2.31. *The following median subalgebra is not empty*

$$Y := \{x \in X \mid \forall (\gamma_k)_{k \in \mathbb{N}} \in \Gamma \subset \Gamma,\ \pi_i(\gamma_k) \to id \textit{ then } \ \gamma_i(x) = x\ \forall i \geqslant N \textit{ for some } N \in \mathbb{N}\}.$$

To prove the claim, one consider a thick halfspace $\mathfrak{h}_0$ such that the restriction of the function f on the set of halfspaces which their walls are "contained" in $\mathfrak{h}_0$ is at distance ϵ from f. One remark then that for any $\gamma \in \Gamma$ such that $\gamma.\mathfrak{h}_0 \cap \mathfrak{h}_0 = \varnothing$ then $||f - \gamma.f||^2 \geqslant 2||f||^2 - 6\epsilon^2$. The action $\Gamma \curvearrowright X$ being Roller non elementary, Roller minimal, without wall inversion and the space X being irreducible, there exist $g_1, g_2 \in \Gamma$ such that $\mathfrak{h}_0$, $\mathfrak{h}_1 := g_1.\mathfrak{h}$ and $\mathfrak{h}_2 := g_2.\mathfrak{h}$ are pairwise strongly separated which are uniquely determined by a point $a \in \mathfrak{h}_0^c \cap \mathfrak{h}_1^c \cap \mathfrak{h}_2^c$ in the sense that for any $(x_0, x_1, x_2) \in \mathfrak{h}_0 \times \mathfrak{h}_1 \times \mathfrak{h}_2$ we have $m(x_1, x_2, x_3) = a$. Then for any sequence $(\gamma_k)_{k \in \mathbb{N}} \in \Gamma$, such that $\pi_i(\gamma_k)$ tends to identity in G_i and $N \in \mathbb{N}$ such that $||\gamma_n.f - f||$, $||\gamma_n(g_1.f) - (g_1.f)||$ and $||\gamma_n(g_2.f) - (g_2.f)||$ are small enough for any $n \geqslant N$, we have $\gamma_n.\mathfrak{h}_i \cap \mathfrak{h}_i \neq \varnothing$ for all $i \in \{0, 1, 2\}$. Therefore, we get $\gamma_n(a) = a$ whenever $n \geqslant N$, which implies that $a \in Y$.

For the last step, it remains to extend the restriction of the action of Γ on the closure of Y to G_i. The closure of Y is a Γ-invariant complete median subalgebra of X. By construction of the subalgebra Y, any sequence $(\gamma_k)_{k \in \mathbb{N}} \subseteq \Gamma$ such that its projection into G_i tends to identity, then its image in $Isom(\bar{Y})$ also tends to the identity, when the latter is endowed with the pointwise convergence topology. This ensures us (see Proposition 4.3 [Sha00]) that the extension of the action of Γ through its projection into G_i, where it projects as a dense subgroup, is a continuous homomorphism from G_i to $Isom(Y)$.

Remark 3.2.32. Let $a \in Y$ be the point arising from the proof of the Claim 3.2.31. As the action $\Gamma \curvearrowright X$ is Roller non elementary and Roller minimal, the orbit of a meets every thick halfspace. Hence the halfspaces $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_3$ that are determined by a are transverse to Y, that is, they constitute a strongly separated facing triple in Y. Therefore, by Proposition 3.2.21, the stabilizer of a is also open in $Isom(Y)$.

3.3 Action of S-arithmetic lattices on finite rank median spaces

Let $\Gamma \leqslant G = \prod_{s \in S} \mathrm{PSL}(2, k_s)$ be a lattice such that at least one k_i is archimedean and the projection of Γ into each factor $\mathrm{PSL}(2, k_i)$ is dense. By the work of Chatterji and Drutu, both $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ acts properly discontinuously on locally compact median space of infinite rank (Theorem 1.2.28). Hence, the lattice Γ acts geometrically on a locally compact median space obtained by the product of the median spaces associated to $\mathbb{H}^2$, $\mathbb{H}^3$ and the $(p+1)$-homogeneous simplicial trees associated to $\mathrm{PSL}(2, \mathbb{Q}_p)$. However, in the finite rank case we have the following :

Theorem 3.3.1. *Let $\Gamma \leqslant G = \prod_{s \in S} \mathrm{PSL}(2, k_s)$ be a lattice such that at least one k_i is archimedean and the projection of Γ into each factor $\mathrm{PSL}(2, k_i)$ is dense. Then there is no proper action of Γ on a complete median space of finite rank.*

Proof. We first note that there is no loss of generality to assume that Γ is irreducible, up to considering a projection into $\prod_{s \in J} \mathrm{PSL}(2, k_s)$ for a subset $J \subset S$ of cardinal greater than 2. Margulis superrigidity theorem ensures that there is at least one $s \in J$ such that k_s is archimedean.

By Selberg's Lemma, there exist a finite index subgroup which is torsion free. Hence, there is no loss of generality if we assume that Γ is torsion free. If the lattice is non-uniform, there exist by virtue of Theorem 3.1.5 a solvable subgroup of Γ which is non virtually abelian. Hence the action cannot be proper by Theorem 3.2.27.

Let us assume now that Γ is uniform. The group Γ being quasi simple and by Theorem 3.2.24, any Roller elementary action of Γ on an irreducible median space fixes a point in X, up to restricting to a finite index subgroup. Thus, we assume that the action of Γ is Roller non elementary. Let us first deal with the case where the action of Γ is Roller minimal.

We set $S_1 = \{i \in S \mid k_i \text{ is archimedean}\}$, $S_2 := S \backslash S_1$ and set $\tilde{G}_1 = \prod_{i \in S_1} \mathrm{PSL}(2, k_i)$, $\tilde{G}_2 = \prod_{i \in S_2} \mathrm{PSL}(2, k_i)$. We note that the projections of Γ into $\tilde{G}_1$ and $\tilde{G}_2$ are both dense. The median space X decomposes into a finite product $X_1 \times .. \times X_k$ where each X_i is irreducible. Up to considering a finite index subgroup of Γ, we may assume that Γ preserves the factorization of X, that is, the representation maps Γ into the product $Isom(X_1) \times .. \times Isom(X_k)$. Let us consider a decomposition of $X = \tilde{X} \times \tilde{X}_1 \times .. \times \tilde{X}_{k'}$, where the $\tilde{X}_i$'s are irreducible and such that the action of Γ is Roller elementary on $\tilde{X}$ and Roller non elementary on each $\tilde{X}_i$. Again up to considering a finite index subgroup, there exist a Γ fixed point $\tilde{x} \in \tilde{X}$ and Γ-invariant closed median subalgebras $Y_i \subseteq \tilde{X}_i$ such that the restriction of the action of Γ on each Y_i extends continuously to G and factors through $\mathrm{PSL}(2, k_{n_i})$ for some $n_i \in S$. As the action of Γ is Roller non-elementary, it cannot factor through $\mathrm{PSL}(2, k_i)$ where k_i is archimedean since the latter is generated by any small neighbourhood of the identity and there are points $x \in Y_i$ such that the stabilizer of x in $Isom(Y_i)$ is an open which contains the identity(see Remark 3.2.32). Hence, we obtain a continuous action of $\tilde{G}_2$ on $Y := \{\tilde{x}\} \times Y_1 \times .. \times Y_{k'}$ which extend the action of Γ. The lattice Γ being irreducible, its projection into $\tilde{G}_2$ is dense, hence it accumulates around the identity. Therefore, the action of Γ on $Y \subseteq X$ is not proper.

It last to consider the case where the action of Γ on X is not Roller minimal. By Proposition 3.2.2, there exist an Γ-invariant closed convex $\tilde{X}$ inside a Γ-invariant component of the Roller compactification of X, such that the restriction of the action of Γ on $\tilde{X}$ is Roller minimal. By the previous reasoning, there exist a Γ-invariant median subspace $Y \subset \tilde{X}$ such that the action of Γ is extended to $\tilde{G}_2$, where Γ lies as a dense subgroup. Let $\xi \in Y$ such that $Stab(x)$ is open in $Isom(Y)$, such point exist see 3.2.32. There exist a compact open subgroup K of $\tilde{G}_2$, which corresponds to the stabilizer in $\tilde{G}_2$ of a finite subset of the product $\prod_{i \in S_2} T_{k_i}$, which is mapped into $Stab_Y(\xi)$. Let us denote $\tilde{\Gamma}$ the intersection of Γ with K. The subgroup $\tilde{\Gamma}$ being dense in K, its commutator subgroup is infinite (the group Γ being torsion free, it is enough to find two elements which does not commute). In the other hand, the image of the subgroup $\tilde{\Gamma}$ by the representation lies in the intersection $Stab_Y(\xi) \cap Stab_X(\xi)$. Hence, we deduce that the intersection of its image with N_ξ, the kernel of the full transfer character map χ_ξ arising in Theorem 3.2.24, is infinite. Therefore, there exist a point in X which is fixed by infinitely many elements of Γ, which completes the proof. □

Chapitre 4

Isometric actions on finite rank median spaces

4.1 An embedding lemma of the convex hull

The following embedding property of the convex hull of two closed convex subsets with no transverse halfspace in common will be a key ingredient in the proof of Theorem D :

Proposition 4.1.1. *Let X be a complete median space of finite rank and let $C_1, C_2 \in X$ be closed convex subsets such that there is no half-space which is transverse to them both. Then the following map*

$$\begin{array}{rcl} f : Conv(C_1, C_2) & \to & (C_1 \times C_2 \times [c_1, c_2], d^1_\ell) \\ x & \mapsto & f(x) = (\pi_{C_1}(x), \pi_{C_2}(x), \pi_{[c_1,c_2]}(x)) \end{array}$$

is an isometric embedding, where π_{C_i} denote the projection onto the closed convex subset C_i, $c_1 = \pi_{C_1}(C_2)$ and $c_2 = \pi_{C_2}(C_1)$

For the proof of Proposition 4.1.1, we need the following lemma :

