Parabolic Hitchin connection

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Connexion de Hitchin parabolique

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Abstract

Zakaria OUARAS

Parabolic Hitchin connection

Let $\mathcal{C}/S$ be a smooth family of projective complex curves of genus $g \geqslant 2$ parameterized by a complex variety S and take a divisor on $\mathcal{C}$ of relative degree N given by N disjoint sections of $\mathcal{C}/S$. For a fixed parabolic type $\alpha_* = (k, \vec{a}, \vec{m})$ we associate $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ the relative moduli space of parabolic rank-r vector bundles of parabolic type α_* with fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$. This moduli space is equipped with a polarization given by Θ_{par} the parabolic determinant line bundle (which depends on the parabolic type α_*).

In this dissertation we study the existence of a projective connection, the so called *Hitchin connection* on the pushforward of Θ_{par} to S with algebro-geometric techniques. The main tool is the notion of *Heat operators* in algebraic geometry introduced by van Geemen and de Jong. We take the quadratic part of the parabolic Hitchin system, we call parabolic symbol (which depends only on the quasi-parabolic type $\vec{m}$ and not on the weights $\vec{a}$). We prove that it is invariant under Hecke modifications and that it satisfies the van Geemen-de Jong criterion for the existence of a heat operator with this symbol. Thus we obtain a projective connection, which turns out to be flat.

To do this we define two Atiyah-type exact sequences in the parabolic context that allow us to prove a deformation theorem for marked curves equipped with a quasi-parabolic vector bundle. Hence we get a factorisation of the Kodaira-Spencer map of the family of moduli spaces $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ along the Kodaira-Spencer map of the family of marked curves $\mathcal{C}/S$ and a description of the Atiyah class of the pullbacks of the determinant line bundles under the forgetful maps to the moduli space $\mathcal{SU}_{\mathcal{C}/S}(r, *)$ of semi-stable rank-r vector bundles with fixed determinant $*$. The key ingredient to conclude is a decomposition of the parabolic determinant bundle Θ_{par} and of the canonical bundle over the moduli space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$.

Key Words: Parabolic vector bundles, Filtered vector bundles, Moduli spaces, Determinant parabolic line bundle, Heat operator, Hitchin connection.

Résumé

Connexion de Hitchin parabolique

Soit $\mathcal{C}/S$ une famille lisse de courbes projectives complexes de genre $g \geqslant 2$ paramétrées par une variété complexe S et prenons un diviseur sur $\mathcal{C}$ de degré relatif N, donné par N sections différentes de la famille $\mathcal{C}/S$. Pour un type parabolique $\alpha_* = (k, \vec{a}, \vec{m})$ fixé, on considère $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ l'espace de modules relatif des fibrés vectoriels paraboliques de rang-r de type parabolique α_* et de déterminant fixe $\delta \in \mathrm{Pic}(\mathcal{C}/S)$. Cet espace de modules est muni d'une polarisation donnée par Θ_{par} le fibré parabolique déterminant (dépendant du type parabolique α_*).

Dans cette thèse nous étudions l'existence d'une connexion projective que nous appelons la *connexion de Hitchin* sur l'image direct de Θ_{par} sur S avec des techniques algébro-géométrique. L'outil principal est la notion *d'opérateur de la Chaleur* en géométrie algébrique introduite par van Geemen-de Jong. On prend la partie quadratique du système de Hitchin parabolique ρ_{par}, que l'on appelle le symbole parabolique (qui ne dépend que du type quasi-parabolique $\vec{m}$ et non des poids $\vec{a}$). Nous prouvons qu'il est invariant sous les modifications de Hecke et qu'il satisfait le critère de van Geemen et de Jong, donc il se relève à un opérateur de la chaleur à valeurs dans Θ_{par} et de symbole ρ_{par}. Donc on obtient une connexion projective sur l'image direct de Θ_{par} sur S, qui s'avère être une connexion plate.

Pour ce faire nous définissons dans le contexte parabolique deux suites exacte de type Atiyah, qui nous permettent de démontrer un théorème de déformation pour des courbes marquées munies d'un fibré vectoriel quasi-parabolique. On obtient donc une factorisation du morphisme de Kodaira-Spencer de la famille des espaces de modules $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ le long du Kodaira-Spencer de la famille des courbes marquées $\mathcal{C}/S$ et une description de la classe d'Atiyah des pull-backs des fibrés déterminants sous les applications d'oubli vers les espaces des modules $\mathcal{SU}_{\mathcal{C}/S}(r, *)$ de fibrés vectoriels semi-stables de rang-r et de déterminant fixe $*$. L'ingrédient clef pour conclure est la décomposition du fibré déterminant parabolique Θ_{par} et du fibré canonique sur l'espace de modules $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$.

Mots-clefs: Fibrés paraboliques, Fibrés filtrés, Espaces de modules, Fibrés déterminants parabolique, Opérateur de la chaleur, Connexion de Hitchin.

Je dédie cette thèse à mes parents

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[1] Ça va me manquer de discuter des maths en rentrant à pied du labo sur la promenade des Anglais.
[2] On a fait un long chemin ensemble: Maroc, M2 Nice puis les quatre années de thése.
[3] Je tiens à notre spectable de blagues de maths.
[4] Que je qualifie de sage, ses analyses de situation me fascinent et j'ai beaucoup appris avec lui.

Contents

Introduction

The main object of study of this thesis is the Hitchin connection on the sheaf of non-abelian theta functions on a family of smooth projective curves. In this manuscript we will extend Hitchin's construction of the connection to the sheaf of parabolic non-abelian theta functions on a family of marked curves.

In the first part of this introduction we start with recalling Hitchin's original work and present in chronological order the main contributions leading to an algebro-geometric construction of Hitchin's connection for parabolic bundles.

In a second part we will outline a new approach based on an interpretation of parabolic bundles as $\mathbb{R}$-filtered bundles and a thorough study of the Picard group of the moduli space of parabolic bundles.

Part I

I-1) Hitchin's original construction: Hitchin in [Hit90a] constructed a projective flat connection in the context of geometric quantization, motivated by the invariant of 3-manifolds introduced in Witten [Wit89] in the study of quantum Chern-Simons theory, which can be seen as a vector space V canonically associated to a closed topological surface X, a compact Lie group G and an integer k, called the level. The vector space V is related to the geometric quantization of a (real) symplectic manifold M. For a given Lie group G one associates the irreducible representations of the fundamental group $\pi_1(X)$ into G modulo conjugation

$$\mathrm{Hom}_{irr}\left(\pi_1(X), G\right)/G$$

which is canonically a symplectic manifold (M, ω) by the Atiyah-Bott-Goldman form [AB83]. Hitchin quantizes this manifold with a Kähler polarization since a conformal structure on the surface X induces a complex structure on M, hence a Kähler polarisation. By the Narasimhan-Seshadri Theorem M is identified with the moduli space of stable holomorphic $G^{\mathbb{C}}$-bundles [5] over the Riemann surface $C = (X, J)$ and for each Kähler polarization the vector space V (defined up to a scalar factor) can be seen as the space of

[5] $G^{\mathbb{C}}$ is the complex Lie group with compact form G

global sections of a pre-quantum line bundle L.

$$(X, J) \rightsquigarrow (M_J, L_J) \rightsquigarrow V_J := \mathrm{H}^0(M_J, L_J)$$

Hitchin gives conditions of existence of a projectively flat connection for a family of Kähler polarisations on M induced by a family of complex structures J on X.

Theorem 0.0.1 (Hitchin [Hit90a], Theorem 1.20) *Given a family of Kähler polarisation on M, such that for each polarisation we have*

1. *The map given by cup-product with the class $[\omega]$ of L*

$$\cup[\omega] : \mathrm{H}^0(M, T_M) \longrightarrow \mathrm{H}^1(M, \mathcal{O}_M)$$

is an isomorphism (This means that there is no global vector field fixing L, i.e. $\mathrm{H}^0(M, \mathcal{D}_M^{(1)}(L)) = \mathrm{H}^0(M, \mathcal{O}_M)$).

2. *For each $s \in \mathrm{H}^0(M, L)$ and a tangent vector I to the base of the family there exists a smoothly varying*

$$A(I, s) \in \mathbb{H}^1(M, \mathcal{D}_M^{(1)}(L) \xrightarrow{\cdot s} L)$$

such that the symbol $-\mathrm{i}\nabla_1(A(I, s))$ equals the Kodaira-Spencer class $[I]$ in $\mathrm{H}^1(M, T_M)$.

Then this defines a projective connection on the bundle of projective spaces $\mathbb{P}(\mathrm{H}^0(M, L))$ over the base of the family.

Here $\mathcal{D}_M^{(1)}(L)$ is the sheaf of first order differential operators on L and ∇_1 its symbol map to T_M. $\mathbb{H}^1$ stands for the first hypercohomology group of the two-term complex $\mathcal{D}_M^{(1)}(L) \xrightarrow{\cdot s} L$ given by evaluation on s which parameterizes the infinitesimal deformations of the triple (M, L, s), for $s \in \mathrm{H}^0(M, L)$ ([Wel83], Proposition 1.2).

Moreover, Hitchin showed that (M, ω) satisfies the condition of the theorem, where (M, L) is the space of flat unitary trace-free connections on the trivial rank-r bundle (case $G = \mathrm{SU}(r)$) over a closed oriented surface X of genus $g \geqslant 2$ (exception $r = g = 2$), which is not a manifold but its smooth locus is equipped with a canonical symplectic form and $L \cong \mathcal{L}^k$ is the k-th power of the ample generator of the Picard group. By the Narasimhan-Seshadri Theorem M is the moduli space of semi-stable rank-r vector bundles with trivial determinant, thus a projective variety. The symplectic form ω is a Kähler form and the inverse of the determinant line bundle provides a pre-quantum line bundle. See [Qui85]. In this case we can describe the map

$$A(-, s) : \mathrm{H}^1(C, T_C) \longrightarrow \mathbb{H}^1(M, \mathcal{D}_M^{(1)}(\mathcal{L}^k) \xrightarrow{\cdot s} \mathcal{L}^k),$$

as follows: take the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}_M^{(1)}(\mathcal{L}^k) & \longrightarrow & \mathcal{D}_M^{(2)}(\mathcal{L}^k) & \longrightarrow & \mathrm{Sym}^2(T_M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot s} & & \downarrow{\scriptstyle \cdot s} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}^k & \longrightarrow & \mathcal{L}^k & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with connecting morphism

$$\delta_1 : \mathrm{H}^0(M, \mathrm{Sym}^2(T_M)) \longrightarrow \mathbb{H}^1(M, \mathcal{D}^{(1)}_M(\mathcal{L}^k) \xrightarrow{\cdot s} \mathcal{L}^k). \tag{0.0.1}$$

Thus the map $A(-, s)$ is given by δ_1 pre-composed with the quadratic part of the Hitchin system ρ

$$\rho : \mathrm{H}^1(C, T_C) \longrightarrow \mathrm{H}^0(M, \mathrm{Sym}^2(T_M)).$$

Hitchin proved that $\frac{1}{r+k}A(-, s)$ satisfies the second condition of Theorem 0.0.1, hence the existence of a projectively flat connection over $\mathcal{V}_k$, the push-forward of $\mathcal{L}^k$ to the Teichmüller space. In [Hit90b] Hitchin generalizes his construction of the connection to the parabolic case over the projective line.

I-2) A first generalization to parabolic bundles: Scheinost and Schottenloher [SS95] generalize Hitchin's construction to any dimension and also deal with the case of parabolic vector bundles in dimension one. Take X a Kähler manifold, and for Lie group G associate the space $\mathrm{H}^1(X, G)$ (non-abelian cohomology). They study the case where $G = \mathrm{SU}(r)$ hence the space $\mathrm{H}^1(X, G)$ can be identified with the moduli space M of semi-stable holomorphic rank-r vector bundles E on X with total Chern class $c(E) = 1$ and $\det(E) = \mathcal{O}_X$. For a compact Kähler manifold this space is a compact, complex variety with singularities and not necessary connected and equipped with a natural Kähler form that represents in some cases the Chern class of natural ample line bundle $\mathcal{L}$ that generalises the determinant line bundle in dimension one.

Theorem 0.0.2 *Let M a smooth connected and simply-connected component of $\mathrm{H}^1(X, \mathrm{SU}(r))$. Let $(I_t, \omega_t)_{t\in B}$ be a family of Kähler structures on M, induced by an algebraic family of Kähler structures of X, such that*

1. *$\omega_t = \omega_M$ is a fixed integral form,*
2. *$(I_t)_{t\in B}$ is a holomorphic family of complex structures on M given by holomorphic map $\varpi : \mathcal{M} \longrightarrow B$.*

Let $\mathcal{L}$ be a universal bundle over $\mathcal{M}$, such that $\mathcal{L}|_{\mathcal{M}_t}$ is the generalized theta line bundle over $M_t = (M, I_t)$ for $t \in B$. Let $c_1(\mathcal{M}) = 0 \bmod[2]$, and $K^{1/2}_{\mathcal{M}}$ a square root of the canonical bundle. Then for all integer $k > 0$:

1. *The direct image sheaf $\mathcal{W}_k := \varpi_*\left(\mathcal{L}^k \otimes K^{1/2}_{\mathcal{M}}\right)$ is locally free.*
2. *There exists a natural projectively flat connection on the vector bundle $\mathcal{W}_k$ on B.*

The twist by $K^{1/2}_{\mathcal{M}}$ is what they call a metaplectic correction.

A special case is the case of *elliptic surfaces* which is related to parabolic vector bundles over curves. We recall the definition

Definition 0.0.3 *An elliptic surface is a smooth compact complex surface X with a fibration (proper surjective holomorphic morphism) onto Riemann surface C*

$$f : X \longrightarrow C,$$

such that the generic fibre is an elliptic curve.

Bauer [Bau91] gives a description of the space $\mathrm{H}^1(X, SU(r))$ for X an elliptic surface in term of objects over the Riemann surface C.

Theorem 0.0.4 *Let $f : X \longrightarrow C$ be an elliptic fibration with*

$$b_1(X) \text{ even, } \chi(\mathcal{O}_X) > 0, \text{ and } kod(X) = 1.$$

Then there is an isomorphism between

- *The moduli space of semi-stable rank-r parabolic bundles of degree 0 on the curve C with N marked points $x_1, x_2, ..., x_N$ and with certain rational weights.*
- *A corresponding component of the moduli space of semi-stable rank-r vector bundles on X with $c(E) = 1$ and $\det(E) = \mathcal{O}_X$.*

Thus by applying Theorem 0.0.2 under the assumption that the canonical bundle admits a square root, they get a projectively flat connection over the pushforward of the generalized determinant line bundle with a metaplectic correction [6]. Bjerre [Bje18] (Theorem 10.1) removed this restriction and proved the existence of the Hitchin connection over the space $\mathcal{W}_k^0 := \varpi_* \left(\mathcal{L}^k\right)$ in Theorem 0.0.2 by working on different moduli spaces of parabolic bundles, and using a general construction of the Hitchin connection in geometric quantization, as done in [And12] and the Hitchin connection in the setting of metaplectic quantization, as done in [AGL12].

I-3) The use of heat operators by van Geemen and de Jong: Hitchin uses methods from differential geometry and Kähler geometry. There exist several works related to algebro-geometric constructions of the Hitchin connection: Faltings [Fal93], Ran [Ran06], Sun-Tsai [ST04]. In this manuscript we mainly use the approach given by van Geemen-de Jong [GdJ98]. One of their main results is an algebraic criterion for the existence of the Hitchin connection. One of the three conditions of their criterion is the following

$$\mu_L \circ \rho = -\kappa_{\mathcal{M}/S},$$

where $\kappa_{\mathcal{M}/S}$ is the Kodaira-Spencer map of a family $\mathcal{M} \longrightarrow S$ parameterized by a variety S an μ_L is a map associated to the line bundle L over $\mathcal{M}$ and ρ a symbol map (for the details see chapter 3). It should be noted that van Geemen-de Jong do not show that the family of moduli spaces of vector bundles satisfies their criterion, this was done later in [BBMP23]. They only construct by different methods Hitchin's connection in rank two, genus two.

[6] Note that we don't know how the generalized determinant bundle introduced in [SS95] and the parabolic determinant bundle are related

I-4) The work of Baier-Bolognesi-Martens-Pauly [BBMP23] give an algebro-geometric construction of the Hitchin connection over the relative moduli space $\mathcal{SU}_{\mathcal{C}/S}(r)$ of semi-stable rank-r vector bundles of trivial determinant over a family of complex projective curves of genus $g \geqslant 2$ (except the case $r = g = 2$) parameterized by a variety S using the so-called trace complexes [BS88], the Bloch-Esnault quasi-isomorphism [BE99], [ST04] to give a description of the Atiyah class of the determinant line bundle $\mathcal{L}$. The setting is the following

$$\begin{array}{ccc} \mathcal{X} = \mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r) & \xrightarrow{p_n} & \mathcal{SU}_{\mathcal{C}/S}(r) \\ \downarrow{\scriptstyle p_w} & & \downarrow{\scriptstyle p_e} \\ \mathcal{C} & \xrightarrow[p_s]{} & S \end{array}$$

They prove that: For a virtual universal bundle $\mathcal{U}$ over $\mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r)$ one has an isomorphism of sheaves

$$R^1 p_{n*}\left(\left[\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}(r)}(\mathcal{U})\right]^\vee\right) \cong \mathcal{A}_{\mathcal{SU}_{\mathcal{C}/S}(r)/S}(\mathcal{L}),$$

where $\mathcal{L}$ is the ample generator of the relative Picard group $\mathrm{Pic}\left(\mathcal{SU}_{\mathcal{C}/S}(r)/S\right)$, $\mathcal{A}_{\mathcal{SU}_{\mathcal{C}/S}(r)/S}(\mathcal{L})$ its Atiyah algebra and $R^1 p_{n*}\left(\left[\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}(r)}(\mathcal{U})\right]^\vee\right)$ the first direct image of the traceless dual of the Atiyah algebra of the virtual universal bundle $\mathcal{U}$ with respect to the projection p_{n*}. As symbol map they take a multiple of the quadratic part of the Hitchin system ρ^{Hit} which is the classical Hitchin symbol, they prove that it satisfies the van Geemen-de Jong criterion for $\mathcal{L}^k$ a positive power of the determinant line bundle $\mathcal{L}$.

$$\mu_{\mathcal{L}^k} \circ \frac{1}{(r+k)}\left(\rho^{Hit} \circ \kappa_{\mathcal{C}/S}\right) = -\kappa_{\mathcal{SU}_{\mathcal{C}/S}(r)/S},$$

where $\kappa_{\mathcal{C}/S}$ and $\kappa_{\mathcal{SU}_{\mathcal{C}/S}(r)/S}$ are the Kodaira-Spencer maps of the family of curves and of the relative moduli space respectively.