Lemma 4.1.2. *Let X be a complete median space of finite rank and let $C_1, C_2 \subseteq X$ such that there is no halfspace which is transverse to both. After setting $c_1 = \pi_{C_1}(C_2)$, $c_2 = \pi_{C_2}(C_1)$ and considering any $x, y \in Conv(C_1, C_2)$ we have :*

$$\mathcal{W}(x,y) = \mathcal{W}(\pi_{C_1}(x), \pi_{C_1}(y)) \sqcup \mathcal{W}(\pi_{C_2}(x), \pi_{C_2}(y)) \sqcup \mathcal{W}(\pi_{[c_1,c_2]}(x), \pi_{[c_1,c_2]}(y)) \qquad (4.1.1)$$

Proof. For any closed convex subset $C \subseteq X$ and any $x, y \in X$, we have :

$$\mathcal{W}(\pi_C(x), \pi_C(y)) \subseteq \mathcal{W}(x,y).$$

This come from the fact that for any $c \in C$, the interval $[c, x]$ contains $\pi_C(x)$. Thus, we have the inclusion of the right hand of the equality (1) into the left side. For the other inclusion, let us consider $x, y \in Conv(C_1, C_2)$ and $\mathfrak{h} \in \mathcal{H}(x,y)$. Let us assume that $\mathfrak{h}$ does not separate $\pi_{C_1}(y)$ from $\pi_{C_1}(x)$ and $\pi_{C_2}(y)$ from $\pi_{C_2}(x)$. By Proposition 1.1.19, if the projections $\pi_{C_1}(x)$, $\pi_{C_1}(y)$, $\pi_{C_2}(x)$ and $\pi_{C_2}(y)$ lie in a halfspace $\mathfrak{h}$, then so do x and y. As the halfspace $\mathfrak{h}$ separates y from x there is no loss of generality if we assume that $\pi_{C_1}(x), \pi_{C_1}(y)$ belong to $\mathfrak{h}^c$ and $\pi_{C_2}(x), \pi_{C_2}(y)$ to $\mathfrak{h}$. As $\pi_{C_1}(y) \in [c_1, y]$ and $\pi_{C_2}(x) \in [c_2, x]$, we necessarily get that $c_1 \in \mathfrak{h}^c$ and $c_2 \in \mathfrak{h}$. We conclude that $\pi_{[c_1,c_2]}(x) \in [c_1, x] \subseteq \mathfrak{h}^c$ and $\pi_{[c_1,c_2]}(y) \in [c_2, y] \subseteq \mathfrak{h}$ (see the Figure 4.1.1 below). Therefore, the halfspace $\mathfrak{h}$ lies in $\mathcal{H}(\pi_{[c_1,c_2]}(x), \pi_{[c_1,c_2]}(y))$.

It is left to show that the sets arising in the right hand of the equality are indeed disjoint. Under our assumtpion that the convex subsets C_1 and C_2 being strongly separated, we already have the disjointness of $\mathcal{W}(\pi_{C_1}(x), \pi_{C_1}(y))$ with $\mathcal{W}(\pi_{C_2}(x), \pi_{C_2}(y))$. A wall which separates two points of the interval $[c_1, c_2]$ must separate c_1 and c_2. The point c_1 being contained in any interval connecting C_1 to C_2, we deduce that any wall in $\mathcal{W}(\pi_{[c_1,c_2]}(x), \pi_{[c_1,c_2]}(y))$ must separate C_1 and C_2. Hence, such wall cannot be in $\mathcal{W}(\pi_{C_1}(x), \pi_{C_1}(y))$ nor in $\mathcal{W}(\pi_{C_2}(x), \pi_{C_2}(y))$. □

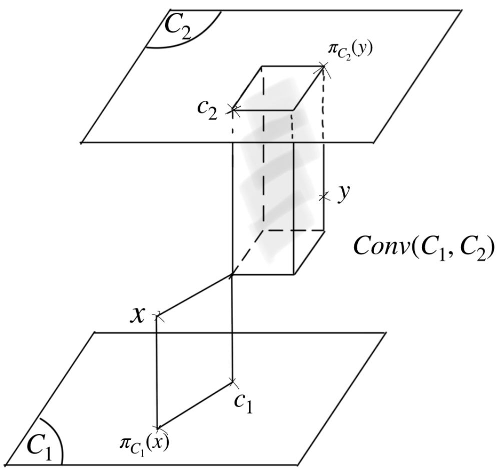

FIGURE 4.1.1 – Any halfspace which separates x and y is either transverse to C_1, to C_2 or to the interval $[c_1, c_2]$.

Proof of Proposition 4.1.1. As there is no halfspace which is transverse to both C_1 and C_2, the projection of C_1 (resp C_2) into C_2 (resp C_1) is a singleton, according to Proposition 1.1.26. Let us set denote them by $c_1 = \pi_{C_1}(C_2)$ and $c_2 = \pi_{C_2}(C_1)$.

By Lemma 4.1.2, we get the following :

$$\mathcal{W}(x, y) = \mathcal{W}(\pi_{C_1}(x), \pi_{C_1}(y)) \sqcup \mathcal{W}(\pi_{C_2}(x), \pi_{C_2}(y)) \sqcup \mathcal{W}(\pi_{[c_1,c_2]}(x), \pi_{[c_1,c_2]}(y)).$$

We deduce then

$$\begin{aligned}\mu(\mathcal{W}(x,y)) &= \mu(\mathcal{W}(\pi_{C_1}(x),\pi_{C_1}(y))) + \mu(\mathcal{W}(\pi_{C_2}(x),\pi_{C_2}(y))) + \mu(\mathcal{W}(\pi_{[c_1,c_2]}(x),\pi_{[c_1,c_2]}(y)))\\ d(x,y) &= d(\pi_{C_1}(x),\pi_{C_1}(y)) + d(\pi_{C_2}(x),\pi_{C_2}(y)) + d(\pi_{[c_1,c_2]}(x),\pi_{[c_1,c_2]}(y))\\ &= d_{\ell^1}(f(x),f(y))\end{aligned}$$

where μ is the measure given by Theorem 1.2.20. □

By considering a stronger assumption in Proposition 4.1.1, we get a local version of the first part of Theorem D :

Proposition 4.1.3. *Let X be a complete median space of finite rank and let $C_1, C_2 \subseteq X$ be two closed convex subsets such that $C_1 \cap C_2 = \{x_0\}$. Then, for any $x \in Conv(C_1, C_2)$, the interval $[x, x_0]$ is isometric to the ℓ^1-product of $[\pi_{C_1}(x), x_0]$ and $[\pi_{C_2}(x), x_0]$.*

Proof. For any $x \in Conv(C_1, C_2)$, we have :

$$\mathcal{H}([x,x_0]) = \mathcal{W}(x,x_0) \overset{1}{=} \mathcal{W}(\pi_{C_1}(x),x_0) \sqcup \mathcal{W}(\pi_{C_2}(x),x_0)$$

where any halfspace in $\mathcal{H}(\pi_{C_1}(x), x_0)$ is transverse to any halfspace in $\mathcal{H}(\pi_{C_2}(x), x_0)$. Hence, by Proposition 1.2.34 the median space $\mathcal{U}(\mathcal{H}([x, x_0]))$ is isometric to the ℓ^1-product of $\mathcal{U}(\mathcal{H}([\pi_{C_1}(x), x_0]))$ with $\mathcal{U}(\mathcal{H}([\pi_{C_2}(x), x_0]))$. The intervals being closed subsets and the median space being complete, we conclude by Theorem 2.2.1 that the interval $[x, x_0]$ is isomorphic to $\mathcal{U}(\mathcal{H}([x, x_0]))$ and that the latter is isomorphic to the ℓ^1-product of the two intervals $[\pi_{C_1}(x), x_0], [\pi_{C_2}(x), x_0]$, by Proposition 1.2.34. □

Median space of rank greater or equal 2 are are not CAT(0) spaces, they are not even uniquely geodesic spaces. Nevertheless, convexity in median space being defined by mean of intervals, convex subsets in median spaces are rigid enough to share many properties that are featured in CAT(0) spaces.

Using Proposition 4.1.4, Lemma 1.1.24 and the fact that gate projection are 1-lipschitz, we deduce the following version of the Sandwich lemma ([BH99] Exercise II.2.12) :

Proposition 4.1.4 (Proposition 2.21 [Fio19])**.** *Let X be a median space and let $C_1, C_2 \subseteq X$ be two closed convex subset. Then $Conv(\pi_{C_1}(C_2), \pi_{C_2}(C_1))$ is isometric to $\pi_{C_1}(C_2) \times [x, \pi_{C_2}(x)]$ where x is any point in $\pi_{C_1}(C_2)$.*

Remark 4.1.5. Proposition 4.1.1 can be extended to the case where C_1 and C_2 admits a common transverse halfspace by taking the projection of the convex hull between C_1 and C_2 into the product $C_1 \times C_2 \times \mathcal{B}(C_1, C_2)$ endowed with the ℓ^1-product metric. The map is not necessarily an isometry, it is a 2-lipschitz embedding. For any two points in $Conv(C_1 \cup C_2)$ separated by halfspaces which are transverse to both C_1 and C_2, the horizontal distance with respect to C_1 and C_2 is counted twice in $C_1 \times C_2 \times \mathcal{B}(C_1, C_2)$.

1. This equality comes from Lemma 4.1.2

4.2 Characterization of compactness by mean of half-spaces

In the first subsection, we recall some results about the convex hull of compact subsets in a median space. The next subsection is devoted to the proof of Theorem B.

4.2.1 Convex hull of compact subsets

It was shown in [Fio20] that any interval in a median space of rank n embeds isometrically into $\mathbb{R}^n$, see Proposition 2.19 therein. A direct consequence is that the convex hull of finite subsets in a finite rank median space are compact. More generally, the convex hull of a compact subset in a complete finite rank median space is also compact, see Lemma 13.2.11 in [Bow22]. In a complete median space, the convex hull of a compact subset is not necessarily compact. In general, even the interval are not necessarily compact, consider for instance intervals in $L^1(\mathbb{R})$. However, under the assumption that the intervals of X are compact, the convex hull between any two convex compact subsets is also compact. Before giving a proof, we will be needing some results.