I-5) Setting of the problem: Let $\pi_s : \mathcal{C} \longrightarrow S$ be a smooth family of projective curves of genus $g \geqslant 2$ parameterized by a projective variety S and let $\sigma_i : S \longrightarrow \mathcal{C}$ for $i \in I = \{1, 2, ..., N\}$ N disjoint sections of π_s, i.e. ,$\forall i \neq j$ and $\forall s \in S$, we have: $\sigma_i(s) \neq \sigma_j(s)$. We note $D = \sum_{i=1}^N \sigma_i(S)$ the associated divisor of relative degree N.
A rank-r parabolic type with respect to D is a triple $\alpha_* := (k, \vec{a}, \vec{m})$, given by

- A quasi-parabolic type $\vec{m} = (\ell_i, m(i))_{i \in I}$, where $\ell_i \in \mathbb{N}^*$, a sequence of integers $m(i) = (m_1(i), m_2(i), ..., m_{\ell_i}(i))$, $m_j(i) \in \mathbb{N}^*$ and satisfies the relation $\sum_{j=1}^{\ell_i} m_j(i) = r$.
- A system of parabolic weights $(k, \vec{a})$, where $k \in \mathbb{N}^*$ and $\vec{a} = (a_j(i))_{\substack{i \in I \\ 1 \leqslant j \leqslant \ell_i}}$ a sequence of integers satisfying for each $i \in I$

$$0 \leqslant a_1(i) < a_2(i) < ... < a_{\ell_i}(i) < k.$$

For a fixed rank-r parabolic type $\alpha_* = (k, \vec{a}, \vec{m})$ with respect to the parabolic divisor D and a relative line bundle $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ (i.e. line bundles over $\mathcal{C}$ relatively to S), we denote by

$$\pi_e : \mathcal{SM}^{par}_{\mathcal{C}/S} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) \longrightarrow S,$$

the relative moduli space of parabolic rank-r vector bundles over $(\mathcal{C}, D)/S$ of determinant δ and parabolic type α_* equipped with the ample line bundle $\Theta_{par} \in \mathrm{Pic}\left(\mathcal{SM}^{par}_{\mathcal{C}/S}/S\right)$ called the parabolic determinant bundle.

$$\begin{array}{ccc} \mathcal{X}^{par} := \mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S} & \xrightarrow{\pi_n} & \mathcal{SM}^{par}_{\mathcal{C}/S} \\ \downarrow \pi_w & & \downarrow \pi_e \\ (\mathcal{C}, D) & \underset{\pi_s := p_s}{\longrightarrow} & S \\ & \underset{\sigma_i}{\longleftarrow} & \end{array} \tag{0.0.2}$$

Question: For any data as above is there for $\nu \in \mathbb{N}^*$ a projective flat connection on the vector bundle $\mathcal{V}^{par}(\alpha_*, \delta, \nu) := \pi_{e*}\left(\Theta^{\nu}_{par}\right)$ over S associated to a heat operator ?.

Motivation: In the theory of conformal blocks, Tsuchiya-Ueno-Yamada [TUY89] constructed for the Lie algebra $sl_r(\mathbb{C})$ the vector space $\mathbb{V}_C(D, \vec{\lambda}, k)$, called the space of conformal blocks, where $\vec{\lambda} = \{\lambda_x\}_{x \in D}$ is an N-tuple of dominant weights of $sl_r(\mathbb{C})$, k is an integer and (C, D) denotes a marked curve. This vector spaces can be glued to a vector bundle of conformal blocks $\mathbb{V}(D, \vec{\lambda}, k)$ over the moduli space $\mathcal{M}_{g,N}$ parameterizing N-pointed curves of genus-g. In [TUY89] they constructed a projective flat connection over the vector bundle $\mathbb{V}(D, \vec{\lambda}, k)$. Moreover Beauville-Laszlo [BL94] proved that the vector spaces V_J in Hitchin's constructions are canonically identified with $\mathbb{V}_C(D, \vec{\lambda}, k)$ over a curve with one point and trivial weight. It is natural to inquire whether the Hitchin connection [Hit90a] and the Tsuchiya-Ueno-Yamada connection [TUY89] coincide. This question was addressed and proven by Laszlo [Las98]. Pauly [Pau96] gave a generalization of Beauville-Laszlo's identification of the space of non-abelian parabolic theta functions $\mathrm{H}^0(\mathcal{SM}^{par}_C, \Theta_{par})$ with the conformal blocks $\mathbb{V}_C(D, \vec{\lambda}, k)$, where the dominant weights $\vec{\lambda}$ and the integer k depend on the parabolic weights. By Tsuchiya-Ueno-Yamada [TUY89] we know that the vector bundle $\mathbb{V}(D, \vec{\lambda}, k)$ is equipped with a flat projective connection, hence it is natural to ask the above question, and then to compare the two connections via the projective isomorphism. This problem has been solved in [BMW21a] and [BMW21b] for semi-simple structure group G which correspond to trivial determinant parabolic bundles.

I-6) Work of Biswas-Mukhopadhyay-Wentworth: In [BMW21a] the authors give a proof of the existence of the Hitchin connection for parabolic G-bundles (we present here the case $G = SL_r$ [7]) following [BBMP23]. Their strategy uses *Galois coverings* as following:

Let h be a Galois covering

$$h : \widehat{C} \longrightarrow (C, D),$$

where $D = \{x_1, x_2, ..., x_N\}$ is a subset of the ramification divisor and let Γ be the Galois group of the covering such that $\widehat{C}/\Gamma = C$. Then we have the following correspondence. See [MS80], [Bho89] and [BR93].

Theorem 0.0.5 ([Ses77]) *There is an isomorphism between the two moduli spaces*

- $\mathcal{SM}_C^{par}(r,d)$ *the moduli space of parabolic rank-r vector bundles over* (C, D) *of fixed degree and parabolic type with certain rational weights.*
- $\mathcal{SU}_{\widehat{C}}^{\Gamma}(r,d)$ *the moduli space of* Γ*-bundles over* $\widehat{C}$ *of fixed local type.*

They use this theorem for trivial determinant vector bundles. Take the forgetful map

$$Q : \mathcal{SM}_C^{par}(r) \cong \mathcal{SU}_{\widehat{C}}^{\Gamma}(r) \longrightarrow \mathcal{SU}_{\widehat{C}}(r),$$

which associates to a parabolic bundle over C the Γ-bundle by the Galois covering and forget the Γ-linearisation to get a rank-r vector bundle over $\widehat{C}$.

Now let $h : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$ be a family of Galois coverings parameterized by a variety S. By [BBMP23] Proposition 4.7.1 over the space $\mathcal{SU}_{\widehat{\mathcal{C}}/S}(r)$ applied to the determinant line bundle $\widehat{\mathcal{L}}$, we have the equation

$$\cup\, [\widehat{\mathcal{L}}] \circ \left(\widehat{\rho}^{Hit} \circ \kappa_{\mathcal{C}/S}\right) = -\kappa_{\mathcal{SU}_{\mathcal{C}/S}(r)/S}, \tag{0.0.3}$$

where $\widehat{\rho}^{Hit}$ is the Hitchin symbol map over the family of curves $\widehat{\mathcal{C}}/S$. Their idea is to use the map Q to transport the equation (0.0.3) to the space $\mathcal{SM}_{\mathcal{C}/S}^{par}(r)$. They prove in [BMW21a] Theorem 5.3 the following equality (metaplectic correction) for $\widehat{\mathcal{L}}_Q := Q^*(\widehat{\mathcal{L}})$

$$\cup[\widehat{\mathcal{L}}_Q] \circ \left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right) = -\kappa_{\mathcal{SM}_{\mathcal{C}/S}^{par}(r)/S},$$

where ρ_{par} is the quadratic part of the parabolic Hitchin system, the parabolic Hitchin symbol map, and $\kappa_{\mathcal{C}/S}$ (resp. $\kappa_{\mathcal{SM}_{\mathcal{C}/S}^{par}(r)/S}$) is the Kodaira-Spencer map of the family of marked curves (resp. the Kodaira-Spencer map of the family of relative moduli space which depends on the Galois cover). To conclude they prove that the map $\mu_{\widehat{\mathcal{L}}_Q}$ is an isomorphism, hence define a modified symbol map for the line bundle $(\widehat{\mathcal{L}}_Q)^a$ for a positive integer a by

$$\rho_{par,\Gamma}^{Hit}(a) := \mu_{(\widehat{\mathcal{L}}_Q)^a}^{-1} \circ \left(\cup[\widehat{\mathcal{L}}_Q] \circ \rho_{par} \circ \kappa_{\mathcal{C}/S}\right). \tag{0.0.4}$$

[7] Correspond to parabolic bundles with trivial determinant.

In a second paper [BMW21b] they prove that the map $\cup[\widehat{\mathcal{L}}_Q]$ is independent of the parabolic weights in the full flag case, using abelianization and parabolic Higgs bundles. Hence the modified symbol map (0.0.4) is independent of the parabolic weights (as ρ_{par} and $\kappa_{\mathcal{C}/S}$ are independent on the weights) using the equality $\widehat{\mathcal{L}}_Q := Q^*(\widehat{\mathcal{L}}) \cong \Theta_{par}^{|\Gamma|/k}$, given in [BR93] (Proposition 4.14), where Γ is the Galois group of the Galois covering $h : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$, $|\Gamma|$ its order and $k :=$l.c.m $\{$denominators of the rational parabolic weights$\}$. This modified symbol can be written for any positive integer $a \in \mathbb{N}^*$ as follow

$$\rho^{Hit}_{par,\Gamma}(a) := |\Gamma| \; \mu^{-1}_{\Theta^a} \circ \left(\cup[\Theta] \circ \rho_{par} \circ \kappa_{\mathcal{C}/S}\right),$$

where Θ is the pull-back of the determinant line bundle by $\mathcal{SM}^{par}_{\mathcal{C}/S}(r) \longrightarrow \mathcal{SU}_{\mathcal{C}/S}(r)$ the forgetful map (forget the parabolic structure) over the family of curves $\mathcal{C}/S$.

We follow the same strategy as in [BBMP23]: first we take as symbol map ρ_{par}, the quadratic part of the parabolic Hitchin system, and we prove the van Geemen-de Jong criterion for the parabolic determinant line bundle, which is the equation

$$\mu_{\Theta_{par}} \circ \left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right) = -(k + r) \; \kappa_{\mathcal{SM}^{par}_{\mathcal{C}}/S}.$$

Our work is independent of the work of [BMW21a]. The objects that we define $\mathcal{A}^{par}_X(E)$ the parabolic Atiyah algebroid and $\mathcal{A}^{par,st}_X(E)$ the strongly parabolic Atiyah algebroid are intrinsically attached to the marked curve and the quasi-parabolic type. Our proof is over $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ the relative moduli space of parabolic rank-r vector bundles with fixed parabolic type α_* and fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$, where as [BMW21a] assume that $\delta = \mathcal{O}_{\mathcal{C}}$.

Part II: Main results

We first define two Atiyah-type algebroids and exact sequences that we denote $\mathcal{A}^{par}_X(E)$ and $\mathcal{A}^{par,st}_X(E)$ called respectively parabolic and strongly parabolic Atiyah algebroids that depends only on the marked curve and the quasi-parabolic structure. See section 4.1 for definitions.

II-1) Filtered bundles

Let α_* be a fixed rank-r parabolic type. Our strategy is to use the description of parabolic bundle as filtered bundles via Hecke modification 1.3. See chapter 2 for more details on filtered bundles.

Definition (Filtered bundles) *Let (C, D) be a marked curve and E a vector bundle over C. A filtered bundle structure on E is given by $E_\bullet = (E_\lambda)_{\lambda \in \mathbb{R}}$ a left continuous decreasing $\mathbb{R}$-filtration of locally free rank-r bundles over C where $E_0 = E$, and such that*

1. *The length of the filtration is finite over* $[0, 1]$.

2. *Periodic: For all $\lambda \in \mathbb{R}$, we have $E_{\lambda+1} = E_\lambda(-D)$.*

A system of weights $\lambda_\bullet$ of a filtered bundle $E_\bullet$ is given by:

1. *The jumping real numbers in the real interval $[0,1]$; that we suppose rational, and*

2. *The lengths of the torsion sheaves E/E_λ at the points of the divisor D for $\lambda \in [0,1]$.*

Yokogawa-Maruyama [Yok91] constructed for a fixed system of weights $\lambda_\bullet$ over a smooth family of marked curves $\pi_s : (\mathcal{C}, D) \longrightarrow S$ a moduli space $\mathcal{M}_\bullet(r, \lambda_\bullet, \delta)$ parameterizing semi-stable rank-r filtered bundles of determinant $\det(E_0) = \delta \in \mathrm{Pic}^d(\mathcal{C}/S)$. For a fixed rank-r parabolic type α_* with respect to the divisor D we can associate a filtered system of weights $\lambda_\bullet$ and get an isomorphism of S-varieties

$$\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) \longrightarrow \mathcal{M}_\bullet(r, \lambda_\bullet, \delta)$$

where we associate to a parabolic bundle $\mathcal{E}_*$ its Hecke filtration

$$\mathcal{E}(-\sigma_i(S)) = \mathcal{H}_i^{\ell_i}(\mathcal{E}) \subset \mathcal{H}_i^{\ell_i-1}(\mathcal{E}) \subset \cdots \subset \mathcal{H}_i^2(\mathcal{E}) \subset \mathcal{H}_i^1(\mathcal{E}) \subset \mathcal{H}_i^0(\mathcal{E}) = \mathcal{E}.$$

where $\mathcal{H}_i^j(\mathcal{E})$ are Hecke modifications of the vector bundle $\mathcal{E}$ with respect to the $(j+1)$-th element of the flag at the i-th section $\sigma_i : S \longrightarrow \mathcal{C}$ of the map π_s (see section 2.2.1). We define for each rational number $\gamma \in \mathbb{Q}$ a shift map $\mathcal{H}_\gamma$, as follows

$$\begin{array}{rccc} \mathcal{H}_\gamma : & \mathcal{M}_\bullet(r, \lambda_\bullet, \delta) & \longrightarrow & \mathcal{M}_\bullet(r, \mathcal{H}_\gamma(\lambda_\bullet), \mathcal{H}_\gamma(\delta)) \\ & \mathcal{E}_\bullet = (\mathcal{E}_\lambda)_{\lambda\in\mathbb{R}} & \longmapsto & \mathcal{E}[\gamma]_\bullet = (\mathcal{E}_{\lambda+\gamma})_{\lambda\in\mathbb{R}} \end{array}$$

where $\mathcal{H}_\gamma(\lambda_\bullet)$ is the shifted system of weights and $\mathcal{H}_\gamma(\delta) := \det(E_\gamma)$.

II-2) A parabolic version of Beilinson-Schechtman-Bloch-Esnault theorem

For $\mathcal{E}_\bullet = (\mathcal{E}_\lambda)_{\lambda\in\mathbb{R}}$ a "virtual universal" filtered bundle over $\mathcal{X}_\bullet := \mathcal{C} \times_S \mathcal{M}_\bullet(r, \lambda_\bullet, \delta)$ (see Definition 1.2.14). For each $\lambda \in \mathbb{R}$ we associate the following exact sequence

$$0 \longrightarrow R^1\pi_{n*}(K_{\mathcal{X}_\bullet/\mathcal{M}_\bullet}) \longrightarrow R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}_\bullet/\mathcal{M}_\bullet}(\mathcal{E}_\lambda)(\mathcal{D})\right]^\vee\right) \longrightarrow R^1\pi_{n*}(\mathrm{parEnd}^0(\mathcal{E}_\lambda)) \longrightarrow 0$$

given by taking the first direct image of the dual of the strongly parabolic trace free Atiyah sequence twisted by the divisor $\mathcal{D} := D \times_S \mathcal{M}_\bullet(r, \lambda_\bullet, \delta)$, we denote its extension class by $\Delta(\lambda)$ and we set $\partial_\lambda := \cup \Delta_\lambda$.

We denote by $\Theta(\lambda)$ the pull-back of the ample generator of the relative group $\mathrm{Pic}\left(\mathcal{SU}_\mathcal{C}(r, \delta(\lambda))/S\right)$ by the classifying rational map

$$\begin{array}{rccc} \phi_\lambda : & \mathcal{M}_\bullet(r, \lambda_\bullet, \delta) & \longrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta(\lambda)) \\ & \mathcal{E}_\bullet & \longmapsto & \mathcal{E}_\lambda \end{array}$$

we set $d(\lambda) = \deg \delta(\lambda)$ and $n(\lambda) = \gcd(r, d(\lambda))$.

We prove a parabolic version of [BBMP23] Theorem 4.4.1.