Lemma 4.2.1. *Let $C \subseteq X$ be a convex subset and a point $x \in [a,b] \subseteq X$, we have :*

$$d(x,C) \leqslant d(a,C) + d(b,C).$$

Proof. Any halfspace which separates C from x must separate it either from a or from b (or from both). Thus we get

$$d(x,C) = \mu(\mathcal{W}(x,C)) \leqslant \mu(\mathcal{W}(a,C)) + \mu(\mathcal{W}(b,C)) = d(a,C) + d(b,C).$$

□

We deduce the following lemma :

Lemma 4.2.2. *Let X be a complete median space. Then the join between any two closed convex subsets is closed.*

Proof. Let us consider two convex subsets $C_1, C_2 \subseteq X$ and let $(x_n)_{n\in\mathbb{N}} \subseteq [C_1, C_2]$ be a sequence of points converging to $x \in X$. Note that each x_n lies in the interval $[\pi_{C_1}(x_n), \pi_{C_2}(x_n)]$. As gate projections are 1-lipschitz, the sequences $(\pi_{C_1}(x_n))_{n\in\mathbb{N}}$ and $(\pi_{C_2}(x_n))_{n\in\mathbb{N}}$ are Cauchy sequences. Thus they converge to $a \in C_1$ and $b \in C_2$ respectively. In the other hand, we have

$$\begin{aligned} d(x,[a,b]) = d(x,m(x,a,b)) &\leqslant d(x,x_n) + d(x_n, m(x_n,a,b)) + d(m(x_n,a,b), m(x,a,b)) \\ &= d(x,x_n) + d(m(x_n,a,b), m(x,a,b)) + d(x_n,[a,b]) \end{aligned}$$

Where the right side tend to zero when n goes to infinity by the continuity of the projection and Lemma 4.2.1. □

Remark 4.2.3. The join between two closed subset of a complete median space of finite rank is not necessarily closed, even if we assume that the subsets are bounded. Take for instance the product of the closed segment of the real line with a star like simplicial tree with infinite edges of length 1. One may consider then a sequence of points such that their projections into the star like simplicial tree run injectively through its vertices and their projections into the closed segment of the real line accumulate around 0 but never attain it.

Proposition 4.2.4. *Let X be a complete median space which have compact intervals. Then the convex hull between any two compact convex subsets is also compact.*

Proof. By Proposition 4.1.1 and Lemma 4.2.2, for any closed convex subset $C_1, C_2 \subseteq X$, their convex hull embeds as a closed subsets into the ℓ^1-product of C_1, C_2 and $\mathcal{B}(C_1, C_2)$, where the latter, by Proposition 4.1.4, is isometric to an interval and a closed convex subset of C_1, which is compact. □

In particular, we have the following :

Corollary 4.2.5. *Let X be a complete medians space with compact intervals. Then the convex hull of any finite subset is compact.*

In the following lemma, we show that the Hausdorff limit of compact subsets is a relatively compact subset :

Lemma 4.2.6. *Let X be a complete metric space and let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact subsets of X which converge, with respect to the Hausdorff metric, to a subset $K \subseteq X$. Then the closure of K is a compact subset of X.*

Démonstration. Note that up to considering the sequence of subsets $\tilde{K}_n = \bigcup_{i=0}^{n} K_n$, there is no loss of generality to assume that the sequence $(K_n)_{n\in\mathbb{N}}$ is ascending. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in K and let us show that it contains a subsequence which converge to a point in X. If there exist K_i such that it contains an infinite subsequence of $(x_i)_{i\in\mathbb{N}}$ then we are done. Let us assume then that each K_i contains finitely many points of $(x_i)_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let $i_n \in \mathbb{N}$ be such that $d_{Haus}(K_{i_n}, K) < \frac{1}{n}$. We consider a sequence $(\tilde{x}_{n,i})_{i\in\mathbb{N}} \subseteq K_{i_n}$ such that $d(\tilde{x}_{n,i}, x_i) < \frac{1}{n}$ for any $i \in \mathbb{N}$. The subset K_n being compact, there exists subsequence $(\tilde{x}_{n,\Phi(i)})_{i\in\mathbb{N}}$ which converges to a point $\tilde{x}_n \in K_{i_n}$. Iterating the same process for each n and considering at each step a subsequence of the previous subsequence, we obtain the following configuration :

- For each n there exist a sequence $(\tilde{x}_{n,i})_{i\in\mathbb{N}} \subseteq K_{i_n}$ such that for any $i \in \mathbb{N}$ we have $d(\tilde{x}_{n,i}, x_{\Phi_n(i)}) < \frac{1}{n}$, where each $\Phi_n : \mathbb{N} \to \mathbb{N}$ is an increasing injective map and $\Phi_{n+1}(\mathbb{N}) \subseteq \Phi_n(\mathbb{N})$.
- Each sequence $(\tilde{x}_{n,i})_{i\in\mathbb{N}}$ converges to a point $\tilde{x}_n \in K_{i_n}$.

Claim 1 : The sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let us show that for any $n, m \in \mathbb{N}$ we have $d(\tilde{x}_n, \tilde{x}_m) < \frac{1}{n} + \frac{1}{m}$. Let us fix $n, m \in \mathbb{N}$ such that $m \geqslant n$ and consider $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that for any integer $i \geqslant N$, we have $d(\tilde{x}_{n,i}, \tilde{x}_n) < \epsilon$ and $d(\tilde{x}_{m,i}, \tilde{x}_m) < \epsilon$. Hence for any $i \in \mathbb{N}$ such that $min(i, \Phi_n^{-1}(\Phi_m(i))) \geqslant N$, we have :

$$\begin{aligned} d(\tilde{x}_n, \tilde{x}_m) &\leqslant d(\tilde{x}_n, \tilde{x}_{n,\Phi_n^{-1}(\Phi_m(i))}) + d(\tilde{x}_{n,\Phi_n^{-1}(\Phi_m(i))}, x_{\Phi_m(i)}) + d(x_{\Phi_m(i)}, \tilde{x}_{m,i}) + d(\tilde{x}_{m,i}, \tilde{x}_m) \\ &\leqslant \epsilon + \frac{1}{n} + \frac{1}{m} + \epsilon \\ &\leqslant \frac{1}{n} + \frac{1}{m} + 2\epsilon. \end{aligned}$$

The ϵ being arbitrary, we conclude that $d(\tilde{x}_n, \tilde{x}_m) < \frac{1}{n} + \frac{1}{m}$.

As the space X is complete, the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ converges to a point $\tilde{x}$.

Claim 2 : The point $\tilde{x}$ is an accumulation point for the sequence $(x_i)_{i\in\mathbb{N}}$.
Let us fix $\epsilon > 0$ and consider $n \in \mathbb{N}$ such that $d(\tilde{x}, \tilde{x}_n) < \epsilon$. For any $i \in \mathbb{N}$ big enough such that $d(\tilde{x}_n, \tilde{x}_{n,i}) \leqslant \epsilon$, we get

$$\begin{aligned} d(\tilde{x}, x_{\Phi_n(i)}) &\leqslant d(\tilde{x}, \tilde{x}_n) + d(\tilde{x}_n, \tilde{x}_{n,i}) + d(\tilde{x}_{n,i}, x_{\Phi_n(i)}) \\ &\leqslant 2\epsilon + \frac{1}{n} \end{aligned}$$

Which proves Claim 2 and finishes the proof of the lemma. □

4.2.2 Proof of Theorem B

Definition 4.2.7. Let X be a complete median space of finite rank and let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace. We call the ***depth*** of $\mathfrak{h}$ in $A \subseteq X$, that we denote by $depth_A(\mathfrak{h})$, the maximum distance between points lying in $\mathfrak{h} \cap A$ and the hyperplane $\hat{\mathfrak{h}}$ bounding $\mathfrak{h}$, i.e. $depth_A(\mathfrak{h}) := sup\{d(x, \hat{\mathfrak{h}}) \mid x \in \mathfrak{h} \cap A\}$.

Before proving Theorem B, we will be needing some lemmas. The following lemma is a strengthening of Lemma 1.2.16 :

Lemma 4.2.8. *Let X be a complete connected median space of rank n and let $a, b \in X$. Then for any small $\epsilon > 0$ which is smaller then $\frac{d(a,b)}{n}$, there exist a pairwise transverse halfspaces $\mathfrak{h}_1, ..., \mathfrak{h}_k \in \mathcal{H}(a, b)$, where $k \leqslant n$, such that for all $i \in \{1, ..., k\}$ we have :*

— $d(\mathfrak{h}_i^c, b) \geqslant \epsilon$.

— $d(a, \bigcap_{i=1}^{k} \mathfrak{h}_i) \geqslant d(a, b) - \frac{n(n+1)}{2}\epsilon$.

Proof. Let us proceed by induction on the rank of X. The lemma is trivially true for complete connected median space of rank 1, that is, in the case of $\mathbb{R}$-trees.

Let us assume then that the lemma is true for complete connected median space of rank $n - 1$. Let us fix $a, b \in X$ and $0 < \epsilon \leqslant \frac{d(a,b)}{n}$. Let us consider $x \in [a, b]$ such that

$d(x,b) = n\epsilon$, such point x exists as the space X is connected. By Lemma 1.2.16, there exists a halfspace $\mathfrak{h} \in \Delta(x,b)$ such that $d(\mathfrak{h}^c, b) \geqslant \frac{d(x,b)}{n} \geqslant \epsilon$ and $d(x,\mathfrak{h}) = 0$. For simplicity let us set $\tilde{a} := \pi_{\hat{\mathfrak{h}}}(a)$. The hyperplane $\hat{\mathfrak{h}}$ being a median space of rank less than n, there exists then a family of pairwise transverse halfspaces $\mathfrak{h}_1, ..., \mathfrak{h}_k \in \mathcal{H}(\tilde{a}, x)$, where $k \leqslant n-1$ and such that for any $i \in \{1,...,k\}$, we have $d(\mathfrak{h}_i^c, x) \geqslant \epsilon$ and $d(\tilde{a}, \bigcap_{i=1}^{k} \mathfrak{h}_i) \geqslant d(\tilde{a}, x) - \frac{n(n-1)}{2}\epsilon$. Note that each halfspace $\mathfrak{h}_i$ is transverse to $\mathfrak{h}$ as it is, with its complementary, the lift of a halfspace of $\hat{\mathfrak{h}}$ with non empty interior (see Proposition 4.3.2 below). As the point x belongs to the interval $[a,b]$, any halfspace which separates x from $\mathfrak{h}_i^c$ must also separate b from $\mathfrak{h}_i^c$. Hence each $\mathfrak{h}_i^c$ is at distance greater than $\epsilon > 0$ from b. It last to show that the intersection of $\mathfrak{h}$ with the $\mathfrak{h}_i$'s is at distance greater than $d(a,b) - \frac{n(n+1)}{2}\epsilon$ from a. For any $y \in (\bigcap_{i=1}^{k} \mathfrak{h}_i) \cap \mathfrak{h}$, the point $\tilde{a} = \pi_{\hat{\mathfrak{h}}}(a) = \pi_{\overline{\mathfrak{h}}}(a)$belongs to the interval $[a,y]$. Thus we have :

$$d(a,y) = d(a,\tilde{a}) + d(\tilde{a},y) \geqslant d(a,\tilde{a}) + d(\tilde{a},x) - \frac{n(n-1)}{2}\epsilon \tag{4.2.1}$$

The equality $d(\tilde{a},x) + d(a,\tilde{a}) = d(a,x) = d(a,b) - n\epsilon$ combined with Inequality (4.2.1) above yields

$$d(a,y) \geqslant d(a,b) - n\epsilon - \frac{n(n-1)}{2}\epsilon = d(a,b) - \frac{n(n+1)}{2}\epsilon.$$

□

Lemma 4.2.9. *Let C be a complete connected median space of finite rank which is bounded. Let $\mathfrak{h} \subset X$ be a halfspace, then for any $\epsilon > 0$ such that there is no two disjoint halfspaces of depth bigger than ϵ contained in $\mathfrak{h}$ and for any $a \in \mathfrak{h}$ such that $d(a,\hat{\mathfrak{h}}) \geqslant depth_c(\mathfrak{h}) - \epsilon$, the convex hull $Conv(\{a\} \cup \hat{\mathfrak{h}})$ it at Hausdorff distance less than $(n(n+1)+1)\epsilon$ from $\mathfrak{h}$, where n is the rank of C.*

Proof. Let us choose a point $a \in \mathfrak{h}$ such that $d(a,\hat{\mathfrak{h}}) \geqslant depth_C(\mathfrak{h}) - \epsilon$. We set $C_{\mathfrak{h}} = Conv(\hat{\mathfrak{h}}, a)$ and take a point $x \in \mathfrak{h}$ lying outside $C_{\mathfrak{h}}$. We consider its projections into $C_{\mathfrak{h}}$ and $[a, \pi_{\hat{\mathfrak{h}}}(a)]$ that we denote by $x_{C_{\mathfrak{h}}} := \pi_{C_{\mathfrak{h}}}(x)$ and $\tilde{x} := m(x, a, \pi_{\hat{\mathfrak{h}}}(a))$ respectively (See Figure 4.2.1) .
Let us first show that $d(\tilde{x}, \hat{\mathfrak{h}}) = d(x_{C_{\mathfrak{h}}}, \hat{\mathfrak{h}})$. As the interval $[\pi_{\hat{\mathfrak{h}}}(a), a]$ lies in $C_{\mathfrak{h}}$, we have :

$$\tilde{x} = m(x,a,\pi_{\hat{\mathfrak{h}}}(a)) = \pi_{[\pi_{\hat{\mathfrak{h}}}(a),a]}(x) = \pi_{[\pi_{\hat{\mathfrak{h}}}(a),a]}(\pi_{C_{\mathfrak{h}}}(x)) = \pi_{[\pi_{\hat{\mathfrak{h}}}(a),a]}(x_{C_{\mathfrak{h}}}).$$

Hence, any halfspace separating $\tilde{x}$ from $\hat{\mathfrak{h}}$, separates a from $\pi_{\hat{\mathfrak{h}}}(a)$, therefore it must separate $x_{C_{\mathfrak{h}}}$ from $\hat{\mathfrak{h}}$ as well by Lemma 1.1.14. In the other hand, note that any halfspace separating any point in $C_{\mathfrak{h}}$ from $\hat{\mathfrak{h}}$ must separate the point a from $\hat{\mathfrak{h}}$ as the convex subset $C_{\mathfrak{h}}$ is the convex hull of $\hat{\mathfrak{h}} \cup \{a\}$. Hence by Lemma 1.1.14, any halfspace separating $x_{C_{\mathfrak{h}}}$ from $\hat{\mathfrak{h}}$

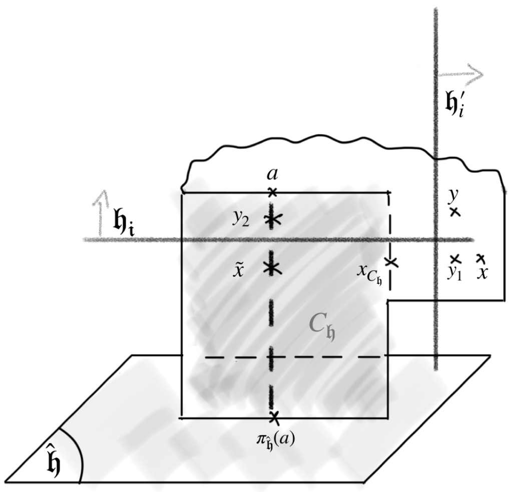

FIGURE 4.2.1 – The configuration arising in the second part of the proof of Lemma 4.2.9

separates also $\tilde{x}$ from $\hat{\mathfrak{h}}$. Therefore, the two walls intervals $\mathcal{W}(\tilde{x}, \hat{\mathfrak{h}})$ and $\mathcal{W}(x_{C_\mathfrak{h}}, \hat{\mathfrak{h}})$ coincide, which implies the equality $d(\tilde{x}, \hat{\mathfrak{h}}) = d(x_{C_\mathfrak{h}}, \hat{\mathfrak{h}})$. We deduce then the following :

$$\begin{aligned} d(x, x_{C_\mathfrak{h}}) &= d(x, \hat{\mathfrak{h}}) - d(x_{C_\mathfrak{h}}, \hat{\mathfrak{h}}) \\ &\leqslant depth_C(H) - d(\tilde{x}, \hat{\mathfrak{h}}) \\ &\leqslant d(a, \hat{\mathfrak{h}}) + \epsilon - d(\tilde{x}, \hat{\mathfrak{h}}) \end{aligned}$$

As $\tilde{x}$ lies in the interval $[a, \pi_\mathfrak{h}(a)]$, then its projection into $\hat{\mathfrak{h}}$ is precisely the point $\pi_\mathfrak{h}(a)$. Hence $d(a, \hat{\mathfrak{h}}) = d(\tilde{x}, \pi_\mathfrak{h}(a))$. Replacing the latter in the inequality above, we get :

$$d(x, x_{C_\mathfrak{h}}) \leqslant d(a, \tilde{x}) + \epsilon \tag{4.2.2}$$

If the distance between x and $C_\mathfrak{h}$ is less than $(n+1)\epsilon$, then there is nothing to show. Let us assume then that $d(x, C_\mathfrak{h}) \geqslant (n+1)\epsilon$, which by Inequality (4.2.2) above, implies also that $d(a, \tilde{x}) \geqslant n\epsilon$. Hence by Lemma 4.2.8, there exist two families of pairwise transverse halfspaces $\{\mathfrak{h}_1, ..., \mathfrak{h}_p\} \subseteq \mathcal{H}(\tilde{x}, a)$ and $\{\mathfrak{h}'_1, ..., \mathfrak{h}'_q\} \subseteq \mathcal{H}(x_{C_\mathfrak{h}}, x)$ such that the halfspaces $\mathfrak{h}_i^c$ and $\mathfrak{h}'^c_j$ are of depth bigger than ϵ and verify the following :

$$d(\tilde{x}, \bigcap_{i=1}^{p} \mathfrak{h}_i) \geqslant d(\tilde{x}, a) - \frac{n(n+1)}{2}\epsilon \quad \text{and} \quad d(x_{C_\mathfrak{h}}, \bigcap_{i=1}^{q} \mathfrak{h}_i) \geqslant d(x_{C_\mathfrak{h}}, x) - \frac{n(n+1)}{2}\epsilon.$$

By assumption, the halfspace $\mathfrak{h}$ does not contain two disjoint halfspace of depth bigger than ϵ. Hence, the halfspaces $\mathfrak{h}_i^c$ and $\mathfrak{h}'^c_j$ are not disjoint for any $i \in \{1, ..p\}$ and $j \in \{1, .., q\}$. In the other hand, any halfspace $\mathfrak{h}$ in $\Delta(\hat{\mathfrak{h}}, \tilde{x}) = \Delta(\hat{\mathfrak{h}}, x_{C_\mathfrak{h}})$ contains the points a and x (note that they do not not necessarily contain the halfspaces $\mathfrak{h}_i$ and $\mathfrak{h}_j$). Hence the intersection $(\bigcap_{\mathfrak{h}'\in\Delta(\hat{\mathfrak{h}},\tilde{x})}\mathfrak{h}')$ contains the interval $[a,x]$. Therefore by Helly's Theorem 1.1.5, the intersection $(\bigcap_{\mathfrak{h}'\in\Delta(\hat{\mathfrak{h}},\tilde{x})}\mathfrak{h}')\cap(\bigcap_{i=1}^{p}\mathfrak{h}_i^c)\cap(\bigcap_{j=1}^{q}\mathfrak{h}_j'^c)$ is not empty. Let us consider a point y in the latter intersection and let y_1 and y_2 be its projections into the interval $[x_{C_\mathfrak{h}},x]$ and $[\tilde{x},a]$ respectively. We claim the following :

$$d(y,\hat{\mathfrak{h}}) \geqslant d(x_{C_\mathfrak{h}},\hat{\mathfrak{h}}) + d(x_{C_\mathfrak{h}},y_1) + d(\tilde{x},y_2) \tag{4.2.3}$$

Indeed, by construction we have the following inclusion :

$$\mathcal{W}(y_1,x_{C_\mathfrak{h}}) \cup \mathcal{W}(y_2,\tilde{x}) \cup \mathcal{W}(x_{C_\mathfrak{h}},\hat{\mathfrak{h}}) \subseteq \mathcal{W}(y,\hat{\mathfrak{h}}).$$