Theorem 4.3.3: *For each $\lambda \in \mathbb{R}$, we have the following isomorphism of short exact sequences over $\mathcal{M}_\bullet(r, \lambda_\bullet, \delta) \cong \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$*

$$\begin{array}{ccccc} R^1\pi_{n*}\left(K_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}/S}}\right) & \hookrightarrow & R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}^{par}}(\mathcal{E}_\lambda)(\mathcal{D})\right]^\vee\right) & \twoheadrightarrow & R^1\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E}_\lambda)\right) \\ \frac{r}{n(\lambda)} \downarrow \simeq & & \downarrow \cong & & \downarrow \cong \\ \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}}} & \hookrightarrow & \mathcal{A}_{\mathcal{SM}^{par}_{\mathcal{C}}/S}(\Theta(\lambda)) & \overset{\nabla_1}{\twoheadrightarrow} & T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \end{array}$$

This isomorphism is equivalent to the equality of extension classes

$$\frac{r}{n(\lambda)}\Delta_\lambda = [\Theta(\lambda)] \in \mathrm{H}^0\left(S, R^1\pi_{e_*}\left(\Omega^1_{\mathcal{M}_\bullet/S}\right)\right).$$

II-3) Description of the parabolic Kodaira-Spencer map

Let $\mathcal{E}_*$ be a virtual universal bundle over $\mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ (see diagram (0.0.2)), take the first direct image of its traceless parabolic Atiyah exact sequence with respect to the map $\pi_e : \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha, \delta) \to S$.

$$0 \longrightarrow T_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S} \longrightarrow R^1\pi_{n*}\left(\mathcal{A}^{0,par}_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}/S}}(\mathcal{E})\right) \longrightarrow R^1\pi_{n*}\left(\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right)\right) \longrightarrow 0,$$

We denote by Φ^{par} the first connecting morphism of the long exact sequence in cohomology with respect to π_{e*}. We prove in Proposition 4.5.4 that the map Φ^{par} commutes with the Kodaira-Spencer maps of the two families

$$\Phi^{par} \circ \kappa_{\mathcal{C}/S} = \kappa_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S},$$

where $\kappa_{\mathcal{C}/S}$ and $\kappa_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S}$ are the Kodaira-Spencer of the family of marked curves and the family of relative moduli space respectively. We call the map Φ^{par} the parabolic Kodaira-Spencer map.

This factorization follows from the deformation theory of the triple (C, D, E_*) given by a smooth marked projective curve of genus $g \geqslant 2$, D a reduced divisor of degree N equipped with a quasi-parabolic rank-r vector bundle E_* of fixed quasi-parabolic type $\vec{m}$. Then we prove that the infinitesimal deformations of $(C, D, (E, F^*_*(E))$ are parameterized by $H^1\left(C, \mathcal{A}^{par}_C(E)\right)$. This result is proven independently in [BDHP22] using Galois covers. Our proof follows [Mar09].

Parabolic Hitchin symbol map: We take for the Hitchin symbol ρ_{par} the quadratic part of the parabolic Hitchin system. We prove that the trace of parabolic endomorphism is invariant under Hecke modification.i.e., for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$, then $tr(\mathcal{H}_i^j(f)) = tr(f)$ for all parabolic endomorphisms f of a parabolic vector bundle E_*. Hence ρ_{par} is invariant under Hecke modifications. Now let ∂ be the first connecting morphism of the long exact sequence for π_e of the sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}/S}} \longrightarrow R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}/S}}(\mathcal{E})\,(\mathcal{D})\right]^\vee\right) \longrightarrow T_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S} \longrightarrow 0,$$

where $\mathcal{D} := \pi_w^*(D) = D \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$.

We prove a parabolic version of Proposition 4.7.1 [BBMP23]. Then

Proposition 4.5.5: *The following diagram commute*

$$\begin{array}{ccc} R^1\pi_{s_*}\left(T_{\mathcal{C}/S}(-D)\right) & \xrightarrow{-\Phi^{par}} & R^1\pi_{e_*}\left(T_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S}\right) \\ & {\scriptstyle\rho_{par}}\searrow \quad \nearrow{\scriptstyle\partial} & \\ & \pi_{e_*}\mathrm{Sym}^2\left(T_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S}\right) & \end{array}$$

ie: $\Phi^{par} + \partial \circ \rho_{par} = 0.$

As a corollary we get the following theorem.

Theorem 4.5.7: *For all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$, we have the equations*

$$\cup\,[\Theta_j(i)] \circ \rho_{par} = -\frac{r}{n_j(i)}\,\Phi_{par}, \quad \text{and} \quad \cup\,[\Theta] \circ \rho_{par} = -\frac{r}{n_j(i)}\,\Phi_{par}.$$

Here, $\Theta_j(i)$ is the pull-back of the ample generator of the relative Picard group $\mathrm{Pic}\left(\mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i))/S\right)$ under the classifying map

$$\begin{array}{cccc} \phi_{j,i} & : \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) & \longrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i)) \\ & \mathcal{E}_* & \longmapsto & \mathcal{H}_i^j(\mathcal{E}) \end{array}$$

and where $\mathcal{H}_i^j(\mathcal{E})$ is the Hecke modification of the vector bundle $\mathcal{E}_*$ with respect to the $(j+1)$-th element of the flag at the i-th section $\sigma_i : S \longrightarrow \mathcal{C}$ of the map π_s, Θ is the pull-back of of the ample generator of the relative Picard group $\mathrm{Pic}\left(\mathcal{SU}_{\mathcal{C}/S}(r, \delta)/S\right)$ under the forgetful map and $n_j(i) := \gcd(r, \deg\, \delta_j(i))$ and $n := \gcd(r, \deg\delta)$.

II-4) Some line bundles over the space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$

In order to obtain the van Geemen-de Jong equation for the line bundle Θ_{par} we need to study the relation between Θ_{par} and the line bundles $\Theta_j(i)$, this is done in the following propositions.

Proposition 4.6.3 (Parabolic determinant bundle and Hecke modifications) *Let $\mathcal{E}_*$ be a family of parabolic rank-r vector bundles with determinant $\delta \in \mathrm{Pic}(\mathcal{C}/S)$ of parabolic type α_* over a smooth family of curves $\pi_s : \mathcal{C} \longrightarrow S$ parameterized by a S-variety T. Then*

$$\lambda_{par}(\mathcal{E}_*)^r = \Theta^a \otimes \left(\bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \Theta_j(i)^{q_j(i)} \right),$$

where, for all $i \in I$ and $j \in \{1,2,...,\ell_i - 1\}$

- *Θ is the pull-back of the ample generator of $\mathrm{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r,\delta)/S)$ by the classifying map ϕ_T and $n = \gcd(r,d)$.*
- *$\Theta_j(i)$ is the pull-back of the ample generator of $\mathrm{Pic}\left(\mathcal{SU}_{\mathcal{C}/S}(r,\delta_j(i))/S\right)$ by the classifying maps $\phi^T_{i,j}$ and $n_j(i) = \gcd(r, d_j(i))$.*
- *$p_j(i) = a_{j+1}(i) - a_j(i)$ and $q_j(i) = n_j(i)p_j(i)$.*
- *$a = n\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\right)$.*

For a virtual universal parabolic bundle over $\mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$ and by the previous proposition, the parabolic theta line bundle over $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$, satisfies the equation

$$\Theta^r_{par} = \Theta^{n\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\right)} \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{n_j(i)p_j(i)}.$$

With the same methods we give a description of the relative canonical line bundle.

Proposition 4.6.5: *The relative canonical bundle of the space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$ is given as follows*

$$K_{\mathcal{SM}^{par}_{\mathcal{C}/S}/S} = \Theta^{-n\left(2+\deg(D) - \sum_{i=1}^{N} \ell_i\right)} \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{-n_j(i)}.$$

Using these decompositions and the van Geemen-de Jong equation over the space $\mathcal{SU}_{\mathcal{C}/S}(r,*)$ of fixed determinant $* \in \mathrm{Pic}(\mathcal{C}/S)$, we prove that the van Geemen-de Jong criterion is fulfilled for the parabolic determinant line bundle over the space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$.

Theorem 4.7.1: *Let $\nu \in \mathbb{N}^*$. The ν-th power of the parabolic determinant line bundle Θ_{par}^{ν} satisfies the van Geemen-de Jong equation, i.e.,*

$$\mu_{\Theta_{par}^{\nu}} \circ \rho_{par} = -(\nu k + r)\Phi_{par}.$$

We define the flag part of the parabolic determinant line bundle Θ_{par}, as follow

$$\mathcal{F}(\alpha_*) := \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \left(\Theta^{-n} \otimes \Theta_j(i)^{n_j(i)}\right)^{p_j(i)},$$

such that

$$\Theta_{par}^{r} = \Theta^{nk} \otimes \mathcal{F}(\alpha_*),$$

and take the cup-product map

$$\cup\, [\mathcal{F}(\alpha_*)] : \pi_{e*}\mathrm{Sym}^2 T_{\mathcal{SM}^{par}/S} \longrightarrow R^1\pi_{e*} T_{\mathcal{SM}^{par}/S}.$$

Then as a corollary of the previous theorems we get that the cup- product with the flag part is zero. Hence the parabolic system of weights $\vec{a}$ does not contribute.

II-5) Main theorems:

Theorem 4.7.5: *Let ν be a positive integer. We consider a smooth family $\pi_s : (\mathcal{C}, D) \longrightarrow S$ of complex projective marked curves of genus $g \geqslant 2$, D a reduced divisor of relative degree N and $\alpha_* = (k, \vec{a}, \vec{m})$ a fixed rank-r parabolic type with respect to the divisor D without trivial parabolic points. We denote by $\pi_e : \mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta) \longrightarrow S$ the relative moduli space of parabolic rank-r vector bundles over $(\mathcal{C}, D)/S$ with determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$, equipped with the parabolic determinant bundle Θ_{par}. Then there exists a unique projective flat connection on the vector bundle $\pi_{e*}(\Theta_{par}^{\nu})$ of non-abelian parabolic theta functions, induced by a heat operator with symbol*

$$\rho_{par}^{Hit}(\nu) := \frac{1}{(\nu k + r)}\left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right).$$

For $D = \varnothing$ and $\alpha_* = k \in \mathbb{N}^*$ the trivial parabolic type, we have the identification of the moduli space $\mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$ with $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$ the moduli space of semi-stable rank-r vector bundles with determinant δ, hence $\Theta_{par}^{r/n} = \mathcal{L}^k$ for $n := \gcd(r, \deg(\delta))$ and $\rho_{par} = \rho^{Hit}$. We obtain the following special case for non-parabolic vector bundles.

Theorem 4.7.6: *Let k be a positive integer. Suppose a smooth family $p_s : \mathcal{C} \longrightarrow S$ of complex projective curves of genus $g \geqslant 2$ (and $g \geqslant 3$ if $r = 2$ and $\deg(\delta)$ even), set $n := \gcd(r, \deg(\delta))$. Let $\mathcal{L}$ be the ample generator of the Picard group of $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$. Then there exists a unique projective flat connection on the vector bundle $p_{e*}(\mathcal{L}^k)$ of non-abelian theta functions, induced by a heat operator with symbol*

$$\rho(k) := \frac{n}{r(k+n)}\left(\rho^{Hit} \circ \kappa_{\mathcal{C}/S}\right).$$

In fact for $k \in \mathbb{N}^*$, *we have* $\Theta_{par}^{r/n} = \mathcal{L}^k$, *hence* $\rho(k) = \rho_{par}^{Hit}(\frac{r}{n})$.

Note that for $\delta = \mathcal{O}_\mathcal{C}$ we recover the classical case proved in [Hit90a].

Document structure

Chapter 1: We recall the definition of parabolic vector bundles, some of their proprieties such as semi-stability conditions and Mehta-Seshadri theorem of the existence of a relative moduli space of semi-stable parabolic vector bundles of fixed parabolic type over a family of smooth projective complex curves. In the second part we study the stability criterion under the forgetful maps, the description of the determinant line bundle and the definition of the parabolic determinant line bundle.

Chapter 2: We present the Yokogawa-Maruyama and Simpson point of view of filtered vector bundles and the identification of their moduli space with the space of parabolic vector bundles. We give some of their properties and we give a study of the group of parabolic transformations and their action on the parabolic weights.

Chapter 3: In this chapter we present the main theorems that we need in our work: the van Geemen-de Jong criterion of existence of Hitchin connection and the flatness criterion. We recall the definitions of Atiyah classes, sheaves of differential operators on vector bundles and associated morphisms and the definition of heat operators.

Chapter 4: We present an algebro-geometric construction of Hitchin's connection in the sheaf of parabolic non-abelian theta functions. We give a decomposition of the parabolic determinant line bundle. We prove a factorisation theorem of the Kodaira-Spencer map of the relative moduli space of parabolic bundles. Using these decompositions, we prove that the parabolic determinant line bundle satisfies van Geemen-de Jong equation over the moduli space of parabolic vector bundles, where we take as symbol map the quadratic part of the parabolic Hitchin system. Hence we construct the existence of a flat projective connection on the sheaf of parabolic non-abelian theta functions. We prove that this connection is flat.

Chapter 1

Parabolic vector bundles and their moduli spaces

1.1 Parabolic bundles

Let C be a smooth projective complex curve of genus $g \geqslant 2$ and $D{=}\{x_1, x_2, ..., x_N\}$ a finite set of points $x_i \in C$. The set D will also be called a parabolic divisor.
We set $I = \{1, 2, ..., N\}$, where $N = \deg(D)$.

Parabolic type of a vector bundle

A parabolic type for a rank-r vector bundle over C with respect to the parabolic divisor D is the following numerical data $\alpha_* = (k, \vec{a}, \vec{m})$ consisting of :

- A quasi-parabolic type $\vec{m} = (\ell_i, m(i))_{i \in I}$, where
  1. $\ell_i \in \mathbb{N}^*$ called the length at the point $x_i \in D$.
  2. a sequence of integers, called the flag type at $x_i \in D$

     $$m(i) = (m_1(i), m_2(i), ..., m_{\ell_i}(i)).$$

     with $m_j(i) \in \mathbb{N}^*$.
  3. we have for every $i \in I$ the relation $\sum_{j=1}^{\ell_i} m_j(i) = r$.
- A system of parabolic weights $(k, \vec{a})$, where $k \in \mathbb{N}^*$ and $\vec{a} = (a_j(i))_{\substack{i \in I \\ 1 \leqslant j \leqslant \ell_i}}$ a sequence of integers satisfying for each $i \in I$

  $$0 \leqslant a_1(i) < a_2(i) < ... < a_{\ell_i}(i) < k.$$

We say that $x_i \in D$ is a trivial point if $\ell_i = 1$, which implies that $m_1(i) = r$ and $m(i) = (r)$.

We say that α_* is full flag parabolic type if $\ell_i = r$ for all $i \in I$, thus $m_j(i) = 1 \ \forall i, j$.

The notion of parabolic vector bundle was introduced by Seshadri in [Ses77].

Definition 1.1.1 (Parabolic vector bundles) *Let E be a rank-r vector bundle over C. A quasi-parabolic structure of quasi-parabolic type $\vec{m} = (l_i, m(i))_{i \in I}$ on E with respect to the parabolic divisor D is given by a filtration of length ℓ_i on the fibre E_{x_i} for each $i \in I$, by linear subspaces*

$$F_*^*(E): \qquad E_{x_i} = F_i^1(E) \supset F_i^2(E) \supset \cdots \supset F_i^{\ell_i}(E) \supset F_i^{\ell_i+1}(E) = \{0\}$$

such that for $j \in \{1, 2..., \ell_i\}$ we have

$$\dim_{\mathbb{C}} \left(F_i^j(E)/F_i^{j+1}(E) \right) = m_j(i).$$

We denote a quasi-parabolic bundle by $(E, F_*^*(E))$.

A parabolic structure on E with respect to the parabolic divisor D is the data $(E, F_*^*(E), \alpha_*)$ where $\alpha_* = (k, \vec{a}, \vec{m})$ a fixed parabolic type and (E, F_*^*) is a quasi-parabolic structure over E of type $\vec{m}$ with respect to the parabolic divisor D. We denote a parabolic vector bundle by E_* and α_* is called its parabolic type.

For all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$, we define the following quotients

$$Gr_i^j(E) := F_i^j(E)/F_i^{j+1}(E),$$

of dimension $m_j(i)$ and

$$Q_i^j(E) := E_{x_i}/F_i^{j+1}(E).$$

We denote their dimensions by

$$r_j(i) = \dim_{\mathbb{C}} Q_i^j(E) = \sum_{q=1}^{j} m_q(i).$$

Note that $r_{\ell_i}(i) = r$.

Definition 1.1.2 (parabolic degree, slope and Euler characteristic)
Let E_* be a parabolic vector bundle over C. We define

1. *The parabolic degree*

$$\text{pardeg}(E) = \deg(E) + \frac{1}{k} \sum_{i=1}^{N} \sum_{j=1}^{\ell_i} m_j(i) a_j(i).$$

2. *The parabolic slope*
$$\mu_{par}(E) = \frac{\text{pardeg}(E)}{\text{rank}(E)}.$$

3. *The parabolic Euler characteristic*
$$\begin{aligned}\chi_{par}(E) &= \chi(E) + \frac{1}{k}\sum_{i=1}^{N}\sum_{j=1}^{\ell_i} m_j(i)a_j(i) \\ &= \deg(E) + \text{rank}(E)(1-g) + \frac{1}{k}\sum_{i=1}^{N}\sum_{j=1}^{\ell_i} m_j(i)a_j(i) \\ &= \text{pardeg}(E) + \text{rank}(E)(1-g).\end{aligned}$$

Definition 1.1.3 (parabolic quotient and subbundles) *Let E_* be a parabolic bundle over C of parabolic type α_* with respect to the divisor D.*

1. *A parabolic subbundle is the following data:*
   (a) *a parabolic vector bundle F_* of parabolic type β_* with respect to the parabolic divisor D such that F is a vector subbundle of E.*
   (b) *for all $i \in I$ and $q \in \{1, 2, ..., l_i(F)\}$, let j is the greatest integer such that*
   $$F_i^q(F) \subset F_i^j(E),$$
   then we have
   $$b_q(i) = a_j(i).$$

2. *A parabolic quotient bundle of E_* is the following data: a parabolic bundle F_* of parabolic type β_* with a surjective morphism $f : E \longrightarrow F$ such that:*
   (a) *for all $i \in I$ and $q \in \{1, 2, ..., l_i(F)\}$ there is an element $j \in \{1, 2, ..., l_i(E)\}$ such that*
   $$f_{x_i}(F_i^j(E)) = F_i^q(F).$$
   (b) *if j is the greatest integer such that the equality holds, we have*
   $$b_q(i) = a_j(i).$$

Remark 1.1.4 *Let E' be a subbundle of a parabolic bundle E_*. Then E' is equipped with a canonical parabolic structure as follows: the filtration $F_i^*(E')$ consists of the distinct terms of the filtration $F_i^j(E) \cap E'_{x_i}$ and the parabolic weights are taken as*
$$a'_*(i) = max\{a_q(i) \ / \ F_i^q(E) \cap E'_{x_i} = F_i^*(E')\}$$
We denote the induced parabolic structure by E'_* and the associated parabolic weights by α'_*. We define the same canonical parabolic structure for quotient bundles.