In the other hand, all the wall intervals arising on the left hand of the inclusion are disjoints, therefore we get :

$$\begin{aligned}\mu(\mathcal{W}(y_1,x_{C_\mathfrak{h}}) \cup \mathcal{W}(y_2,\tilde{x}) \cup \mathcal{W}(x_{C_\mathfrak{h}},\hat{\mathfrak{h}})) &= \mu(\mathcal{W}(y_1,x_{C_\mathfrak{h}})) + \mu(\mathcal{W}(y_2,\tilde{x})) + \mu(\mathcal{W}(x_{C_\mathfrak{h}}),\hat{\mathfrak{h}}))\\ &= d(y_1,x_{C_\mathfrak{h}}) + d(y_2,\tilde{x}) + d(x_{C_\mathfrak{h}},\hat{\mathfrak{h}})\\ &\leqslant \mu(\mathcal{W}(y,\hat{\mathfrak{h}})) = d(y,\hat{\mathfrak{h}})\end{aligned}$$

Having the inequality 4.2.3 in hand, we get :

$$\begin{aligned}depth_C(H) &\geqslant d(y,\hat{\mathfrak{h}})\\ &\geqslant d(x_{C_\mathfrak{h}},\hat{\mathfrak{h}}) + d(y_1,x_{C_\mathfrak{h}}) + d(y_2,\tilde{x})\\ &\geqslant d(x_{C_\mathfrak{h}},\hat{\mathfrak{h}}) + d(x_{C_\mathfrak{h}},x) - \frac{n(n+1)}{2}\epsilon + d(\tilde{x},a) - \frac{n(n+1)}{2}\epsilon\end{aligned}$$

As $d(x_{C_\mathfrak{h}},\hat{\mathfrak{h}}) = d(\tilde{x},\hat{\mathfrak{h}})$, we get :

$$depth_C(H) \geqslant d(\tilde{x},\hat{\mathfrak{h}}) + d(\tilde{x},a) + d(x_{C_\mathfrak{h}},x) - n(n+1)\epsilon = d(\hat{\mathfrak{h}},a) + d(x_{C_\mathfrak{h}},x) - n(n+1)\epsilon.$$

We deduce then the following :

$$d(x_{C_\mathfrak{h}},x) \leqslant depth_C(H) - d(\hat{\mathfrak{h}},a) + n(n+1)\epsilon \leqslant (n(n+1)+1)\epsilon.$$

Which finishes the proof. □

Proof of Theorem B. Let us first remark that there is no loss of generality to assume that C is a closed convex subset. Indeed, the complete space X being of finite rank, the convex hull of C is compact if and only C is compact (see Lemma 13.2.11 [Bow22]). In the other hand, Remark 1.2.17 implies that if a halfspaces is of depth less than ϵ in C, then it is of depth less than $n.\epsilon$ in $Conv(C)$ where n is the rank of the space X. Hence, the condition (3) holds with respect to C if and only if it holds with respect to $Conv(C)$.

The implication $1 \Rightarrow 2$ is obvious. Let us first show the implication $2 \Rightarrow 3$. For a fixed $\epsilon > 0$ there exist $x_0, x_1, ..., x_{n_\epsilon} \in C$ such that the subset C is at Hausdorff distance less than ϵ from $\bigcup_{i=1}^{n_\epsilon} [x_0, x_i]$. Let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace transverse to C. If $\mathfrak{h}$ does not contain any of the x_i, then it must be of depth less than ϵ in C. Thus any halfspace transverse to C of depth bigger than ϵ must separate x_0 from some x_i. Therefore there is only finitely many pairwise disjoint halfspaces transverse to C and of depth bigger than ϵ.

We now prove the implication $3 \Rightarrow 1$. By Lemma 4.2.6, it is enough to show that under the conditions of statement (3), the set C is the Hausdorff limit of some sequence of compact subsets. Let us fix ϵ and consider a family $\mathcal{H}_\epsilon$ of maximal cardinal of pairwise disjoint halfspaces transverse to C and of depth bigger than ϵ in C. The maximality implies that each halfspace $\mathfrak{h} \in \mathcal{H}_\epsilon$ does not contain two disjoint halfspaces of depth bigger than ϵ in $C \cap \mathfrak{h}$. As we are considering only halfspaces which are transverse to the convex subset C, we may forget about the ambient space X and assume that each halfspace is a halfspace of C, up to taking the intersection with the latter (see Remark 1.1.15 and Proposition 4.3.1). Under the assumption of statement (3), the set $\mathcal{H}_\epsilon$ is finite. Let us set $C_\epsilon = Conv(\bigcup_{\mathfrak{h} \in \mathcal{H}_\epsilon} \hat{\mathfrak{h}})$ and first show that it is at Hausdorff distance less than $n\epsilon$ from $C \setminus \bigcup_{\mathfrak{h} \in \mathcal{H}_\epsilon} \mathfrak{h}$. Let $x \in C$ be a point lying outside all of the halfspace $\mathfrak{h} \in \mathcal{H}_\epsilon$. Note then that any halfspace separating x from C_ϵ is disjoint from any halfspace in $\mathcal{H}_\epsilon$. Hence by the maximality of the family $\mathcal{H}_\epsilon$, any halfspace separating x from C_ϵ is of depth less than ϵ in C. Therefore, we conclude by Lemma 1.2.16 that the point x is at distance at most $n\epsilon$ from C_ϵ.

For each $\mathfrak{h} \in \mathcal{H}_\epsilon$, we choose a point $a_\mathfrak{h} \in \mathfrak{h}$ such that $d(a_\mathfrak{h}, \hat{\mathfrak{h}}) \geqslant depth_C(\mathfrak{h}) - \epsilon$. We set $C_\mathfrak{h} := Conv(\{a_\mathfrak{h}\} \cup \hat{\mathfrak{h}})$ and use Lemma 4.2.9 to conclude that it is at Hausdorff distance less than $(n(n+1)+1)\epsilon$ from $\mathfrak{h}$.

Let us set $\tilde{C}_\epsilon = \bigcup_{\mathfrak{h} \in \mathcal{H}_\epsilon} C_\mathfrak{h} \cup C_\epsilon$. We have shown that $\tilde{C}_\epsilon$ is at Hausdorff distance less than $(n(n+1)+1)\epsilon$ from C. It last to show that it is compact. We proceed by induction on the rank of C. Note that when the rank of C is 1, then the hyperplane corresponds to a point. Hence by Proposition 4.2.4, the subset $\tilde{C}$ is compact as it is a finite union of the convex hull of compact subsets. Let us assume now that the rank of C is equal n and that the implication $3 \Rightarrow 1$ is true for median space of rank less or equal $n-1$. Since we have assumed Condition (3) to be true, it is verified by each hyperplane $\hat{\mathfrak{h}}$. Therefore each hyperplane $\hat{\mathfrak{h}}$ is compact. We conclude then by Proposition 4.2.4 that $\tilde{C}_\epsilon$ is compact, which finishes the proof.

□

For the general case when the rank is infinite, it is harder to manipulate halfspaces as they may all be dense in the space and even if it is the case, one can no longer use an argument by induction on the rank of the space. Let us give a criterion of local compactness in the infinite rank case :

Proposition 4.2.10. *Let X be a complete median space with compact intervals and let $K \subseteq X$ be a closed subset. If the outer measure of the set of transverse halfspaces to K is finite,*
i.e $\bar{\mu}(\mathcal{H}(K)) < +\infty$, where μ is the canonical measure associated to $\mathcal{H}(X)$, then K is compact.

Let us first make the following remarks :

Remark 4.2.11.
— If X be a complete median space and $K \subseteq X$ a subset such that $\bar{\mu}(\mathcal{H}(K)) < +\infty$, then the convex hull of K is bounded. This is first due to the fact that the set of halfspaces which are transverse to K is the same as the set of halfspaces which are transverse to the convex hull of K. In the other hand, having a sequence of points which is unbounded give rise to a sequence of wall interval with an arbitrarily big measure.

— The converse of Proposition 4.2.10 is false, even in the finite rank case. One may consider a star like tree obtained from the concatenation of the intervals $[0, \frac{1}{n}]$ at $\{0\}$.

Proof of Proposition 4.2.10. Let $K \subseteq X$ be a closed subset such that $\bar{\mu}(\mathcal{H}(K)) = M$. By Remark 4.2.11, there is no loss of generality if we consider the closure of the convex hull of K. Let us first remark that for any $x, y \in X$, the set of halfspaces which separate x from y is exactly the same as the set of halfspaces separating $\pi_K(x)$ from $\pi_K(y)$. Hence, for any $\epsilon > 0$, there exist $x_1, y_1, ..., x_{n_\epsilon}, y_{n_\epsilon} \in K$ such that $\mathcal{W}(x_i, y_i)$ and $\mathcal{W}(x_j, y_j)$ are disjoint for any $i \neq j$ and :

$$\mu(\bigcup_{i=1}^{n_\epsilon} \mathcal{W}(x_i, y_i)) = \mu(\sum_{i=1}^{n_\epsilon} d(x_i, y_i)) \geqslant M - \epsilon.$$

Let consider the convex hull of all the point $C_\epsilon = Conv(\{x_1, y_1, ..., x_{n_\epsilon}, y_{n_\epsilon}\})$ and a point $x \in K$. Then any halfspace separates x from C_ϵ if and only if it separates x from all the points x_i and y_i. Thus, we must have $\mu(\mathcal{W}(x, C_\epsilon)) \leqslant \epsilon$. Let us then consider a sequence $C_{\frac{1}{n}}$ defined as above. We may assume that the sequence $(C_{\frac{1}{n}})_{n\in\mathbb{N}}$ is ascending with respect to the inclusion. By Proposition 4.2.5, each subset $C_{\frac{1}{n}}$ is compact. In the other hand, the sequence $(C_{\frac{1}{n}})_{n\in\mathbb{N}}$ converges with respect to the Hausdorff metric to $C = \bigcup_{n\in\mathbb{N}}^{i} C_{\frac{1}{n}}$. As each $C_{\frac{1}{n}}$ is compact, the subset C is totally bounded. Thus its closure is a compact which contains the closed subset K. We conclude that the subset K is also compact. ☐

4.3 Transitive actions on median spaces of finite rank and local compactness

4.3.1 Trace of halfspaces on convex sets

Throughout this section X is a complete connected median space of finite rank.

Let us denote the set of hyperplanes of X by $\hat{\mathcal{H}}(X)$ and deduce from Remark 1.1.15 that any hyperplane in a closed convex subset is induced from a hyperplane of the ambient space as stated in the following :

Proposition 4.3.1. *Let $C \subseteq X$ be a closed convex subset. We have then :*

$$\hat{\mathcal{H}}(C) = \{\hat{\mathfrak{h}} \cap C \mid \hat{\mathfrak{h}} \in \hat{\mathcal{H}}(X) \text{ and separates two points of } C\}.$$

Proof. By Remark 1.1.15, it is enough to show that for any halfspace $\mathfrak{h}$ which is transverse to C we have $\overline{\mathfrak{h} \cap C} = \bar{\mathfrak{h}} \cap C$. The inclusion $\overline{\mathfrak{h} \cap C} \subseteq \bar{\mathfrak{h}} \cap C$ is obvious. For the converse, let us take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{h}$ which converges to a point $x \in C$. We denote by $y_n := \pi_C(x_n)$ their projection into the closed convex set C. As $\mathfrak{h} \cap C$ is not empty, we get that $(y_n)_{n\in\mathbb{N}} \subseteq \mathfrak{h} \cap C$. Due to the continuity of the projection, the sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathfrak{h}$ converges to $\pi_C(x) = x$. □

Let $C \subseteq X$ be a convex subset and $\mathfrak{h} \subseteq X$ a halfspace. We call the ***trace*** of the hyperplane $\hat{\mathfrak{h}}$ on C the intersection $\hat{\mathfrak{h}} \cap C$.

Proposition 4.3.2. *Let $C \subseteq X$ be a closed convex subset and $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace. Then the lift of any halfspace T of $\hat{\mathfrak{h}} \cap C$ to the ambient space X is a halfspace transverse to $\mathfrak{h}$ assuming that T and its complementary are of non empty interior inside $\hat{\mathfrak{h}} \cap C$.*

Proof. Let $T \in \mathcal{H}(\hat{\mathfrak{h}} \cap C)$ be an halfpsace of the trace of the hyperplane $\hat{\mathfrak{h}}$ on the convex subset C. To show that the lift of T to X is transverse to $\mathfrak{h}$, it is enough to show that there exist points in $\mathfrak{h}$ and $\mathfrak{h}^c$ which projects into T and another ones which projects into $T^c \cap C \cap \hat{\mathfrak{h}}$. The halfspace T being inside the trace of the hyperplane $\hat{\mathfrak{h}}$, for any point x in the interior of T or its complementary inside $\hat{\mathfrak{h}} \cap C$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ inside $\mathfrak{h} \cap C$ which converge to x. Taking the index n big enough, the point x_n projects inside a small neighborhood of x in T. □

Remark 4.3.3. The proposition will no longer be true if we drop the assumption on T and its complementary being both of non empty interior. One may consider for instance the following subspace of $(\mathbb{R}^2, \ell^1)$:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid y - x \geqslant 0\}.$$

We take the convex subset C to be the half line $x = 0$ and T, a halfspace of C, to be the trace of the halfspace $\mathfrak{h} := \{(x, y) \in X \mid y > 0\}$. The convex subset C is also the hyperplane which bounds the halfspace defined by the inequality $x > 0$. The lift of $(0, 0)$, the complement of T inside C, is not transverse to $\mathfrak{h}$ as it is disjoint from it.

Proposition 4.3.4. *Let $C \subseteq X$ be a convex subset which isometric to an n-cube $([-\epsilon, \epsilon], \ell^1)$ where n is the rank of X. Let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace in X which is disjoint from C. If its hyperplane $\hat{\mathfrak{h}}$ intersects the interior of C, then there exist $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}(X)$ such that they constitute with the halfspace $\mathfrak{h}$ a facing triple in X and $\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}}_1 \cap \hat{\mathfrak{h}}_2 \neq \varnothing$.*

Proof. Let us identify C with $[-\epsilon, \epsilon]^n$. By Proposition 4.3.2, the rank of $\hat{\mathfrak{h}} \cap C$ is smaller than n. Hence the latter is contained in a hyperplane of C, let us say the hyperplane given by the equation $x_1 = 0$. Then the lift to the ambient space of the two halfspaces $\mathfrak{H}_1 = \{(x_1, ..., x_n) \in C \mid x_1 > 0\}$ and $\mathfrak{H}_2 = \{(x_1, ..., x_n) \in C \mid x_1 < 0\}$ give us the desired halfspaces. Indeed, Lemma 1.1.14 tell us that $\overline{\mathfrak{h}}$ projects into $\overline{\mathfrak{h}} \cap C = \hat{\mathfrak{h}} \cap C$, which is outside $\mathfrak{H}_1$ and $\mathfrak{H}_2$. □

Finally we deduce the following corollary in the case where the median space admits a transitive action :

Corollary 4.3.5. *Let X be a complete connected median space of rank n. Let $x \in X$ and let $([-\epsilon, \epsilon]^n, \ell^1) \cong C \subseteq X$ be an isometrically embedded n-cube centred at x. If $\mathcal{H}_x(X)$ does not contain a facing triple, then it coincides with $\mathcal{H}_x(C)$.*

Proof. By Remark 1.1.15, it is enough to show that any halfspace in $\mathcal{H}_x(X)$ is transverse to the n-cube C. Let us consider a $\mathfrak{h} \in \mathcal{H}_x(X)$ branched at x. The set $\mathcal{H}_x(X)$ does not contain a facing triple and the intersection $\hat{\mathfrak{h}} \cap C$ is not empty as both contain the point x, hence Proposition 4.3.4 implies then that the halfspace $\mathfrak{h}$ is necessarily transverse to C. □

4.3.2 Proof of Theorem D

Proof of the first claim of Theorem D : Let X be a complete connected median space of finite rank which admits a transitive action. Let us fix a maximal pairwise transverse family of halfspaces $\mathcal{H} = \{\mathfrak{h}_1, ..., \mathfrak{h}_n\}$ in $\mathcal{H}(X)$. Let us set the following

$$D_i := \bigcap_{j \neq i} \hat{\mathfrak{h}}_j \tag{4.3.1}$$

Let us show that each D_i is a strongly convex isometric embedding of an $\mathbb{R}$-tree.

Proposition 4.3.6. *Each D_i, endowed with the induced metric of X, is a complete connected median space of rank 1.*

Let us first show the following lemma :

Lemma 4.3.7. *Let X be a complete connected median space of finite rank. We assume that there exist two transverse halfspaces $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq X$. Then there exist two transverse halfspaces such that both them and their complements are of non empty interior inside X.*

Proof. Let us consider two transverse halfspaces $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq X$. By Proposition 1.2.10 and up to considering the complement, we may assume that they are both open in X. Let us consider a point $x \in \mathfrak{h}_1 \cap \mathfrak{h}_2$. Let us set the following $x_i := \pi_{\mathfrak{h}_i^c}(x)$ and $x_0 := \pi_{\mathfrak{h}_1^c \cap \mathfrak{h}_2^c}(x)$. Note that $x_0 = \pi_{\mathfrak{h}_1^c}(x_2) = \pi_{\mathfrak{h}_2^c}(x_1)$ by Lemma 1.1.14. The halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ being both open, the points x_1 and x_2 are distinct from x_0. Let us set $C := Conv([x_0, x_1] \cup [x_0, x_2])$ and set $\tilde{x} := \pi_C(x)$. Note that $\pi_{\mathfrak{h}_i^c}(\tilde{x}) = x_i$. As $[x_0, x_1] \cap [x_0, x_2] = \{x_0\}$, Proposition 4.1.3 implies that the interval $[\tilde{x}, x_0]$ is isometric to the ℓ^1-product $[x_0, x_1] \times [x_0, x_2]$. Hence, the lift to X of any halfspaces $\mathfrak{H}_1 \in \mathcal{H}([x, x_1])$ and $\mathfrak{H}_2 \in \mathcal{H}([x, x_2])$, such that $\mathfrak{H}_i$ and $\mathfrak{H}_i^c$ are of non empty interior in $[x, x_i]$, yields two transverse halfspaces such that both them and their complements are of non empty interior in X. □

Proof of Proposition 4.3.6. By Helly's Theorem 1.1.5, each D_i is a non empty closed convex subset of X and it intersects both $\mathfrak{h}_i$ and $\mathfrak{h}_i^c$. Hence, each D_i is of rank bigger then or equal one. It is left to show that it is of rank smaller then two. If there exist two transverse halfspaces in D_i, Lemma 4.3.7 ensures that there is no loss of generality if we assume them to be with their complement inside D_i of non empty interior. In the other hand, Proposition 4.3.2 implies that the lift of such halfspaces to the ambient space X yields halfspaces which are transverse to each $\mathfrak{h}_j$ where $j \neq i$, which would contradict the maximality of the family $(\mathfrak{h}_i)$. □

Following the same argument of the proof of Proposition 4.3.6, we note that for $i \neq j$, we have $D_i \cap D_j = \bigcap_{k=1}^{n} \hat{\mathfrak{h}}_k = \{a_0\}$ for some $a_0 \in X$.

In the following, we show that any point is the center of an isometrically embedded n-cube.

Lemma 4.3.8. *Under the assumptions that the median space X is complete and admits a transitive action, there exists $\epsilon > 0$ such that any $x \in X$ is the center of an isometrically embedded convex n-cube $([-\epsilon, \epsilon]^n, \ell^1)$ centred at x.*

Proof. By the transitivity assumption of the isometry group of X, it is enough to show the existence of an isometrically embedded n-cube in X. By Helly's Theorem 1.1.5, the intersection $\bigcap_{i=1}^{n} \mathfrak{h}_i$ is not empty. Let us consider a point a in the latter intersection. Again by Helly's Theorem, the intersections $\mathfrak{h}_i \cap D_i$ are not empty and do not contain a_0, hence the projection of a into D_i avoid a_0. Let us set $a_i := \pi_{D_i}(a)$. By Proposition 4.3.6, each interval $[a_0, a_i]$ is isometric to closed interval of $\mathbb{R}$. Let us denote by C the convex hull between the $\mathbb{R}$-trees D_i. By Lemma 4.1.3, the interval $[a_0, \pi_C(a)]$ is isometric to the ℓ^1-product of the intervals $[a_0, a_i]$, thus completing the argument. □

Let us assume now that the set $\mathcal{H}_{a_0}(X)$ does not contain a pairwise disjoint triple of halfspaces. Remark that the transitivity assumption implies then that for any $x \in X$ the set $\mathcal{H}_x(X)$ does not contain such triple.