When we consider subbundles (resp. quotient bundles) of a parabolic bundle E_*, they will be equipped with the canonical parabolic structures.

Definition 1.1.5 (parabolic and strongly parabolic endomorphisms) *Let E_* be a parabolic bundle over C and let $f \in \mathrm{End}(E)$ then*

1. *f is a parabolic endomorphism if for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ we have*
$$f_{x_i}\left(F_i^j(E)\right) \subset F_i^j(E).$$
We denote the sheaf of parabolic endomorphism by $\mathrm{parEnd}(E)$.

2. *f is a strongly parabolic endomorphism if for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ we have*
$$f_{x_i}\left(F_i^j(E)\right) \subset F_i^{j+1}(E).$$
We denote the sheaf of parabolic endomorphism by $\mathrm{SparEnd}(E)$.

The sheaves introduced above are locally free and by definition we have the following sheaf inclusions
$$\mathrm{SparEnd}(E) \hookrightarrow \mathrm{parEnd}(E) \hookrightarrow \mathrm{End}(E). \tag{1.1.1}$$

Remark 1.1.6 *The definitions of parabolic and strongly-parabolic endomorphisms depend only on the quasi-parabolic structure and not on the the system of parabolic weights.*

Proposition 1.1.7 ([Yok91]) *Let E_* be a parabolic vector bundle over C with respect to the parabolic divisor D. Then we have a canonical isomorphism of locally free sheaves*
$$\mathrm{parEnd}(E)^\vee \cong \mathrm{SparEnd}(E) \otimes \mathcal{O}_C(D).$$
This isomorphism is given by the non-degenerate trace paring
$$\begin{array}{rccc} \mathrm{Tr} & : & \mathrm{parEnd}(E) \otimes \mathrm{SparEnd}(E) & \longrightarrow & \mathcal{O}_C(-D) \\ & & \phi \otimes \psi & \longmapsto & \mathrm{Tr}(\phi \circ \psi). \end{array}$$
And by dualizing (1.1.1) we get
$$\mathrm{End}(E)^\vee \cong \mathrm{End}(E) \hookrightarrow \mathrm{SparEnd}(E)(D) \hookrightarrow \mathrm{parEnd}(E)(D).$$

1.2 Moduli spaces of parabolic bundles

To construct the moduli space of parabolic vector bundles over a curve C, we need a notion of semi-stability and stability which will depend on the parabolic type α_*. In this section we recall the definition of stability and the theorem of existence of a coarse moduli space parameterizing parabolic bundles. The main reference is [Ses82].

Definition 1.2.1 (stability) *A parabolic bundle E_* over a curve C is said to be parabolic stable (resp. parabolic semi-stable) with respect to the parabolic type α_* if for all proper vector subbundles F equipped with the canonical parabolic structure we have*
$$\mu_{par}(F_*) < \mu_{par}(E_*) \quad (\text{resp. } \leqslant).$$

We will from now on abbreviate parabolic (semi-)stable by (semi-)stable when considering parabolic vector bundles.

Remark 1.2.2 *Let E_* be a rank-r parabolic bundle with respect to a parabolic divisor D.*

1. *Suppose there is $i_0 \in I$ such that $x_{i_0} \in D$ is a trivial point. Then there a natural parabolic structure on E denoted E'_* with respect to the parabolic divisor $D \backslash \{x_{i_0}\}$ of type α'_* (we forget the weight on x_{i_0}). We have the following relation*
$$\text{pardeg}(E'_*) = \text{pardeg}(E_*) - r\frac{a_1(i_0)}{k},$$
hence
$$\mu_{par}(E'_*) = \mu_{par}(E_*) - \frac{a_1(i_0)}{k}$$
and we have the following equivalence
$$E'_* \text{ stable (resp. semi} - \text{stable)} \iff E_* \text{ stable (resp. semi} - \text{stable)}.$$

2. *Let L be a line bundle over C, thus $E \otimes L$ can be equipped with a canonical parabolic structure induced by the parabolic structure of E_*, we denoted this structure by $(E \otimes L)_*$ which is also of parabolic type α_* and we have the equivalence*
$$(E \otimes L)_* \text{ stable (resp. semi stable)} \iff E_* \text{ stable (resp. semi stable)}.$$

Proposition 1.2.3 (Jordan-Hölder filtration) *Let E_* be a semi stable parabolic bundle over C. There exists a filtration of E by subbundles called the Jordan-Hölder filtration*
$$0 = E_{p+1} \subset E_p \subset \ldots \subset E_2 \subset E_1 \subset E_0 = E \tag{1.2.1}$$
such that for all $1 \leqslant i \leqslant p$, the vector bundle E_i/E_{i+1} equipped with the canonical parabolic structure is stable and
$$\mu_{par}((E_i/E_{i+1})_*) = \mu_{par}(E_*).$$
We define the parabolic graded bundle
$$Gr(E_*) := \oplus_{i=1}^{p}(E_i/E_{i+1}),$$
equipped with the canonical parabolic structure.

It can be shown that the isomorphism class of $Gr(E_*)$ does not depend on the filtration (1.2.1).

Definition 1.2.4 (S-equivalence) *We say that two semi-stable parabolic rank-r vector bundles E_* and E'_* over the curve C of parabolic type α_* with respect to the parabolic divisor D are S-equivalent (and we denoted $E_* \sim_S E'_*$) if their associated graded $Gr(E_*)$ and $Gr(E'_*)$ are isomorphic as parabolic vector bundles. Moreover if E_* is stable then $Gr(E_*) = E_*$.*

Mehta and Seshadri constructed the moduli space of semi-stable parabolic vector bundles over a smooth projective complex curve C.

Theorem 1.2.5 (Mehta-Seshadri [MS80]) *For a fixed parabolic type α_*, there is a coarse moduli space $\mathcal{M}_C^{par} := \mathcal{M}_C^{par}(r, \alpha_*, d)$ which is a projective irreducible normal variety, parameterizing semi-stable parabolic rank-r vector bundles and of degree-d modulo S-equivalence over the curve C. Moreover, the subspace $\mathcal{M}_s^{par} \subset \mathcal{M}^{par}$ of stable parabolic bundles is an open subset included in the smooth locus.*

Assuming $g \geqslant 2$ (see remark 4.7.7 for the cases of genus=0,1), the dimension of the moduli space $\mathcal{M}_C^{par} = \mathcal{M}_C^{par}(r, \alpha_*, d)$ is

$$\dim_{\mathbb{C}} \mathcal{M}_C^{par}(r, \alpha_*, d) = r^2(g-1) + 1 + \frac{1}{2}\sum_{i=1}^{N}\left(r^2 - \sum_{j=1}^{\ell_i} m_j(i)^2\right).$$

For a line bundle $\delta \in \mathrm{Pic}^d(C)$ we define the subspace

$$\mathcal{SM}_C^{par}(r, \delta) = \{E_* \in \mathcal{M}_C^{par}(r, \alpha_*, d) \ / \ \det(E) \cong \delta\}$$

parameterizing rank-r S-equivalence classes of vector bundles over C of determinant δ, which is also projective irreducible normal variety of dimension

$$\dim_{\mathbb{C}} \mathcal{SM}_C^{par}(r, \delta) = (r^2-1)(g-1) + \frac{1}{2}\sum_{i=1}^{N}\left(r^2 - \sum_{j=1}^{\ell_i} m_j(i)^2\right).$$

Remark 1.2.6

1. *The dimension does not depend on the system of weights.*
2. *For $i \in I = \{1, 2, ..., N\}$ the integer $\frac{1}{2}\left(r^2 - \sum_{j=1}^{\ell_i} m_j(i)^2\right)$, is the dimension of a flag variety of type $m(i)$.*

1.2.1 Families of parabolic vector bundles

Let $\mathcal{T}$ be a Noetherian scheme over $\mathbb{C}$ and C a smooth projective complex curve equipped with a parabolic divisor D. We also fix a parabolic type α_* with respect to the parabolic divisor D and let $\delta \in \mathrm{Pic}^d(C)$.

Definition 1.2.7 *A family of parabolic rank-r vector bundles of fixed degree d (resp. fixed determinant δ) over C parameterized by $\mathcal{T}$ is a locally free sheaf $\mathcal{E}$ over $C \times \mathcal{T}$ together with the following data: for each $i \in I$, we give a filtration of the locally free sheaf*

$$\mathcal{E}_{x_i} := \mathcal{E}|_{\{x_i\}\times\mathcal{T}}$$

by locally free subsheaves

$$\mathcal{E}_{x_i} = F_i^1(\mathcal{E}) \supset F_i^2(\mathcal{E}) \supset \cdots \supset F_i^{\ell_i}(\mathcal{E}) \supset F_i^{\ell_i+1}(\mathcal{E}) = \{0\},$$

such that

$$\operatorname{rank}\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right) = m_j(i),$$

and for all closed points $t \in \mathcal{T}$ the rank-r vector bundle $\mathcal{E}_t := \mathcal{E}|_{C\times\{t\}}$ is of degree d (resp. determinant δ) and equipped with the induced parabolic structure is a semi-stable parabolic bundle of type α_* with respect to the parabolic divisor D.

Two families of parabolic bundles $\mathcal{E}_*$ and $\mathcal{E}'_*$ of parabolic type α_* parameterized by $\mathcal{T}$ are equivalent if there is an invertible sheaf L over $\mathcal{T}$ such that

$$\mathcal{E}_* \cong \mathcal{E}'_* \otimes p_2^*(L),$$

where $p_2 : C \times \mathcal{T} \longrightarrow \mathcal{T}$ the second projection map.

We get a functor

$$\begin{array}{rcl} \underline{\mathcal{M}^{par}} := \underline{\mathcal{M}^{par}(r, \alpha_*, d)} \;\; : \;\; \mathbb{C}-\textit{schemes} & \longrightarrow & \textit{Set} \\ \mathcal{T} & \longmapsto & \underline{\mathcal{M}^{par}}(\mathcal{T}), \end{array}$$

which associate to a Noetherian $\mathbb{C}$-scheme $\mathcal{T}$ the set of equivalent families of parabolic rank-r vector bundles over C parameterized by $\mathcal{T}$ of parabolic type α_* with respect to the parabolic divisor D and of fixed degree d.

For a line bundle $\delta \in \operatorname{Pic}^d(C)$ we define a sub-functor

$$\underline{\mathcal{SM}^{par}(r, \alpha_*, \delta)} \subset \underline{\mathcal{M}^{par}(r, \alpha_*, d)},$$

which associate to a noetherian $\mathbb{C}$-scheme $\mathcal{T}$ the set of equivalent families of parabolic rank-r vector bundles over C parameterized by $\mathcal{T}$ of parabolic type α_* and fixed determinant δ, i.e. for any Noetherian $\mathbb{C}$-scheme $\mathcal{T}$ we have

$$\underline{\mathcal{SM}^{par}(r, \alpha_*, \delta)}(\mathcal{T}) := \{\mathcal{E}_* \in \underline{\mathcal{M}^{par}(r, \alpha_*, d)}(\mathcal{T}) \;\; / \;\; \det(\mathcal{E}_t) \cong p_1^*(\delta_t) \;\; \forall t \in \mathcal{T}\},$$

where $p_1 : C \times \mathcal{T} \longrightarrow C$ is the first projection map.

Theorem 1.2.8 [MS80] *The functors*

$$\underline{\mathcal{M}^{par}(r, \alpha_*, d)} \textit{ and } \underline{\mathcal{SM}^{par}(r, \alpha_*, \delta)},$$

are representable respectively by the varieties $\mathcal{M}_C^{par}(r, \alpha_*, d)$ and $\mathcal{SM}_C^{par}(r, \alpha_*, \delta)$ given in the Theorem 1.2.5.

By representable we mean that there is a functorial morphism

$$\psi : \underline{\mathcal{M}^{par}}(\mathcal{T}) \longrightarrow Mor(\mathcal{T}, \mathcal{M}^{par}(r, \alpha_*, d)),$$

which is universal in the following sense :

1. for each algebraic variety $\mathcal{N}$ and a functorial morphism

$$\phi : \underline{\mathcal{M}^{par}}(\mathcal{T}) \longrightarrow Mor(\mathcal{T}, \mathcal{N}),$$

   there is a unique morphism

$$f : \mathcal{M}^{par}(r, \alpha_*, d) \longrightarrow \mathcal{N},$$

   making the following commute

$$\begin{array}{ccc} \underline{\mathcal{M}^{par}}(\mathcal{T}) & \xrightarrow{\psi} & Mor(\mathcal{T}, \mathcal{M}^{par}(r, \alpha_*, d)) \\ & \searrow^{\phi} & \downarrow f \\ & & Mor(\mathcal{T}, \mathcal{N}) \end{array}$$

   The pair $(\mathcal{M}^{par}(r, \alpha_*, d), \psi)$ is uniquely determined by this condition.

2. We note by $\mathcal{S}(r, \alpha_*, d)$ the set of isomorphism classes of semi-stable parabolic rank-r vector bundles of degree d and parabolic type α_*. Then ψ induces a map of sets

$$\psi : \mathcal{S}(r, \alpha_*, d) \cong \underline{\mathcal{M}^{par}}(\mathrm{Spec}(\mathbb{C})) \longrightarrow \mathcal{M}^{par}(r, \alpha_*, d).$$

The second point means that the map surjects and fibers are S-equivalence classes of semi-stable parabolic bundles. Same for the functor $\underline{\mathcal{SM}^{par}(r, \alpha_*, \delta)}$.

1.2.2 Relative moduli spaces

In this subsection we will recall the existence of a relative version of the moduli spaces of semi-stable parabolic vector bundles over a family of smooth projective complex curves equipped with a family of parabolic divisors.

Let $\pi_s : \mathcal{C} \longrightarrow S$ be a smooth family of projective curves of genus $g \geqslant 2$, parameterized by an algebraic variety S over $\mathbb{C}$ and let

$$\sigma_i : S \longrightarrow \mathcal{C}, \qquad i \in I = \{1, 2, ..., N\},$$

be N-sections of π_s, such that

$$\forall i \neq j \in I \text{ and } \forall s \in S, \text{ we have: } \sigma_i(s) \neq \sigma_j(s).$$

We denote by

$$D := \sum_{i \in I} \sigma_i(S),$$

the associated divisor (as the relative dimension of the map π_s is one), which will be seen as a family of parabolic degree N divisors parameterized by the variety S and let $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ (See [FGI+05] for definition of relative Picard groups).

Let $\pi_e : \mathcal{T} \longrightarrow S$ be a S-variety. A relative family of parabolic rank-r vector bundles and fixed degree d (resp. determinant δ) over $\mathcal{C}/S$ of parabolic type α_* parameterized by $\mathcal{T}/S$ is a locally free sheaf $\mathcal{E}$ over $\mathcal{C} \times_S \mathcal{T}$ together with the following data:

- For each $i \in I$, we give a filtration of the vector bundle $\mathcal{E}_{\sigma_i} := \mathcal{E}|_{\sigma_i(S) \times_S \mathcal{T}}$ over $\sigma_i(S) \times_S \mathcal{T} \cong \mathcal{T}$ by subbundles as follow

  $$\mathcal{E}_{\sigma_i} = F_i^1(\mathcal{E}) \supset F_i^2(\mathcal{E}) \supset \cdots \supset F_i^{\ell_i}(\mathcal{E}) \supset F_i^{\ell_i+1}(\mathcal{E}) = \{0\},$$

  $$0 \leqslant a_1(i) < a_2(i) < \ldots < a_{\ell_i}(i) < k,$$

  such that for each $j \in \{1, 2, \ldots, \ell_i\}$ we have

  $$\mathrm{rank}\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right) = m_j(i).$$

  Thus we get a parabolic structure over $\mathcal{E}$, denoted by $\mathcal{E}_*$.

- For each $t \in \mathcal{T}$ we set $\mathcal{C}_t := \pi_s^{-1}(\pi_e(t))$. Then the vector bundle $\mathcal{E}_*|_{\mathcal{C}_t}$ is a semi-stable parabolic bundle of parabolic type α_* of degree d (resp. determinant $\delta_t := \delta|_{\mathcal{C}_t} \in \mathrm{Pic}^d(\mathcal{C}_t)$) with respect to the parabolic divisor

  $$D_t := \sum_{i \in I} \sigma_i(\pi_e(t)).$$

We define the same notion of equivalence of relative families as before. We get a functor

$$\underline{\mathcal{M}^{par}_{\mathcal{C}/S}} := \underline{\mathcal{M}^{par}_{\mathcal{C}/S}(r, \alpha_*, d)} \; : \; \begin{array}{ccc} S-schemes & \longrightarrow & Set \\ \mathcal{T} & \longmapsto & \underline{\mathcal{M}^{par}_{\mathcal{C}/S}}(\mathcal{T}), \end{array}$$

which associates to a Noetherian S-scheme $\mathcal{T}$ the set of equivalent families of parabolic rank-r vector bundles over $\mathcal{C}/S$ parameterized by the scheme $\mathcal{T}/S$ of parabolic type α_* and fixed degree d.

As before we define a sub-functor

$$\underline{\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)} \; \subset \; \underline{\mathcal{M}^{par}_{\mathcal{C}/S}(r, \alpha_*, d)},$$

parameterizing parabolic rank-r vector bundles over $\mathcal{C}/S$ of type α_* with respect to the parabolic divisor D and of fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$.