Proposition 4.3.9. *Each D_i, as defined in the beginning of Subsction 4.3.2 Equality (4.3.1), is isometric to the real line.*

Proof. By Proposition 4.3.6, we already know that each D_i is a closed convex subset of rank 1, hence it is isometric to a complete connected $\mathbb{R}$-tree. Under the assumption that the set $\mathcal{H}_{a_0}(X)$ does not contain any facing triple, Remark 1.1.15 and Helly's Theorem 1.1.5 imply that no D_i can contain a facing triple. Hence, each D_i is isometric to an interval of the real line. To conclude that it is isometric to the real line, it is enough to show that any point in D_i lies in the interior of an interval in D_i. Let us consider a point $x \in D_i$. By Lemma 4.3.8, there exists an embedded n-cube $C \cong]-\epsilon,\epsilon[^n$ in X centred at the point x. Under the assumption that there is no facing triple in $\mathcal{H}_x(X)$, Corollary 4.3.5 implies that the halfspace $\mathfrak{h}_j$, for any $j \neq i$, are transverse to the n-cube. We have then $(\bigcap_{j\neq i} \hat{\mathfrak{h}}_j) \cap C \cong]-\epsilon,\epsilon[\subseteq D_i$. As D_i identifies with an interval of $\mathbb{R}$, the latter intersection intersection is open in D_i, which finish the argument. □

Now, we have all the ingredients needed to prove the first part of Theorem D :

Proposition 4.3.10. *The convex hull of the lines D_i contains X and is isometric to $(\mathbb{R}^n, \ell^1)$.*

Proof. We set $C := Conv(D_1 \cup ... \cup D_n)$. The set C is a closed subset of the complete median space X, as it is the convex hull of finitely many closed convex subsets, hence it is also complete. By Proposition 4.1.1, it embeds isometrically, through the projections onto the D_i's, as a closed subset of $\prod_{i=1}^{n} D_i \cong \mathbb{R}^n$. It is enough to show that the embedding is open to conclude that it is surjective. In our way proving that, we prove also that C contains X. Let us take a point $x \in X$ and consider the family $\mathfrak{h}_{i,l} := \pi_{D_i}^{-1}(]-\infty, \pi_{D_i}(\widetilde{x})[)$, $\mathfrak{h}_{i,r} := \pi_{D_i}^{-1}([\pi_{D_i}(\widetilde{x}), +\infty[)$, where $\widetilde{x} := \pi_C(x)$ and each D_i is identified with $\mathbb{R}$. By Remark 1.1.15, each $\mathfrak{h}_{i,l}$ and $\mathfrak{h}_{i,r}$ is a halfspace of X. By Lemma 4.3.8, there exists an n-cube centered at $\tilde{x}$. Thus by Corollary 4.3.5, the family of halfspaces $\{\mathfrak{h}_{1,l}, \mathfrak{h}_{1,r}, ..., \mathfrak{h}_{n,l}, \mathfrak{h}_{n,r}\}$ and their complementary in X constitutes all the elements of $\mathcal{H}_{\tilde{x}}$. In one hand, this implies that the projection map $(\pi_{D_1}, ..., \pi_{D_n})$ is open. In the other hand, by Lemma 1.2.14 we get :

$$\bigcap_{i=1}^{n} (\mathfrak{h}_{i,l} \cap \mathfrak{h}_{i,r}) = \{\widetilde{x}\}.$$

As the projections onto each D_i factor through the projection onto C, that is $\pi_{D_i}(x) = \pi_{D_i}(\pi_C(x)) = \pi_{D_i}(\tilde{x})$, the point x lie in $(\mathfrak{h}_{i,l} \cap \mathfrak{h}_{i,r})$ for any $i \in \{1, ..., n\}$. Hence, we get $x = \tilde{x}$, which proves that $C = X$ and complete the proof. □

Proof of the second part of Theorem D The idea of the proof is to show that under the assumption of the existence of a facing triple in $\mathcal{H}_{a_0}$ and a transitive action on X, there exist infinitely many pairwise disjoint halfspaces with depth uniformly bounded below inside any ball centred at a_0. Let us first show that any halfpaces in $\mathcal{H}_{a_0}$ is of positive depth inside any ball centred at a_0.

Proposition 4.3.11. *Let X be a complete connected median space of finite rank which admits a transitive action. Then for any halfspace $\mathfrak{h} \in \mathcal{H}_{a_0}$ and $r > 0$, we have $depth_{B(a_0,r)}(\mathfrak{h}) > 0$.*

Proof. Let $\mathfrak{h} \in \mathcal{H}_{a_0}$ be a halfspace containing the point a_0 in its hyperplane. If the halfspace $\mathfrak{h}$ is open, its depth inside any ball centered at a_0 is positive. Let us assume then that the halfspace $\mathfrak{h}$ is closed. By Proposition 4.1.3, there exists an isometrically embedded n-cube $C \cong [-\epsilon, \epsilon]^n$ where n is the rank of the space X. The action being transitive, we can assume that the n-cube is centred at a_0. If the halfspace $\mathfrak{h}$ is transverse to the n-cube C, then it gives rise to a halfspace of C which contains a_0 and which is of positive depth inside C. If $\mathfrak{h}$ is not transverse to C, then it will contain it. Let us consider then the trace of the hyperplane $\hat{\mathfrak{h}}$ which bounds the halfspace $\mathfrak{h}$, on the n-cube, that is, its intersection with the latter. Let us denote it by $\hat{C}$. It is a convex subset which contains a_0. The rank of X being n, Proposition 4.3.2 implies that the convex subset $\hat{C}$ is of rank less than $n-1$. Again, there exist then points inside C which are at positive distance from $\hat{C}$, hence from $\hat{\mathfrak{h}}$. □

Proof of the second part of Theorem D. Let us fix a point $a_0 \in X$ and show that under the assumptions of Theorem D, any neighbourhood of a_0 contains infinitely many disjoint halfspaces of depth bigger than some $\epsilon > 0$ inside the latter neighbourhood. We will conclude then by Theorem B that the space is not locally compact.

The action of $Isom(X)$ being transitive, by Lemma 4.3.8, there exists an isometrically embedded n-cube $C \cong ([-\eta, \eta]^n, \ell^1)$ centred at a_0. Let us parametrize the n-cube by $x_1, ..., x_n$ and identify a_0 with $(0, ..., 0)$. Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{H}_{a_0}$ be a facing triple. By Proposition 4.3.11, each of the $\mathfrak{h}_i$ is of positive depth inside any ball centred at x_0. For any point x inside the n-cube, there exists an isometry $g \in Isom(X)$ which maps a_0 to x. At least, the image of one of the $\mathfrak{h}_i$'s by the isometry g is disjoint from the n-cube. Hence, for any $r \in]0, \eta[$ there exists a halfspace $\mathfrak{h}_r \in \mathcal{H}_{(r,...,r)}$ which is disjoint from the n-cube and of depth bigger than some uniform ϵ. The trace of the hyperplane $\hat{\mathfrak{h}}_r$ on the n-cube C is a convex subset of rank less than n containing the point $(r, ..., r)$. Hence, it is contained in a hyperplane of the n-cube C given by an equation of the form $x_{i_r} = r$. Thus, there exists an infinite subset $I \subseteq]0, \eta[$ such that for any $r_1, r_2 \in I$, the trace of the hyperplane $\hat{\mathfrak{h}}_{r_1}$ on the n-cube is disjoint from the trace of the hyperplane $\hat{\mathfrak{h}}_{r_2}$. By Lemma 1.1.14, if the halfspaces $\mathfrak{h}_{r_1}$ and $\mathfrak{h}_{r_2}$ intersect, then projection of their intersection into the n-cube C lies inside $\hat{\mathfrak{h}}_{r_1} \cap \hat{\mathfrak{h}}_{r_2}$. Hence, for any such r_1 and r_2, the halfspaces $\mathfrak{h}_{r_1}$ and $\mathfrak{h}_{r_2}$ are disjoint. Therefore, the set $(H_r)_{r \in I}$ give us the desired infinite family of pairwise disjoint halfspaces of depth bigger than some uniform $\epsilon > 0$.

□

4.3.3 A comment on a weaker assumption in Theorem D

One may weaken the assumption in Theorem D and assume the action of $Isom(X)$ to be topologically transitive instead of transitive, that is, it admits a dense orbit. The same strategy works, we divide the theorem into condition on the existence of a facing triple in the neighbourhood of a points. Theorem D adapts into the following :

Theorem 4.3.12. *Let X be a complete connected median space of rank n which admits a topologically transitive action. If there exist $x \in X$ and $\epsilon > 0$ such that the set of halfspaces $\mathcal{H}(B(x,\epsilon))$ contains no facing triple then the space X is isomorphic to $(\mathbb{R}^n, \ell^1)$.*

The second part reformulates into the following :

Proposition 4.3.13. *Let X be a complete connected median space of rank n which admits a topologically transitive action. If there exists $x \in X$ such that for any $\epsilon > 0$, the set of halfspaces $\mathcal{H}(B(x,\epsilon))$ contains a facing triple, then the space X is not locally compact.*

The proof of Proposition 4.3.13 follows exactly the path of the proof of the second part of Theorem D. Regarding the proof of Theorem 4.3.12. One shows that under its assumption, we obtain the same result as in Corollary 4.3.5, that is, the space X is locally isometrically modelled on $(\mathbb{R}^n, \ell^1)$. The rest of the arguments follow more or less the same path.

4.4 Actions with discrete orbit

The aim of this section is to show that an isometric action which is Roller non elementary, Roller minimal and minimal on a complete locally compact median space of finite rank has discrete orbits.

We first remark that under the minimality assumption, every halfspaces are thick. We say that an action is ***minimal*** if the convex hull of any orbit is the whole space.

Proposition 4.4.1. *Let X be a complete connected median space which admits a minimal action. Then any halfspace is thick.*

Proof. Let us assume that X is of rank n, there exist then a n-cube isometrically embedded into X. Let us denote by a_0 its center. Let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace in X. Under the assumption of the existence of a minimal action, there exist an isometry which maps the center of the cube a_0 into $\mathfrak{h}$. We obtains embedded n-cube which has it center inside $\mathfrak{h}$. The rank of X being n, at least a lift of one canonical halfspace of the n-cube which contains the center, contains the halfspace $\mathfrak{h}^c$. We consider then a point inside the n-cube which is at positive distance from the corresponding halfspace. the latter point is necessarily at positive distance from $\mathfrak{h}^c$. □

Any complete connected median space X is geodesic (see Lemma 13.3.2 [Bow22]). Hence by Hopf-Rinow Theorem, showing that a median space X is not locally compact is equivalent to find a closed ball which is not compact.