Maruyama and Yokogawa constructed a relative version of the moduli space of semi-stable parabolic vector bundles over a smooth family of projective curves in [Yok93],[MY92] and [Yok95].

Theorem 1.2.9 *The functors* $\underline{\mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d)}$ *and* $\underline{\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)}$*, are representable by proper* S*-schemes that we denote respectively by*

$$\mathcal{M}^{par}_{\mathcal{C}/S} := \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d) \text{ and } \mathcal{SM}^{par}_{\mathcal{C}/S} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta).$$

Their closed points parameterize relative S*-equivalence classes of rank-*r *semi-stable parabolic vector bundles of fixed type* α_* *and degree* d *respectively fix determinant* δ *over the family of curves* $\pi_s : \mathcal{C} \longrightarrow S$*. As a* S*-schemes they are equipped with a surjective proper maps*

$$\begin{aligned} \tilde{\pi}_e :& \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d) \longrightarrow S, \\ \pi_e :& \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta) \longrightarrow S. \end{aligned}$$

And for each $s \in S$ *we get*

$$\tilde{\pi_e}^{-1}(s) = \mathcal{M}^{par}_{\mathcal{C}_s}(r,\alpha_*,d) \text{ and } \pi_e^{-1}(s) = \mathcal{M}^{par}_{\mathcal{C}_s}(r,\alpha_*,\delta_s),$$

the moduli space of semi-stable rank-r *parabolic bundles of parabolic type* α_* *and degree-*d *(resp. determinant* δ_s*) over the curve* $\mathcal{C}_s = \pi_s^{-1}(s)$ *with respect to the parabolic divisor* $D_s = \sum_{i \in I} \sigma_i(s)$*.*

We also define the following fiber products over S of the relative moduli spaces with the family of curves

$$\begin{array}{ccc} \mathcal{X}^{par} := \mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta) & \xrightarrow{\pi_n} & \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} \times_S \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d) & \xrightarrow{\pi_n} & \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d) \\ \big\downarrow{\scriptstyle \pi_w} & & \big\downarrow{\scriptstyle \pi_e} \\ \mathcal{C} & \xrightarrow[\pi_s]{} & S \end{array}$$

Definition 1.2.10 (Universal family) *A universal parabolic vector bundle (or a parabolic Poincaré bundle) over* $\mathcal{C} \times_S \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d)$ *is a family* $\mathcal{E}_*$ *of parabolic vector bundles of rank-*r *and degree* d *of parabolic type* α_* *over the family of curves* $\mathcal{C}/S$ *parameterized by the moduli space* $\mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d)$*, such that*

$$\forall\, [E_*] \in \mathcal{M}^{par}_{\mathcal{C}/S}(r,\alpha_*,d),$$

we have

$$\mathcal{E}_*|_{\mathcal{C}_{E_*}} \sim_S E_*,$$

over the curve

$$\mathcal{C}_{E_*} = \pi_s^{-1}\left(\pi_e\left([E_*]\right)\right).$$

Same definition over the variety $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$, for a line bundle $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$.

Remark 1.2.11

1. *The universal parabolic bundle may not exist.*
2. *When it exists, the universal parabolic bundle is unique up to equivalence of families.*
3. *In fact, existence of universal family is equivalent to the isomorphism of functors*

$$\underline{\mathcal{M}^{par}_{\mathcal{C}/S}(r, \alpha_*, d)}(-) \simeq \mathcal{H}om\left(-, \mathcal{M}^{par}_{\mathcal{C}/S}(r, \alpha_*, d)\right).$$

In this case we say that the moduli space is a fine moduli space.

Proposition 1.2.12 ([BY99], Proposition. 3.2) *The moduli space of parabolic-stable bundles is fine if and only if we have:* $\gcd\{d, m_j(i) | i \in I, 1 \leqslant j \leqslant \ell_i\} = 1$.

As universal bundles do not exists in general, we define a weaker notion.

Proposition 1.2.13 ([NS75], Proposition. 2.4) *We fix a parabolic type α_*. Then there is a non-singular S-variety $\mathcal{SM}'$ equipped with a family of parabolic stable rank-r vector bundles $\mathcal{E}_*$ of parabolic type α_* with determinant δ, such that the map*

$$\begin{array}{cccc} \psi : & \mathcal{SM}' & \longrightarrow & \mathcal{SM}^{par,stab}_{\mathcal{C}/S}(r, \alpha_*, \delta) \\ & t & \longmapsto & \left[\mathcal{E}_*|_{\pi_n^{-1}(t)}\right] \end{array}$$

is étale and surjective.

Definition 1.2.14 (Virtual universal bundle) *The family given in the above proposition is what we call a virtual universal parabolic bundle of parabolic type α_* over the variety $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$.*

Remark 1.2.15 *As $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ is a good quotient of a Hilbert quotient scheme, denoted $\mathcal{Z}^{ss}$, then there is a universal bundle $\mathcal{E}_*$ on $\mathcal{C} \times_S \mathcal{Z}^{ss}$. $\mathcal{E}_*$ may not descend to $\mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}$, but objects such $\mathrm{End}^0(\mathcal{E})$,$\mathrm{parEnd}^0(\mathcal{E})$ and $\mathcal{A}^0(\mathcal{E})$ etc. , descend. Recall that a sheaf $\mathcal{F}$ on $\mathcal{C} \times_S \mathcal{Z}^{ss}$ descends to $\mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}$ if the action of scalar automorphisms of $\mathcal{F}$ relative to $\mathcal{Z}^{ss}$ is trivial. Hence without confusion we will be assuming existence of universal bundle $\mathcal{E}_*$ over $\mathcal{C} \times_S \mathcal{SM}^{par}$ that we call a virtual universal bundle.*

1.3 Hecke modifications

Let $E_* \longrightarrow C$ be a rank-r parabolic vector bundle of parabolic type α_* with respect to the parabolic divisor D of determinant $\delta \in \mathrm{Pic}^d(C)$. We associate the following exact sequences

$$0 \longrightarrow \mathcal{H}_i^j(E) \hookrightarrow E \longrightarrow Q_i^j(E) \longrightarrow 0$$

where for all $i \in I$ and $j \in \{1, 2, 3, ..., \ell_i\}$

$$Q_i^j(E) := E_{x_i}/F_i^{j+1}(E)$$

the quotient sheaf supported on x_i of length

$$r_j(i) = \sum_{q=1}^{j} m_q(i).$$

The subsheaves $\mathcal{H}_i^j(E)$ are locally free of rank-r and their determinant is given by

$$\delta_j(i) := \delta \otimes \mathcal{O}_C \left(-r_j(i)x_i\right).$$

We denote their degree by

$$d_j(i) := \deg\, \delta_j(i) = d - r_j(i),$$

and we set for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ the integers

$$n_j(i) = \gcd(r, d_j(i)). \quad \text{and} \quad n = \gcd(r, d).$$

Definition 1.3.1 (Hecke modifications) *We call the vector bundle $\mathcal{H}_i^j(E)$ the Hecke modification of the parabolic bundle E_* with respect to the subspace $F_i^{j+1}(E) \subset E_{x_i}$ for all $i \in I$ and $j \in \{1, 2, 3, ..., \ell_i\}$. We set $\mathcal{H}_i^0(E) = E$.*

Proposition 1.3.2 (Hecke filtrations) *Let E_* be a parabolic rank-r vector bundle with respect to the parabolic divisor D. Then for each $i \in I$ the Hecke modifications over $x_i \in D$, satisfy the following inclusions for all $j \in \{1, 2, ..., \ell_i\}$*

$$E(-x_i) = \mathcal{H}_i^{\ell_i}(E) \subset \mathcal{H}_i^{\ell_i-1}(E) \subset \cdots \subset \mathcal{H}_i^2(E) \subset \mathcal{H}_i^1(E) \subset \mathcal{H}_i^0(E) = E.$$

Proof. We take the Hecke modifications over a point $x_i \in D$ for $i \in I$ and let $j \in \{1, 2, ..., \ell_i\}$, the j-th Hecke exact sequence

$$0 \longrightarrow \mathcal{H}_i^j(E) \hookrightarrow E \twoheadrightarrow Q_i^j(E) \longrightarrow 0,$$

where the last arrow is given by the composition

$$E \xrightarrow{ev_{x_i}} E_{x_i} \twoheadrightarrow Q_i^j(E) = E_{x_i}/F_i^{j+1}(E).$$

The inclusions

$$F_i^{j+1}(E) \supset F_i^{j+2}(E)$$

give surjective maps

$$Q_i^{j+1}(E) \twoheadrightarrow Q_i^j(E).$$

Then we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_i^{j+1}(E) & \hookrightarrow & E & \twoheadrightarrow & Q_i^{j+1}(E) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}_i^{j}(E) & \hookrightarrow & E & \overset{p}{\twoheadrightarrow} & Q_i^{j}(E) & \longrightarrow & 0
\end{array}$$

$$q : \mathcal{H}_i^{j+1}(E) \longrightarrow E$$

As the right diagram commutes and the map $p \circ q = 0$, we get that the image of the map q is in the subsheaf $\mathcal{H}_i^j(E)$. So as a conclusion, we get a filtration by rank-r locally free subsheaves

$$E(-x_i) = \mathcal{H}_i^{\ell_i}(E) \subset \mathcal{H}_i^{\ell_i-1}(E) \subset \cdots \subset \mathcal{H}_i^2(E) \subset \mathcal{H}_i^1(E) \subset \mathcal{H}_i^0(E) = E.$$

□

Remark 1.3.3 *By the last proposition a rank-r parabolic structure with respect to a parabolic divisor D is equivalent to the following data: $(E, \mathcal{H}_*^*(E), \alpha_*)$ such that*

- *E a rank-r vector bundle over C.*
- *$\alpha_* = (k, \vec{a}, \vec{m})$ is a parabolic type with respect to the divisor D.*
- *For all $i \in I$, we give a filtration by rank-r locally free subsheaves*

  $$E(-x_i) = \mathcal{H}_i^{\ell_i}(E) \subset \mathcal{H}_i^{\ell_i-1}(E) \subset \cdots \subset \mathcal{H}_i^2(E) \subset \mathcal{H}_i^1(E) \subset \mathcal{H}_i^0(E) = E,$$

  such that the torsion sheaves

  $$\mathcal{H}_i^j(E)/\mathcal{H}_i^{j+1}(E)$$

  are supported at $x_i \in D$ and of length

  $$\text{length}\left(\mathcal{H}_i^j(E)/\mathcal{H}_i^{j+1}(E)\right) = m_j(i).$$

1.4 Line bundles over the moduli spaces $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$

In this section we recall the description of the Picard group of the relative moduli space of semi-stable vector bundles of fixed rank and determinant and also its ample generator and its canonical line bundle. See [FGI+05] for relative Picard groups.

Let $\pi_s : \mathcal{C} \longrightarrow S$ be a smooth family of projective complex curves of genus $g \geqslant 2$. We suppose that the parabolic divisor is of degree one given by one section σ of the family π_s and that the parabolic type is trivial (see Remark 1.4.2 below for a definition). Note that the trivial parabolic structure is just the structure of a vector bundles and in this case parabolic semi-stability (resp. stability) coincide with semi-stability (resp. stability) of vector bundles. Thus the relative moduli space $\mathcal{M}^{par}_{\mathcal{C}/S}(r, 0_*, d)$ in Theorem 1.2.5 of rank r and degree d parabolic bundles coincides with the coarse relative moduli space of semi-stable rank-r and degree-d vector bundles that we denote by

$$\mathcal{U}_{\mathcal{C}/S}(r,d) := \mathcal{M}^{par}_{\mathcal{C}/S}(r, 0_*, d),$$

which is of dimension

$$\dim_{\mathbb{C}} \mathcal{U}_{\mathcal{C}/S}(r,d) = r^2(g-1)+1.$$

If we fix a line bundle $\delta \in \mathrm{Pic}(\mathcal{C}/S)$, we define a subvariety of $\mathcal{U}_{\mathcal{C}/S}(r,d)$ which is a coarse moduli space parameterizing semi-stable rank-r vector bundles with determinant δ, given as follow

$$\mathcal{SU}_{\mathcal{C}/S}(r,\delta) := \{\mathcal{E} \in \mathcal{U}_{\mathcal{C}/S}(r,d) \ / \ \det(\mathcal{E}_s) \cong \delta_s, \quad \forall s \in S\}.$$

$\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$ is an irreducible normal variety over S.

Proposition 1.4.1 *We have the following*

1. *The subspace $\mathcal{U}^s_{\mathcal{C}/S}(r,d)$ of stable vector bundles is a smooth open subset.*

2. *If r and d are coprime i.e. n=1, we have $\mathcal{U}_{\mathcal{C}/S}(r,d) = \mathcal{U}^s_{\mathcal{C}/S}(r,d)$, thus $\mathcal{U}_{\mathcal{C}/S}(r,d)$is a smooth variety over S.*

3. *In the case of genus-2 curves, rank-2 and even degree vector bundles, the moduli space $\mathcal{U}_{\mathcal{C}/S}(2,0)$ is smooth and isomorphic to a $\mathbb{P}^3_{\mathbb{C}}$-bundle over the variety S.*

4. *Except the previous case, we have that the smooth locus of the moduli space $\mathcal{U}_{\mathcal{C}/S}(r,d)$ coincides with the stable locus $\mathcal{U}^s_{\mathcal{C}/S}(r,d)$.*

The determinant map induces a morphism over S

$$\begin{array}{rcccc} \det & : & \mathcal{U}_{\mathcal{C}/S}(r,d) & \longrightarrow & \mathrm{Pic}^d(\mathcal{C}/S) \\ & & E & \longmapsto & \det(E). \end{array}$$

For a line bundle $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$, we get

$$\mathcal{SU}_{\mathcal{C}/S}(r,\delta) \cong \det{}^{-1}(\delta).$$

which is of dimension

$$\dim_{\mathbb{C}} \mathcal{SU}_{\mathcal{C}/S}(r,\delta) = (r^2-1)(g-1).$$

We also define the following fiber products $\mathcal{X}$ and $\mathcal{X}'$ over S of the relative moduli spaces with the family of curves

$$\begin{array}{ccc} \mathcal{X} \cong \mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta) & \xrightarrow{p_n} & \mathcal{SU}_{\mathcal{C}/S}(r,\delta) \\ \downarrow & & \downarrow \\ \mathcal{X}' \cong \mathcal{C} \times_S \mathcal{U}_{\mathcal{C}/S}(r,d) & \xrightarrow{p_n} & \mathcal{U}_{\mathcal{C}/S}(r,d) \\ \downarrow{\scriptstyle p_w} & & \downarrow{\scriptstyle p_e} \\ \mathcal{C} & \xrightarrow[p_s=\pi_s]{} & S \end{array}$$

Remark 1.4.2

- *By trivial parabolic type over a parabolic divisor D we mean that each point in D is trivial (see definition 1.1) and that the system of weights is trivial in the following sense*

  $$a_1(i) = 0 \text{ for all } i \in I \text{ and } k = 1.$$

  We denote a trivial parabolic type by 0_*.

- *Taking a parabolic divisor of degree one with a trivial parabolic type is just to use the constructions in the subsection 1.2.2 otherwise modulo the deformation of the parabolic divisor it is equivalent to an empty parabolic divisor thus no parabolic structure.*

1.4.1 Generalized theta divisor

Let $r \geqslant 2$ and $d \in \mathbb{Z}$. We note $n = \gcd(r, d)$. Let F be a vector bundle over the curve C such that for all vector bundles E over C of rank r and relative degree d (with respect to the map π_s), we have

$$\chi(E \otimes F) = 0,$$

which is equivalent by Riemann-Roch to

$$\begin{aligned} \deg(E \otimes F) + \operatorname{rank}(E \otimes F)(1-g) &= 0, \\ \operatorname{rank}(F)\ d + r\ \deg(F) + r\ \operatorname{rank}(F)(1-g) &= 0, \\ \operatorname{rank}(F)\,(d + r(1-g)) + r\ \deg(F) &= 0, \end{aligned}$$

Hence we get the relation

$$\deg(F) = -\frac{\operatorname{rank}(F)(d + r(1-g))}{r},$$
$$\deg(F) = -\frac{\operatorname{rank}(F)\chi(E)}{r}.$$

As we want F with the smallest rank with this propriety, it is sufficient to take

$$\operatorname{rank}(F) = \tfrac{r}{n} \quad \text{and} \quad \deg(F) = -\tfrac{\chi(E)}{n}.$$

By [Hir88] we can find such a vector bundle F for which there is a vector bundle E over C such that

$$H^0(E \otimes F) = H^1(E \otimes F) = 0.$$

In this condition for $\delta \in \operatorname{Pic}^d(C)$ we set

$$\Theta^s_{F,\delta} := \{E \in \mathcal{SU}_C(r,\delta) \ / \ H^0(E \otimes F) \neq 0\},$$

we denote its closure by $\Theta_{F,\delta}$.

Drezet and Narasimhan give the description of the ample generator of the Picard group $\operatorname{Pic}(\mathcal{SU}_C(r,\delta))$ and they describe the canonical bundle to the moduli space $\mathcal{SU}_C(r,\delta)$ for a line bundle δ over the curve C.

Theorem 1.4.3 ([DN89], Theorems B and F) *We have the following proprieties*

1. *$\Theta_{F,\delta}$ is a relative hypersurface, which is called the theta divisor.*
2. *The line bundle $\mathcal{L} := \mathcal{O}(\Theta_{F,\delta})$ is independent of F.*
3. *$\operatorname{Pic}(\mathcal{SU}_C(r,\delta))$ is isomorphic to $\mathbb{Z}$ and it is generated by $\mathcal{L}$.*
4. *The dualizing sheaf of $\mathcal{SU}_C(r,\delta)$ is*

$$K_{\mathcal{SU}_C(r,\delta)} \cong \mathcal{O}(-2n\Theta_{F,\delta}) = \mathcal{L}^{-2n}.$$

where $n = \gcd(r,d)$.

1.4.2 Determinant line bundle

Let $\mathcal{E}$ be a family of semi-stable rank-r vector bundles over $\mathcal{C} \times_S \mathcal{T}$ with fixed determinant $\delta \in \operatorname{Pic}^d(\mathcal{C}/S)$ parameterized by a S-variety $\mathcal{T}$. We get a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} \times_S \mathcal{T} & \xrightarrow{p_n} & \mathcal{T} \\ \downarrow{\scriptstyle p_w} & & \downarrow{\scriptstyle p_e} \\ \mathcal{C} & \xrightarrow[\pi_s = p_s]{} & S \end{array}$$

Definition 1.4.4 (Determinant line bundle[KD76]) *Let $\mathcal{E} \longrightarrow \mathcal{C} \times_S \mathcal{T}$ be a family of vector bundles as above. We define*

$$\det R^{\bullet} p_{n_*}(\mathcal{E}) := (\det p_{n_*}(\mathcal{E}))^{-1} \otimes \det R^1 p_{n_*}(\mathcal{E}),$$

which is an element of $\mathrm{Pic}(\mathcal{T}/S)$ *as $\mathcal{T}$ is a S-variety and we call it the determinant line bundle associated to $\mathcal{E}$ with respect to the map $p_n : \mathcal{C} \times_S \mathcal{T} \longrightarrow \mathcal{T}$.*

We have the following lemmas that summarise some proprieties of the determinant line bundle (see [KD76], see also [Pau98]).

Lemma 1.4.5

- *For any short exact sequence of vector bundles over $\mathcal{C} \times_S \mathcal{T}$*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

we have the equality

$$\det R^{\bullet} p_{n_*}(\mathcal{E}) = \det R^{\bullet} p_{n_*}(\mathcal{E}') \otimes \det R^{\bullet} p_{n_*}(\mathcal{E}'').$$

- *Let $\sigma : S \longrightarrow \mathcal{C}$ be a section of the map p_s and let ι_σ be the closed immersion*

$$\sigma(S) \times_S \mathcal{T} \cong \mathcal{T} \hookrightarrow \mathcal{C} \times_S \mathcal{T}.$$

For a vector bundle $\mathcal{F}$ over $\mathcal{T}$, we have

$$\det R^{\bullet} p_{n_*}(\iota_{\sigma_*} \mathcal{F}) = (\det \mathcal{F})^{-1},$$

where $\iota_{\sigma_*} \mathcal{F}$ is the push-forward of the vector bundle $\mathcal{F}$ by the map ι_σ.

Lemma 1.4.6 (Serre duality)
The determinant line bundle of $\mathcal{E}$ with respect to the map p_n satisfies Serre duality

$$\det R^{\bullet} p_{n_*}(\mathcal{E}) \cong \det R^{\bullet} p_{n_*}\left(\mathcal{E}^{\vee} \otimes p_w^* K_{\mathcal{C}/S}\right),$$

where $K_{\mathcal{C}/S}$ is the relative canonical line bundle of the family $\mathcal{C}/S$. We note that

$$p_w^* K_{\mathcal{C}/S} \cong K_{\mathcal{C} \times_S \mathcal{T}/\mathcal{T}}.$$

Lemma 1.4.7 *Let $\mathcal{E}$ be a vector bundle over $\mathcal{C} \times_S \mathcal{T}$ and F a vector bundle over $\mathcal{C}/S$. If* $\det(\mathcal{E}_t)$ *is independent of $t \in \mathcal{T}$, then for $\sigma : S \longrightarrow \mathcal{C}$ a section of p_s we have*

1. $\det\left(\mathcal{E}|_{\sigma(S) \times_S \mathcal{T}}\right)$ *is independent of the section σ.*
2. *If we denote υ the relative degree of the bundle F i.e. $\upsilon := \deg(F|_{p^{-1}(s)})$, then*

$$\det R^{\bullet} p_{n_*}(\mathcal{E} \otimes p_w^* F) \cong (\det R^{\bullet} p_{n_*}(\mathcal{E}))^{\mathrm{rank}(F)} \otimes (\det \mathcal{E}_\sigma)^{-\upsilon},$$

where $\mathcal{E}_\sigma = \mathcal{E}|_{\sigma(S) \times_S \mathcal{T}}$.

Lemma 1.4.8 *For any base change by a S-morphism $f : \mathcal{T}' \longrightarrow \mathcal{T}$, we have an isomorphism*

$$\det R^{\bullet} p'_{n_*}(f^*(\mathcal{E})) \cong \det R^{\bullet} p_{n_*}(\mathcal{E}),$$

where $p'_n : \mathcal{C} \times_S \mathcal{T}' \longrightarrow \mathcal{T}'$ is the projection map over $\mathcal{T}'$.

Remark 1.4.9 *Let $\mathcal{L}_{\mathcal{T}}$ be a line bundle over $\mathcal{T}$. Then the projection formula leads to the isomorphism*

$$\det R^{\bullet} p_{n_*}\left(\mathcal{E} \otimes p_n^*(\mathcal{L}_{\mathcal{T}})\right) \cong \det R^{\bullet} p_{n_*}(\mathcal{E}) \otimes \mathcal{L}_{\mathcal{T}}^{-\chi(\mathcal{E}_t)},$$

where $\mathcal{E}_t$ is the restriction of $\mathcal{E}$ to the curve $p_n^{-1}(t)$. Note that the Euler-characteristic $\chi(\mathcal{E}_t)$ does not depend on t.

From now on and for simplicity we will denote the determinant line bundle with respect to a morphism $\pi : \mathcal{C} \times_S \mathcal{T} \longrightarrow \mathcal{T}$ as follow

$$\lambda_{\pi}(\mathcal{E}) := \det R^{\bullet} \pi_*(\mathcal{E}),$$

and we omit the reference to the map if the context is clear.

Suppose $S = Spec(\mathbb{C})$. Let $\mathcal{E}$ a virtual universal bundle over $\mathcal{SU}_C(r, \delta)$, then

- The ample generator of the Picard group is expressed as follow

  $$\mathcal{L} = \lambda(\mathcal{E} \otimes p_w^*(F)).$$

  where F is a vector bundle given in the section 1.4.1.

- The canonical bundle satisfy the equalities [LS97]

  $$\mathcal{L}^{-2n} = K_{\mathcal{SU}_C(r,\delta)} = \lambda\left(\mathrm{End}^0(\mathcal{E})\right)^{-1}.$$

In the relative case, by Theorem 1.4.3 there is a relative line bundle $\mathcal{L}$ such that we have an isomorphism

$$\mathrm{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r, \delta)/S) \cong \mathbb{Z}\mathcal{L},$$

where $\mathcal{L}$ a relative ample generator (determined modulo line bundles over S). By relative ample we mean for each closed point $s \in S$ we have

$$\mathcal{L}_s \text{ is the ample generator of the group } \mathrm{Pic}(\mathcal{SU}_{\mathcal{C}_s}(r, \delta_s)).$$

1.5 Classifying maps

Let $\mathcal{E}_*$ be a family of rank-r parabolic vector bundles of fixed parabolic type α_* over $\mathcal{C} \times_S \mathcal{T}$ with parabolic structure at the divisor D with fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ parameterized by a S-variety $\mathcal{T}$. As the semi-stability is an open condition we get rational maps from $\mathcal{T}$ over S, we assume the existence of parabolic bundles having semi-stable underlying vector bundles.

- To the relative moduli space of parabolic semi-stable rank-r vector bundles with fix determinant δ and parabolic type α_*

$$\begin{array}{rcl} \psi_{\mathcal{T}} : \mathcal{T} & \dashrightarrow & \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) \\ t & \longmapsto & [\mathcal{E}_{t*}] \end{array}$$

  where $[\mathcal{E}_{t*}]$ is the S-equivalence class of the semi-stable parabolic bundle $\mathcal{E}_{t_*} := \mathcal{E}_*|_{\mathcal{C}_t}$ over the curve $\mathcal{C}_t := \pi_s^{-1}(\pi_e(t))$.

- To the relative moduli spaces of semi-stable rank-r vector bundles with fixed determinant $\delta_j(i) \in \mathrm{Pic}^{d_j(i)}(\mathcal{C}/S)$ for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ by associating the Hecke modifications (see subsection 1.3)

$$\begin{array}{rcl} \phi^{\mathcal{T}}_{i,j} : \mathcal{T} & \dashrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i)) \\ t & \longmapsto & [\mathcal{H}^j_i(\mathcal{E}_t)] \end{array}$$

  defined as follow

$$\mathcal{H}^j_i(\mathcal{E}_t) := \ker\{\mathcal{E} \longrightarrow Q^j_i(\mathcal{E})\}.$$
$$\delta_j(i) := \delta\left(-r_j(i)\sigma_i(S)\right).$$

- The forgetful rational map (we forget the parabolic structure)

$$\begin{array}{rcl} \phi_{\mathcal{T}} : \mathcal{T} & \dashrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta) \\ t & \longmapsto & [\mathcal{E}_t] \end{array}$$

We call these maps the classifying morphisms.

Remark 1.5.1

1. *Suppose $\mathcal{T}$ is a Noetherian integral separated locally factorial and regular in codimension one S-scheme. Let $\mathcal{V}$ an open subset with compliment $\mathcal{V}^c$, then*

$$\mathrm{codim}(\mathcal{V}^c, \mathcal{T}) \geqslant 2 \Longrightarrow \mathrm{Pic}(\mathcal{T}/S) \simeq \mathrm{Pic}(\mathcal{V}/S).$$

   See [Har13] Chapter 2. Section 6, for the definitions and proprieties.

2. *Let $\mathcal{T}$ be a S-scheme satisfying the proprieties of the previous remark, then*

   (a) *If k is large enough, which is equivalent to the existence of a reel number ε small enough such that*

$$\frac{1}{k}\left(\sum_{i=1}^{N}\sum_{j=1}^{\ell_i} m_j(i)\ a_j(i)\right) < \varepsilon$$

   Then there exists an open subset $\mathcal{V} \subset \mathcal{T}$ satisfying $\mathrm{codim}(\mathcal{V}^c, \mathcal{T}) \geqslant 2$ where the maps ϕ is defined and the pull-back $\phi^*_{\mathcal{T}}(\mathcal{L})$ extends from $\mathcal{V}$ to $\mathcal{T}$. i.e., one has the implication

$$E_* \text{ parabolic stable} \Longrightarrow E \text{ is semi-stable (as vector bundle)}$$

(b) *If k is large enough, such that for all $i \in I$ and $j \in \{1, 2, \ldots, \ell_i\}$, there is an open subsets $\mathcal{V}_j(i) \subset \mathcal{T}$ and $\operatorname{codim}(\mathcal{V}_j(i)^c, \mathcal{T}) \geqslant 2$ where the maps $\phi_{i,j}$ are defined. Then the pull-backs ${\phi_{i,j}^{\mathcal{T}}}^* (\mathcal{L}_{i,j})$ extends to all the space $\mathcal{T}$.*

If $\mathcal{T} = \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ is smooth,[1] then it satisfies all the conditions in the first point in the previous remark. Then for k large enough, there is a big open subset such that:

$$E_* \text{ is parabolic semi-stable} \Longrightarrow E \text{ and } \mathcal{H}^j_i(E) \text{ are semi-stable.}$$

The pull-backs under the classifying morphisms $\phi_{i,j}, \phi$ (we drop the reference to the parameter space) of the ample generators of $\operatorname{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i))/S)$ and $\operatorname{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r, \delta)/S)$ respectively extends to all the space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$. We denote them by $\Theta_j(i)$ and Θ respectively.

Theorem 1.5.2 ([NR93]) *Let $\mathcal{E}$ be a relative family of rank-r vector bundles with fixed determinant $\delta \in \operatorname{Pic}^d(\mathcal{C}/S)$ parameterized by a S-scheme $\mathcal{T}$ over the family $p_s : \mathcal{C} \longrightarrow S$ of curves, then we have*

$$\phi^*_{\mathcal{T}}(\mathcal{L}) = \lambda(\mathcal{E})^{\frac{r}{n}} \otimes \det(\mathcal{E}_\sigma)^{\aleph},$$

where

$$\aleph = \frac{d + r(1-g)}{n} \quad \text{and} \quad n = \gcd(r, d).$$

$\phi_{\mathcal{T}} : \mathcal{T} \longrightarrow \mathcal{SU}_{\mathcal{C}/S}(r, \delta)$ is the classifying morphism to the relative moduli space of semi-stable rank-r bundles of determinant δ and $\sigma : S \longrightarrow \mathcal{C}$ a section of the map p_s.

Remark 1.5.3 *The determinant line bundle depends on the choice of the relative family in the following sense: for a line bundle $L \in \operatorname{Pic}(\mathcal{T}/S)$ we have*

$$\lambda(\mathcal{E} \otimes p_n^* L) = (\det p_{n*}(\mathcal{E} \otimes p_n^* L))^{-1} \otimes \det R^1 p_{n*}(\mathcal{E} \otimes p_n^* L),$$

by the projection formula we get

$$\lambda(\mathcal{E} \otimes p_n^* L) = (\det(p_{n*}(\mathcal{E}) \otimes L))^{-1} \otimes \det\left(R^1 p_{n*}(\mathcal{E}) \otimes L\right),$$

hence

$$\lambda(\mathcal{E} \otimes p_n^* L) = (\det p_{n*}(\mathcal{E}))^{-1} \otimes \det R^1 p_{n*}(\mathcal{E}) \otimes L^{-(d+r(1-g))},$$

which gives the following relation

$$\lambda(\mathcal{E} \otimes p_n^* L) = \lambda(\mathcal{E}) \otimes L^{-n\aleph}.$$

So the line bundle

$$\lambda(\mathcal{E})^{\frac{r}{n}} \otimes \det(\mathcal{E}_\sigma)^{\aleph},$$

[1] See [BY99] for more details.

is independent of the choice of the equivalence class of the family $\mathcal{E}$.ie. for any line bundle $L \in \mathrm{Pic}(\mathcal{T}/S)$ we have the relation

$$\begin{aligned}\lambda(\mathcal{E} \otimes p_n^* L)^{\frac{r}{n}} \otimes \det\left(\mathcal{E}_\sigma \otimes L\right)^{\aleph} &= \lambda(\mathcal{E})^{\frac{r}{n}} \otimes L^{-r\aleph} \otimes \det\left(\mathcal{E}_\sigma\right)^{\aleph} \otimes L^{r\aleph} \\ &= \lambda(\mathcal{E})^{\frac{r}{n}} \otimes \det\left(\mathcal{E}_\sigma\right)^{\aleph}.\end{aligned}$$

Remark 1.5.4 *By applying Theorem 1.5.2 for $\mathcal{T} = \mathcal{SU}_{\mathcal{C}/S}(r, \delta)$, we get for any virtual universal bundle $\mathcal{E}$ a relative ample generator $\mathcal{L}$ of the Picard group given by the formula*

$$\mathcal{L} := \lambda(\mathcal{E})^{\frac{r}{n}} \otimes \det(\mathcal{E})^{\aleph}.$$

1.6 Parabolic determinant line bundle

Let $\mathcal{E}_*$ be a relative family of parabolic rank-r vector bundles over $\mathcal{C} \times_S \mathcal{T}$ of determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ and fixed parabolic type α_* over a smooth family of curves $(\mathcal{C}, D)/S$ parameterized by a S-variety $\mathcal{T}$. Let $\pi_n : \mathcal{C} \times_S \mathcal{T} \longrightarrow \mathcal{T}$ be the projection map.

We assume the following condition

$$(\bigstar) \qquad \left(kd + \sum_{i=1}^{N} \sum_{j=1}^{\ell_i} m_j(i) a_j(i)\right) \in r\mathbb{Z}.$$

Definition 1.6.1 [BR93] *We define the parabolic determinant line bundle as following*

$$\lambda_{par}(\mathcal{E}_*) := \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i} \left\{\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right)^{-a_j(i)}\right\} \otimes \det(\mathcal{E}_\sigma)^{\frac{k\chi_{par}}{r}},$$

which is a line bundle over $\mathcal{T}/S$, where

- *$\mathcal{E}_\sigma := \mathcal{E}|_{\sigma(S) \times_S \mathcal{T}}$ for some section σ of the map $\pi_s : \mathcal{C} \longrightarrow S$.*
- *The determinant line bundle bundle with respect to the map π_n :*

$$\lambda(\mathcal{E}) := \det R^{\bullet} \pi_{n*}(\mathcal{E}) := (\det \pi_{n*} \mathcal{E})^{-1} \otimes \det R^1 \pi_{n*}(\mathcal{E}).$$

- $\chi_{par} = d + r(1-g) + \frac{1}{k} \sum_{i=1}^{N} \sum_{j=1}^{\ell_i} m_j(i) a_j(i)$.

Remark 1.6.2 *If $(\bigstar)$is not satisfied, the bundle $\lambda_{par}(\mathcal{E})$ is not well-defined over $\mathcal{T}$. In this case, we take as a definition its r-th power which is a line bundle over $\mathcal{T}$.*

Theorem 1.6.3 ([Pau96]) *Let $\mathcal{E}_*$ be a relative family of rank-r parabolic vector bundles over $\mathcal{C} \times_S \mathcal{T}$ with fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ and of parabolic type α_* parameterized by a S-scheme $\mathcal{T}$ over a family of smooth projective curves $\pi_s : \mathcal{C} \longrightarrow S$ equipped with a family of degree N parabolic divisors given by N-sections of the map π_s. Then there is a relatively ample line bundle*

$$\Theta_{par} \in \mathrm{Pic}\left(\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha, \delta)/S\right)$$

such that

$$\psi^*_{\mathcal{T}}(\Theta_{par}) = \lambda_{par}(\mathcal{E}_*),$$

where $\psi_{\mathcal{T}} : \mathcal{T} \longrightarrow \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ is the classifying morphism to the relative moduli space of semi-stable rank-r parabolic vector bundles with determinant δ and parabolic type α_*.