Lemma 4.4.2. *Let X be a median algebra of finite rank and let $\mathcal{H} \subseteq \mathcal{H}(X)$ be an infinite subset of halfspaces such that any $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{H}$ are either transverse or disjoint. Then there exists an infinite subset $\mathcal{H}' \subseteq \mathcal{H}$ of pairwise disjoint halfspaces.*

Proof. Let us consider the dual graph Γ of $\mathcal{H}$, that is, the non oriented graph whose vertices are the halfspaces of $\mathcal{H}$ and two vertices are joined by and edge if the halfspaces labelling the vertices are transverse, i.e. $\Gamma := (V, E)$ such that $V = \mathcal{H}$ and $(\mathfrak{h}_1, \mathfrak{h}_2) \in E$ if and only if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are transverse. Thus, finding an infinite family of pairwise disjoint halfspaces in $\mathcal{H}$ translates into finding an infinite subset A of the graph Γ consisting of vertices which are pairwise non adjacent. As the rank of the space X is finite, the graph Γ corresponds to the 1-skeleton of finite dimensional simplicial complex. The set of vertices being infinite, the graph is unbounded with regard to its combinatorial metric. Therefore such subset A exists. □

Let G be a group acting by isometries on a median space X. We denote by $Stab_G(x)$ the subgroup of G consisting of isometries which stabilize the point x. If $G = Isom(X)$ we simply write $Stab(x)$. We have the following proposition :

Proposition 4.4.3. *Let X be a complete connected locally compact median space of finite rank and let $x_0, x \in X$. If all the halfspace of $\mathcal{H}_x$ are thick then the orbit $Stab(x_0).x$ is finite.*

Before proving the proposition, we will be needing some results. We have the following lemma which states that any point $x_0 \in X$ is determined by the couple $x \in X$ and $\mathcal{H}_x \cap \mathcal{H}(x_0, x)$:

Lemma 4.4.4. *Let X be a complete connected median space of finite rank. Let us consider $x, x_0 \in X$ and set $C := \bigcap_{\mathfrak{h} \in \mathcal{H}_x \cap \mathcal{H}(x_0,x)} \mathfrak{h}$. We have then $\pi_C(x_0) := x$.*

Note that the convex subset C is closed by Remark 1.2.12, hence the nearest point projection onto C exists.

Proof. By Remark 1.1.15 and Proposition 1.2.14, we have :

$$C \cap [x_0, x] = \Big(\bigcap_{\mathfrak{h} \in \mathcal{H}_x \cap \mathcal{H}(x_0,x)} \mathfrak{h}\Big) \cap [x_0, x] = \{x\}.$$

We conclude by Lemma 1.1.14 that $\pi_C(x_0) = x$. □

Lemma 4.4.5. *Let X be a complete connected median space of finite rank and let $x \in X$. Then for any isometry $g \in Stab(x_0)$ and a closed halfspace $\mathfrak{h} \in \mathcal{H}(X)$ such that $x_0 \in \mathfrak{h}^c$, the halfspaces $\mathfrak{h}$ and $g.\mathfrak{h}$ are either transverse or disjoint.*

Proof. Let us consider $g \in Stab(x_0)$ and a closed halfspace $\mathfrak{h} \in \mathcal{H}'$. As both $\mathfrak{h}^c$ and $(g.\mathfrak{h})^c$ contains x_0, it is enough to show that we have $g.\mathfrak{h} \subseteq \mathfrak{h}$ if and only if $g.\mathfrak{h} = \mathfrak{h}$. Note that the same conclusion will yield with regards to the case when $\mathfrak{h} \subseteq g.\mathfrak{h}$ as we have $g.\mathfrak{h} \subseteq \mathfrak{h}$ if and only if $\mathfrak{h} \subseteq g^{-1}.\mathfrak{h}$. Let us assume then that $g.\mathfrak{h} \subseteq \mathfrak{h}$. We set $\tilde{x}_0 := \pi_{\mathfrak{h}}(x_0)$ and first show that $\pi_{g.\mathfrak{h}}(x_0) = \tilde{x}_0$. In one hand, We have $\tilde{x}_0 \in [\pi_{g.\mathfrak{h}}(x_0), x_0]$, which implies that :

$$d(\pi_{g.\mathfrak{h}}(x_0), x_0) = d(\pi_{g.\mathfrak{h}}(x_0), \tilde{x}_0) + d(\tilde{x}_0, x).$$

In the other hand, we have

$$d(x_0, \tilde{x}_0) = d(g.x_0, g.\tilde{x}_0) = d(x_0, \pi_{g.\mathfrak{h}}(g(x_0))) = d(x_0, \pi_{g.\mathfrak{h}}(x_0)).$$

Hence, we have $d(\tilde{x}_0, \pi_{g.\mathfrak{h}}(x_0)) = 0$.

Finally, we deduce that for any point $a \in \mathfrak{h}$ we have $\tilde{x}_0 \in [a, x_0]$. As $x_0 \in (g.\mathfrak{h})^c$ and $\tilde{x}_0 \in (g.\mathfrak{h})$, the point a cannot lie outside $g.\mathfrak{h}$. Therefore, we do have $g.\mathfrak{h} = \mathfrak{h}$. □

Proof of Proposition 4.4.3. Let us consider $x_0, x \in X$ and $g \in Stab(x_0)$. We denote by $\mathcal{H}'_x$ the set of minimal halfspaces in $\mathcal{H}_x \cap \mathcal{H}(x_0, x)$. By Lemma 4.4.4, any point $x \in X$ is determined by the point x_0 and the set $\mathcal{H}'_x$. Hence, it is enough to show that the orbit of any halfspace by $Stab(x_0)$ is finite. By Lemma 4.4.5, the union of the orbit of each halfspace in $\mathcal{H}'_x$ under $Stab(x_0)$ constitutes a family of halfspaces which are either transverse or disjoint. The space X being assumed to be locally compact, the finiteness of the latter family is ensured by Lemma 4.4.2 and Theorem B. □

Proof of Theorem A. By Proposition 4.4.1, the minimality assumption on the action implies that all halfspaces of X are thick. Let us show that if X admits an action which is Roller minimal and Roller non elementary with a non discrete orbit, then it is not locally compact. Let us set $G := Isom(X)$ and let $x_0 \in X$ such that $G.x_0$ is non discrete. By Proposition 3.2.20, there exists a facing triple of thick halfspaces $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \in \mathcal{H}(X)$ which are uniquely determined by a point $x \in \mathfrak{h}_1^c \cap \mathfrak{h}_2^c \cap \mathfrak{h}_3^c$ in the sense that for any $x_i \in \mathfrak{h}_i$, we have $m(x_1, x_2, x_3) = x$. Let us fix $R > 0$ and let $K \subseteq Isom(X)$ such that $d(x_0, g_i.x_0) \leqslant R$ for any $g \in K$. As the orbit of x_0 under G is not discrete, the subset K is infinite. If $K.x$ is finite, this implies that the orbit of x_0 under $Stab(x)$ is infinite. Hence, by Proposition 4.4.3 the space X would not be locally compact. Let us assume then that $K.x$ is infinite. By Proposition 1.2.13 and Theorem B, it is enough to find an infinite subset in $\mathcal{H} := K.\mathfrak{h}_1 \cup K.\mathfrak{h}_2 \cup K.\mathfrak{h}_3$ which consists of halfspaces which are pairwise disjoint. If $\mathcal{H}$ does not contain an infinite chain then by considering the minimal element of each maximal chain, one obtain a subfamily of halfspaces which are either transverse of disjoint. Hence, by Lemma 4.4.2, there exist an infinite subfamily of pairwise disjoint halfspaces. Let us assume then that there exists an infinite countable chain $\mathcal{H}_1 \subseteq \mathcal{H}$. As $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are disjoints, the chain $\mathcal{H}_1$ is given by $(g_{1,n}.\mathfrak{h}_{i_n})_{n\in\mathbb{N}}$ where $g_{1,n} \in K$ and $i_n \in \{1, 2, 3\}$. Let us set $\mathcal{H}' := \bigcup_{i\in\mathbb{N}}(g_{1,n}.\mathfrak{h}_1 \cup g_{1,n}.\mathfrak{h}_2 \cup g_{1,n}.\mathfrak{h}_3)$ and note that $\mathcal{H}'\backslash\mathcal{H}_1$ is infinite. Again, If the subset $\mathcal{H}'\backslash\mathcal{H}_1$ does not contain an infinite chain then we are done. Let us assume then that $\mathcal{H}'\backslash\mathcal{H}_1$ contains an infinite chain $\mathcal{H}_2$. Such chain is given by $(g_{2,n}.\mathfrak{h}_{j_n})_{n\in\mathbb{N}}$ where $g_{2,n} \in K$ and $j_n \in \{1, 2, 3\}$. Note that as the halfspaces $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are pairwise strongly separated, for any isometry $g \in G$, the halfspace $g.\mathfrak{h}_i$ cannot intersect two halfspaces in $\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3\}$. Hence, if for each $i \in \{1, 2, 3\}$ such that $g.\mathfrak{h}_i$ intersects a halfspace in $\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3\}$, then there exists a permutation $\sigma \in S_3$ such that $g.\mathfrak{h}_i$ intersects only the halfspace $\mathfrak{h}_{\sigma(i)}$. In the latter case, we necessarily have $g(x) = x$ as for any $x_i \in \mathfrak{h}_i \cap \mathfrak{h}_{\sigma(i)}$ we have $m(x_1, x_2, x_3) = x$ and $m(x_1, x_2, x_3) = g(x)$. As we are considering the isometries $g \in K$ such that $g(x) \neq x$, we conclude that the infinite subset $\bigcup_{i\in\mathbb{N}}(g_{2,n}.\mathfrak{h}_1 \cup g_{2,n}.\mathfrak{h}_2 \cup g_{2,n}.\mathfrak{h}_3)\backslash(\mathcal{H}_1 \cup \mathcal{H}_2)$ consists of pairwise disjoint halfspaces, which completes the proof. □

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