If we apply this theorem to a virtual universal parabolic bundle $\mathcal{E}_*$ over $\mathcal{T} = \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ we get the expression of a relative ample line bundle

$$\Theta_{par} = \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i} \left\{ \det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right)^{-a_j(i)} \right\} \otimes \det(\mathcal{E}_\sigma)^{\frac{k\chi_{par}}{r}}.$$

Chapter 2

The Yokogawa-Maruyama point of view of parabolic vector bundles

In this chapter we give the Yokogawa point of view on parabolic vector bundles and their moduli space. Simpson in[Sim90] gives another description of parabolic vector bundles as filtered bundles which can be generalized to higher dimension. Maruyama and Yokogawa in [MY92], [Yok91] and [Yok93] give the construction of the relative moduli space of semi-stable parabolic vector bundles with the new description and they prove that the moduli space they constructed is isomorphic to the moduli spaces of semi-stable parabolic bundles.

Let C a complex projective smooth curve and $D = \sum_{i=1}^{N} x_i$ a reduced divisor of degree N on C and set $I = \{1, 2, ..., N\}$.

Definition 2.0.1 (Filtered vector bundles [Sim90]) *A filtered rank-r bundle over the marked curve (C, D) is a rank-r vector bundle E over C together with filtrations $E_\bullet = (E_{\lambda,i})_{\substack{i,\in I \\ \lambda\in\mathbb{R}}}$, satisfying for all $i \in I$ the following conditions*

1. *Local freeness: $E_{\lambda,i}$ are locally free of rank-r, $\forall\lambda \in \mathbb{R}$ and $E_{0,i} = E$.*
2. *Decreasing: $E_{\lambda,i} \subset E_{\beta,i}$ for all $\lambda \geqslant \beta$.*
3. *Left continuous: for $\varepsilon > 0$ sufficiently small real number, $E_{\lambda-\varepsilon,i} = E_{\lambda,i}$.*
4. *Finiteness: the length of the filtration for $0 \leqslant \lambda \leqslant 1$ is finite.*
5. *Periodicity: for all real number λ, we have $E_{\lambda+1,i} = E_{\lambda,i}(-x_i)$.*

Definition 2.0.2 (System of weights) *Let $(E_{\lambda,i})_{i\in I,\lambda\in\mathbb{R}}$ be a filtered vector bundle with respect to the divisor D. Then we define the system of weights on x_i for $i \in I$ as the ordered jumping real numbers in the real interval $[0, 1]$ ie. $0 \leqslant \lambda \leqslant 1$ such that*

$$\textit{for } \varepsilon > 0 \textit{ small enough we have } E_{\lambda,i} \neq E_{\lambda+\varepsilon,i}.$$

We will assume that the jumping numbers are rational numbers. So we get for each $i \in I$ an ordered sequence of rational numbers

$$0 \leqslant \lambda_1(i) < \lambda_2(i) < \ldots < \lambda_{\ell_i}(i) < 1,$$

where ℓ_i is the number of jumps at the point x_i.

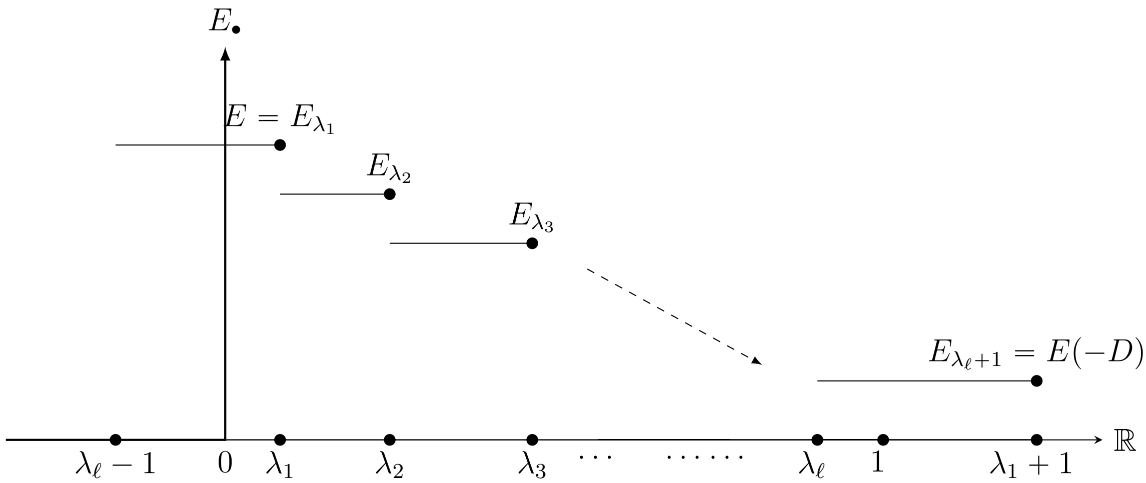
We set $m_j(i) := \operatorname{length}\left(E_{\lambda_j,i}/E_{\lambda_{j-1},i}\right)$ the length of the torsion sheaf $E_{\lambda_j,i}/E_{\lambda_{j-1},i}$ supported on x_i. We call $m_j(i)$ the multiplicity of the weight $\lambda_j(i)$.

Finally we set $\lambda_\bullet := (\lambda_j(i), m_j(i))_{\substack{i\in I \\ 1\leqslant j\leqslant \ell_i}}$ the vector of weights and multiplicities.

Remark 2.0.3 *By definition 2.0.1, we set*

$$E_\lambda = \bigcap_{i=1}^{N} E_{\lambda,i}. \tag{2.0.1}$$

So we get a filtration $E_\bullet := (E_\lambda)_{\lambda\in\mathbb{R}}$ of E by vector subbundles satisfying the first five points in definition 2.0.1 and for the periodicity we get for each $\lambda \in \mathbb{R}$, $E_{\lambda+1} = E_\lambda(-D)$.
We can illustrate a filtered vector bundle in the following graph



Definition 2.0.4 (Morphisms of filtered bundles) *Let $E_\bullet = (E_\lambda)_{\lambda\in\mathbb{R}}$ and $F_\bullet = (F_\lambda)_{\lambda\in\mathbb{R}}$ be two filtered vector bundles over the smooth marked curve (C, D). A morphism of filtered bundles is a family of $\mathcal{O}_C$-linear morphisms*

$$f_\lambda : E_\lambda \longrightarrow F_\lambda$$

such that for all $\lambda \geqslant \beta$ the diagram commute

$$\begin{array}{ccc} E_\lambda & \xrightarrow{f_\lambda} & F_\lambda \\ \bigcap & & \bigcap \\ E_\beta & \xrightarrow[f_\beta]{} & F_\beta \end{array}$$

We denote the sheaf of morphisms of filtered bundles by $\mathcal{H}om(E_\bullet, F_\bullet)$.

Definition 2.0.5 (filtered subbundles) *A filtered bundle* $F_\bullet$ *is a filtered subbundle of a filtered bundle* $E_\bullet$ *if the following conditions holds*

1. F *is a sub bundle of* E.
2. $F_\lambda \subset E_\lambda$ *for all* $\lambda \in \mathbb{R}$.
3. *If* $F_\lambda \subset E_\beta$ *for some* $\beta > \lambda$, *Then* $F_\lambda = F_\beta$.

Equivalently, let $E_\bullet$ *be a filtered bundle and* F *a subbundle of* $E := E_0$, *If we put* $F_\lambda := E_\lambda \cap F$, *then we get a structure of filtered bundle* $F_\bullet$ *over* F *induced by the structure of* $E_\bullet$.

Definition 2.0.6 (quotient of filtered bundles) *Let* $E_\bullet$ *be a filtered bundle over the marked curve* (C, D) *a quotient of filtered bundle, is the following data*

1. *A filtered bundle* $F_\bullet$.
2. *A surjective filtered bundles morphism* $f_\bullet : E_\bullet \longrightarrow F_\bullet$.

$$\text{i.e. } \forall \alpha \in \mathbb{R}, \text{ the map } f_\alpha : E_\lambda \longrightarrow F_\lambda \text{ is surjective.}$$

3. *If we have* $f_\lambda(E_\lambda) \subset F_\beta$ *for some* $\beta > \lambda$, *then* $F_\lambda = F_\beta$.

Remark 2.0.7 *Every vector bundle* E *can be equipped by a natural filtered structure, given by*

$$E_\lambda := E(-\lfloor \lambda \rfloor D) \quad \forall \lambda \in \mathbb{R}.$$

This structure is called the special filtered structure.

Definition 2.0.8 (degree and slope) *Let* $E_\bullet = (E_\lambda)_{\lambda \in \mathbb{R}}$ *be a filtered rank-r bundle over the marked curve* (C, D). *Then we define the*

1. *Filtered degree*

$$\deg(E_\bullet) = \int_0^1 \deg(E_\lambda) \mathrm{d}\lambda.$$

2. *Filtered slope*

$$\mu(E_\bullet) = \frac{\deg(E_\bullet)}{r}.$$

2.1 Moduli space of filtered vector bundles

In this subsection we give Yokogawa and Maruyama's [MY92] construction of the moduli space of filtered vector bundles for a rational fixed system of weights (see Definition 2.0.2).

Definition 2.1.1 (Stability) *Let $E_\bullet = (E_\lambda)_{\lambda \in R}$ be a filtered vector bundle over the marked curve (C, D). Then $E_\bullet$ is said to be stable (resp. semi-stable) if for all filtered proper sub-bundles $F_\bullet$, we have*

$$\mu(F_\bullet) \leqslant \mu(E_\bullet) \quad (\text{resp. } <).$$

Let $\pi_s : \mathcal{C} \longrightarrow S$ be a smooth family of projective curves of genus $g \geqslant 2$, parameterized by an algebraic variety S over $\mathbb{C}$ and let

$$\sigma_i : S \longrightarrow \mathcal{C}, \qquad i \in I = \{1, 2, ..., N\},$$

be N sections of π_s such that

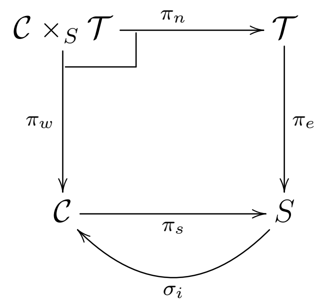
$$\forall i \neq j \in I \text{ and } \forall s \in S, \text{ we have: } \sigma_i(s) \neq \sigma_j(s).$$

We denote by

$$D := \sum_{i \in I} \sigma_i(S),$$

the associated divisor (as the relative dimension is one). We consider the couple $(\mathcal{C}, D)$ as a family of marked curves parameterized by the variety S and let $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ a relative line bundle of degree d over $\mathcal{C}$.

Let $\pi_n : \mathcal{T} \longrightarrow S$ be a S-variety. We get a Cartesian diagram



Definition 2.1.2 (Relative family of filtered bundles) *A relative family of filtered rank-r vector bundles over $(\mathcal{C}, D)$ parameterized by a S-variety $\mathcal{T}$ is the following data: a filtered rank-r vector bundle $\mathcal{E}_\bullet = (\mathcal{E}_\lambda)_{\lambda \in \mathbb{R}}$ over $\mathcal{C} \times_S \mathcal{T}$ with respect to the divisor D, such that for all $t \in \mathcal{T}$ we have*

$$\mathcal{E}_\bullet|_{\mathcal{C}_s} \textit{ is a semi-stable filtered bundle over the marked curve } (\mathcal{C}_s, D_s).$$

where $s = \pi_e(t)$, $\mathcal{C}_s := \pi_s^{-1}(s)$ and $D_s := D|_{\mathcal{C}_s}$.

Theorem 2.1.3 [Yok91] *For a fixed system of weights $\lambda_\bullet$. There is a coarse moduli space $\mathcal{M}_\bullet(r, \lambda_\bullet, d)$ which is a projective irreducible normal variety, parameterizing semi-stable filtered rank-r vector bundles of degree d with fixed system of weights $\lambda_\bullet$ modulo S-equivalence over the smooth family of marked projective curves $(\mathcal{C}, D)$. Moreover, the subspace $\mathcal{M}^s_\bullet(r, \lambda_\bullet, d) \subset \mathcal{M}_\bullet(r, \lambda_\bullet, d)$ of stable filtered bundles is an open subset and coincides with the smooth locus.*

For $\delta \in \mathrm{Pic}^d(\mathbb{C}/S)$ we denote by

$$\mathcal{M}_\bullet(r, \lambda_\bullet, \delta) := \{E_\bullet \in \mathcal{M}_\bullet(r, \lambda_\bullet, d) \quad / \quad \det(E_0) = \delta\} \subset \mathcal{M}_\bullet(r, \lambda_\bullet, d),$$

the subvariety of filtered bundles with determinant δ.

Remark 2.1.4 *We will not give the proof of this theorem as we will see that filtered bundles correspond to parabolic bundles and that this moduli space equals the moduli space of semi-stable parabolic bundles.*

2.2 Filtered bundles as Parabolic bundles

Proposition 2.2.1 *Over a smooth marked curve (C, D). Filtered rank-r vector bundle is equivalent to a parabolic rank-r vector bundle with respect to the same divisor D.*

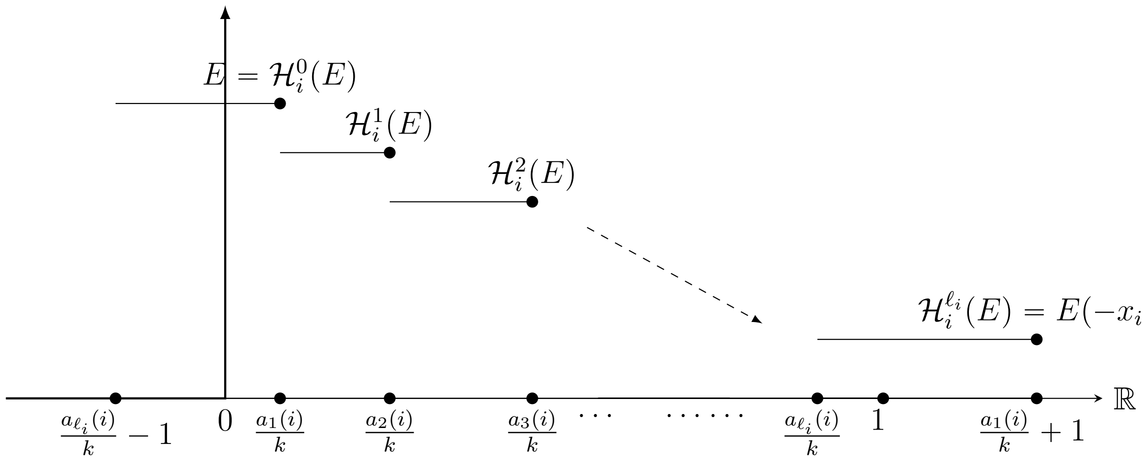
Proof. Let $D = \{x_1, x_2, ..., x_N\}$ be a parabolic divisor over the curve C. Let E_* be a parabolic rank-r vector bundle over C of type $\alpha_* = (k, \vec{a}, \vec{m})$ with respect to the parabolic divisor D. We take its Hecke filtrations 1.3.2 for each $i \in I$

$$E(-x_i) = \mathcal{H}_i^{\ell_i}(E) \subset \mathcal{H}_i^{\ell_i - 1}(E) \subset \cdots \subset \mathcal{H}_i^2(E) \subset \mathcal{H}_i^1(E) \subset \mathcal{H}_i^0(E) = E,$$

$$k > a_{\ell_i}(i) > ... > a_2(i) > a_1(i) \geqslant 0.$$

We set

$$a_0(i) = a_{\ell_i}(i) - k.$$
$$a_{\ell_i + 1}(i) = a_1(i) + k.$$



And we associate to each $\lambda \in \mathbb{R}$ and $j \in \{0, 1, ..., \ell_i + 1\}$ such that

$$\frac{a_{j-1}(i)}{k} < \lambda - \lfloor \lambda \rfloor \leqslant \frac{a_j(i)}{k},$$

the vector bundles

$$E_{\lambda,i} = \mathcal{H}_i^{j-1}(E)(-\lfloor \lambda \rfloor x_i).$$

We get filtrations $(E_{\lambda,i})_{i\in I,\lambda\in\mathbb{R}}$ that satisfy the definition 2.0.1. So a filtered bundle with respect to the divisor D. Note that the system of weight is given by the parabolic weights $\vec{a} = (a_j(i))_{\substack{i\in I \\ 1\leqslant j\leqslant \ell_i}}$ and the multiplicities are the quasi-parabolic type $\vec{m} = (m_j(i))_{\substack{i\in I \\ 1\leqslant j\leqslant \ell_i}}$.

Conversely, let $(E_{\lambda,i})_{\lambda\in\mathbb{R},i\in I}$ be a filtered rank-r vector bundle over the curve C with respect to the divisor D. By definition for each $i \in I$ there is a finite filtration for $\lambda \in [0, 1]$ that we denote as follow

$$E(-x_i) = E_{\lambda_{\ell_i},i} \subset E_{\lambda_{\ell_i-1},i} \subset ... \subset E_{\lambda_1,i} \subset E_{0,i} = E,$$

where the $\alpha_{j,i} \in \mathbb{Q}$ are the jumps for $i \in I$ and satisfies

$$1 > \lambda_{\ell_i,i} > \lambda_{\ell_i-1,i} > ... > \lambda_{1,i} \geqslant \lambda_{0,i} = 0.$$

We set for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$

$$\begin{cases} F_i^j(E) := \ker(E_{x_i} \longrightarrow E/E_{\lambda_{j-1},i}). \\ F_i^{\ell_i+1}(E) = \ker\left(E \longrightarrow \frac{E}{E_{\lambda_{\ell_i},i}} = \frac{E}{E(-x_i)}\right) = \{0\}. \end{cases}$$

and the numerical data:

$$(k, \vec{a}, \vec{m}) = \begin{cases} k := \ell.m.c\{\textit{the denominators of the } \lambda_{j,i}\}_{i,j}. \\ a_j(i) := k\lambda_{j,i}. \\ m_j(i) := \dim_{\mathbb{C}}(F_i^j(E)/F_i^{j+1}(E)). \end{cases}$$

So this data defines a parabolic structure on E of parabolic type $\alpha_* = (k, \vec{a}, \vec{m})$ over the divisor D. □

Remark 2.2.2 *In [MY92] the following equality is proved*

$$deg(E_\bullet) := \int_0^1 \deg(E_\lambda)\mathrm{d}\lambda = \mathrm{pardeg}(E_*) + \mathrm{rank}(E)\,\deg(D).$$

Proposition 2.2.3 *Let $E_\bullet = (E_\lambda)_{\lambda\in\mathbb{R}}$ be a filtered vector bundle over the marked curve (C, D). Then we have the equivalence*

$$E_\bullet \textit{ is stable (resp. semi-stable) as filtered bundle}$$
$$\Updownarrow$$
$$E_* \textit{ the associated parabolic bundle is stable (resp. semi-stable).}$$

By Propositions 2.2.1, 2.2.3 and the above remark, we get the following theorem.

Theorem 2.2.4 *Let $\pi_s : \mathcal{C} \longrightarrow S$ be a smooth family of projective complex curves equipped with a divisor $D = \sum_{i \in I} \sigma_i(S)$ of relative degree N given by N sections $(\sigma_i)_{i \in I}$ of the map π_s. For fixed parabolic weights $\alpha_* = (k, \vec{a}, \vec{m})$ we have an isomorphism of S-schemes*

$$\begin{array}{rcccc} \varpi & : & \mathcal{M}_\bullet(r, \lambda_\bullet, d) & \longrightarrow & \mathcal{M}^{par}_{\mathcal{C}/S}(r, \alpha_*, d) \\ & & E_\bullet & \longmapsto & \varpi(E_\bullet) = E_* \end{array}$$

where to the system of filtered rational weights $\lambda_\bullet$ is given as follow

$$1 > \lambda_{\ell_i, i} > \lambda_{\ell_i - 1, i} > \ldots > \lambda_{1,i} \geqslant \lambda_{0,i} = 0,$$

we associate the system of parabolic weights, for all $i \in I$ and $j \in \{1, 2, \ldots, \ell_i\}$

$$a_j(i) := k\lambda_{j,i} \textit{ and } k := \ell.m.c\{\textit{the denominators of the } \lambda_{j,i}\}_{\forall i, \forall j}.$$

Classifying maps Let $\mathcal{E}_\bullet$ be a family of filtered rank-r bundles over the smooth family of marked curves $(\mathcal{C}, D)$ over S parameterized by a S-variety $\mathcal{T}$ with fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ and fixed weights, we get for each $\lambda \in \mathbb{R}$ a rational map to the moduli space of semi-stable rank-r vector bundles of fixed determinant

$$\begin{array}{rcccc} \phi^{\mathcal{T}}_\lambda & : & \mathcal{T} & \dashrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta(\lambda)) \\ & & t & \longmapsto & \mathcal{E}_\lambda|_{\mathcal{C}_t} \end{array}$$

where for each $t \in \mathcal{T}$ we associate the curve

$$\mathcal{C}_t := \pi_n^{-1}(t) = \pi_s^{-1}(\pi_e(t))$$

and for each $\lambda \in \mathbb{R}$ we associate the line bundle

$$\delta(\lambda) := \det(\mathcal{E}_\lambda) \in \mathrm{Pic}^{d(\lambda)}(\mathcal{C}/S)$$

we set $n(\lambda) = \gcd(r, d(\lambda))$.

If $\mathcal{T} = \mathcal{M}_\bullet(r, \lambda_\bullet, \delta)$ is the moduli space of filtered bundles by Remark 1.5.1 the maps $\phi^{\mathcal{T}}_\lambda$ are defined over big open spaces that depend on $\lambda \in \mathbb{R}$ so the pull-backs of any line bundle extends and we denote by $\Theta(\lambda)$ the pull-back of ample generator of the relative Picard group of the moduli space $\mathcal{SU}_{\mathcal{C}/S}(r, \delta(\lambda))$ by the map $\phi^{\mathcal{T}}_\lambda$. Note that $\phi^{\mathcal{T}}_0$ coincides with the map $\phi_{\mathcal{T}}$ given in Subsection 1.5.

2.3 Some proprieties of filtered bundles

Let (C, D) be a smooth marked projective curve. Let $E_\bullet = (E_\lambda)_{\lambda\in\mathbb{R}}$ be a filtered vector bundle over (C, D) and let $\gamma \in \mathbb{R}$ we define the γ-shift filtered bundle $E[\gamma]_\bullet$ by

$$E[\gamma]_\lambda := E_{\lambda+\gamma}, \quad \forall\lambda \in \mathbb{R}.$$

Definition 2.3.1 (Tensor product) *Let $E_\bullet$ and $F_\bullet$ be two filtered vector bundles. For each $\lambda \in \mathbb{R}$, we set*

$$(E_\bullet \otimes F_\bullet)_\lambda := Span\left(\bigsqcup_{\lambda_1+\lambda_2=\lambda} E_{\lambda_1} \otimes F_{\lambda_2}\right).$$

Proposition 2.3.2 [Yok95] *Let $E_\bullet$ and $F_\bullet$ be two filtered bundles. Then the tensor product and the shift operation commutes, i.e. For $\gamma \in \mathbb{R}$, we have*

$$(E[\gamma]_\bullet \otimes F_\bullet)_\bullet \cong (E_\bullet \otimes F[\gamma]_\bullet)_\bullet \cong (E_\bullet \otimes F_\bullet)[\gamma]_\bullet.$$

Definition 2.3.3 [Yok95] *Let $E_\bullet$ and $F_\bullet$ two filtered bundles. For each $\lambda \in \mathbb{R}$ we set*

$$\mathcal{H}om(E_\bullet, F_\bullet)_\lambda := \mathcal{H}om(E_\bullet, F[\lambda]_\bullet).$$

Remark 2.3.4 *If we denote by E_* the associated parabolic bundle to $E_\bullet$, we get*

$$\mathcal{H}om(E_\bullet, E_\bullet)_0 = \mathrm{parEnd}(E_*).$$

Proposition 2.3.5 [Yok95] *Let $E_\bullet$ and $F_\bullet$ be two filtered bundles, then*
For each $\gamma \in \mathbb{R}$, there are natural isomorphisms

$$\mathcal{H}om(E_\bullet, F_\bullet)[\gamma]_\bullet \cong \mathcal{H}om(E[-\gamma]_\bullet, F_\bullet)_\bullet \cong \mathcal{H}om(E_\bullet, F[\gamma]_\bullet)_\bullet.$$

To define the dual of a filtered bundle and a notion of filtered morphisms that corresponds on the parabolic side to strongly-parabolic morphisms, we define the following operation on filtered bundles. Let $E_\bullet$ be a filtered bundle over (C, D)

- We associate for each $\lambda \in \mathbb{R}$ the following filtered bundle

  $$\widehat{E}_\lambda := \underset{\beta>\lambda}{lim} E_\beta$$

  which is a right-continuous filtered bundle denoted $\widehat{E}_\bullet$. Thus a parabolic bundle by Proposition 2.2.1. In fact if the filtration of $E_\bullet$ is

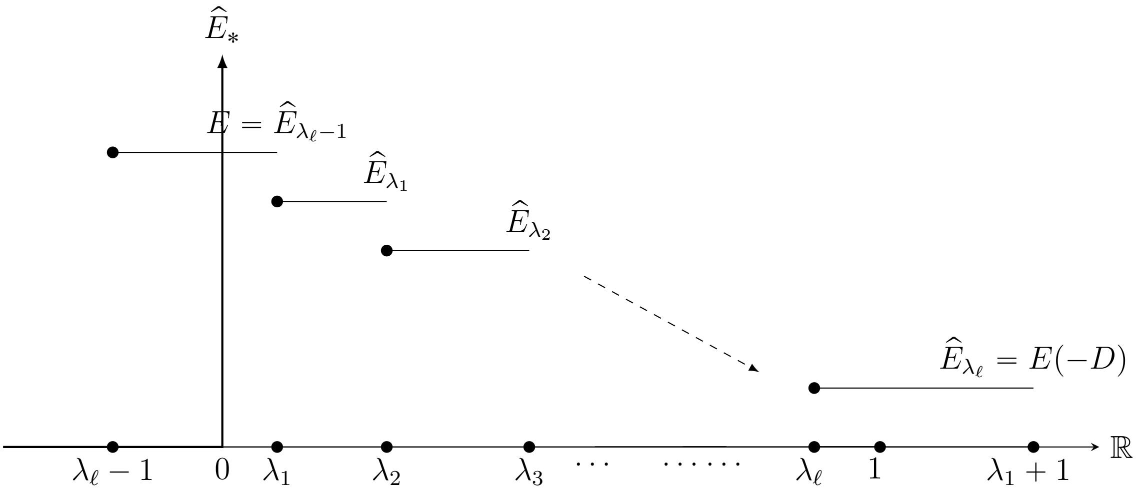
  $$E(-D) = E_{\lambda_\ell} \subset E_{\lambda_{\ell-1}} \subset ... \subset E_{\lambda_1} \subset E_0 = E,$$

with weights

$$1 > \lambda_\ell > \lambda_{\ell-1} > ... > \lambda_1 \geqslant 0.$$

Then $\widehat{E}_*$ is given by the filtration

$$\begin{cases} E_\lambda = E_{\lambda_j - 1} \text{ for all } \lambda_{j-1} \leqslant \lambda < \lambda_j \quad \text{for } j \in \{1, 2, ..., \ell+1\} \\ \text{where } \lambda_0 = \lambda_\ell - 1 \text{ and } \lambda_{\ell+1} = \lambda_1 + 1. \end{cases}$$



- We define the dual filtered bundle $E_\bullet^\vee$ by

$$E_\bullet^\vee := \mathcal{H}om(E_\bullet, \mathcal{O}_C)_\bullet$$

where the trivial line bundle is equipped with the special structure.

Remark 2.3.6 *If we denote by E_* the associated parabolic bundle to $E_\bullet$, we get*

$$\mathcal{H}om(E_\bullet, \widehat{E}_\bullet)_0 = \text{SparEnd}(E_*).$$

Proprieties 2.3.7 *Let $E_\bullet$ be a filtered bundle. Then we have*

1. *For a vector bundle F equipped with the special structure 2.0.7, we have*

$$\mathcal{H}om(E_\bullet, F)_\lambda \cong \mathcal{H}om(\widehat{E}\,[-1]_{-\lambda}\,, F) \cong \mathcal{H}om(\widehat{E}_{-\lambda}(D), F).$$

2. *$E_\bullet^{\vee\vee}$ is canonically isomorphic to $E_\bullet$.*

3. *There is a canonical isomorphism*

$$\mathcal{H}om(E_\bullet, F_\bullet)_\bullet \cong (E_\bullet^\vee \otimes F_\bullet)_\bullet\,.$$

Remark 2.3.8 *Let $E_\bullet$ a filtered bundle over the marked curve (C, D).*

1. *We note that the underlying bundle of* $E_\bullet^\vee$ *is* $E^\vee(-D)$.

2. *We have the following isomorphism of filtered bundles*

$$\mathcal{H}om(E_\bullet, E_\bullet)_\bullet^\vee \cong \mathcal{H}om(E_\bullet, E_\bullet)_\bullet,$$

but, in general

$$\mathcal{H}om(E_\bullet, E_\bullet)^\vee \ncong \mathcal{H}om(E_\bullet, E_\bullet).$$

2.4 Parabolic transformation group of $\mathcal{SM}_C^{par}(r, \alpha_*, \delta)$

Let C be a smooth projective complex curve and $D = \{x_1, x_2, ..., x_N\}$ a degree N reduced divisor. Let E_* be a parabolic bundle of parabolic type α_* with respect to the divisor D. We recall that for each $i \in I$ and $j \in \{1, , 2, ..., \ell_i\}$ the Hecke modification $\mathcal{H}_i^j(E)$ is equipped with a natural quasi-parabolic structure induced by the structure of E_* as follows

1. Over $C \backslash \{x_i\}$ we have an isomorphism of sheaves

$$f : \mathcal{H}_i^j(E)|_{C\backslash\{x_i\}} \longrightarrow E|_{C\backslash\{x_i\}}.$$

2. For $q \neq i$, we take the pullback by f of the filtration over x_q

$$F_q^*(\mathcal{H}_i^j(E)) := f^{-1}(F_q^*(E)).$$

3. At the point x_i we associate

$$0 \longrightarrow \mathcal{H}_i^j\left(F_i^{j+1}(E)\right) \hookrightarrow \mathcal{H}_i^j(E)|_{x_i} \xrightarrow{\quad f \quad} E_{x_i} \longrightarrow E_{x_i}/F_i^{j+1}(E) \longrightarrow 0$$

$$\mathcal{H}_i^j(E)|_{x_i} \searrow F_i^{j+1}(E) \nearrow E_{x_i}, \qquad 0 \nearrow F_i^{j+1}(E) \searrow 0$$

We call the linear subspace

$$\mathcal{H}_i^j\left(F_i^{j+1}(E)\right) := \ker(\mathcal{H}_i^j(E)|_{x_i} \longrightarrow F_i^{j+1}(E))$$

the Hecke transform of $F_i^{j+1}(E)$.

4. Take the filtration at x_i

$$E|_{x_i} = F_i^1(E) \supset F_i^2(E) \supset \cdots F_i^j(E) \supset \cdots \supset F_i^{\ell_i}(E) \supset F_i^{\ell_i+1}(E) = \{0\}$$

as the image of f is in $F_i^{j+1}(E)$, we associate

- for $s \geqslant j+1$, we take the pull-back by f

$$\mathcal{H}_i^j(E)|_{x_i} = f^{-1}\left(F_i^{j+1}(E)\right) \supset \cdots f^{-1}\left(F_i^s(E)\right) \supset \cdots \supset f^{-1}\left(F_i^{\ell_i}(E)\right) \supset f^{-1}(\{0\})$$

- for $s \leqslant j+1$, we get a filtration of $E|_{x_i}/F_i^{j+1}(E)$

$$\frac{E|_{x_i}}{F_i^{j+1}(E)} = \frac{F_i^1(E)}{F_i^{j+1}(E)} \supset \cdots\cdots \frac{F_i^s(E)}{F_i^{j+1}(E)} \supset \cdots\cdots \supset \frac{F_i^j(E)}{F_i^{j+1}(E)} \supset \frac{F_i^{j+1}(E)}{F_i^{j+1}(E)} = \{0\}$$

Note that

$$\mathcal{H}_i^j\left(F_i^{j+1}(E)\right) \cong \frac{E|_{x_i}}{F_i^{j+1}(E)} \otimes \mathcal{O}_C(-x_i)|_{x_i}$$

thus, $\mathcal{H}_i^j(E)|_{x_i}$ gets the induced filtration of same length as that of $E|_{x_i}$.

Hence $\mathcal{H}_i^j(E)$ gets a quasi-parabolic structure of same type as E_* over the same divisor.

Definition 2.4.1 (Basic transformations [AG21]) *A basic transformation of a quasi-parabolic vector bundle is a tuple $T = (\tau, s, L, H)$ consisting on*

- *An automorphism $\tau : C \longrightarrow C$ and $\tau(D) = D$.*
- *A sign $s \in \{1, -1\}$.*
- *A line bundle $L \in \mathrm{Pic}(C)$.*
- *Hecke modifications. That can be expressed for full flag quasi-parabolic structure as follows: Let $H = \sum_{i=1}^{N} h_i x_i$ be an effective divisor supported on D such that $0 \leqslant h_i \leqslant (r-1)$ for all $i \in I$.*

Given a quasi-parabolic vector bundle E_*, then a basic transformation T acts as follow

$$T(E_*) := \begin{cases} \tau^*\left(L \otimes \mathcal{H}_H(E_*)\right) & s = 1 \\ \tau^*\left(L \otimes \mathcal{H}_H(E_*)\right)^\vee & s = -1. \end{cases}$$

where we define the transformation $\mathcal{H}_H(E_*)$ as follow

$$\mathcal{H}_H := \overline{\mathcal{H}}_1^{h_i} \circ \overline{\mathcal{H}}_2^{h_2} \circ \cdots \circ \overline{\mathcal{H}}_N^{h_N}.$$

and $\overline{\mathcal{H}}_i^{h_i}$ is given by

$$\overline{\mathcal{H}}_i^{h_i} := \underbrace{\mathcal{H}_i^2 \circ \mathcal{H}_i^2 \circ \cdots \circ \mathcal{H}_i^2}_{h_i}$$

$\mathcal{H}_i^2$ is the standard Hecke modification at the point x_i with respect to the subspace $F_i^2(E)$.

If $\det(E) = \delta$, we set $T(\delta) := \det\left(T(E_*)\right)$ the determinant of the transformation.

Example 2.4.2 (Full flag case) *we have*

1. $\overline{\mathcal{H}}_i^{\ell_i} = E_* \otimes \mathcal{O}_C(-x_i)$.
2. $$T(\delta) := \det\left(T(E_*)\right) = \begin{cases} \tau^*\left(L^r \otimes \delta(-H)\right) & s = 1 \\ \tau^*\left(L^r \otimes \delta(-H)\right)^\vee & s = -1. \end{cases}$$

Parabolic system of weights under basic transformations

Let E_* be a parabolic rank-r bundle over the curve C of a full flag parabolic type $\alpha_* = (k, \vec{a}, \vec{m})$ with respect to the parabolic divisor $D = \sum_{i \in I} x_i$. (see Alfaya-Gomez[AG21]).

1. Let $\tau : C \longrightarrow C$ be an automorphism and $\tau(D) = D$, then
$$\tau^*(a_j(i)) := a_j(\tau^{-1}(i)).$$
we denote the associated parabolic type by $\tau(\alpha_*)$.
2. Take the parabolic dual $E_*^\vee$, then we associate the weights
$$a_j^\vee(i) := 1 - a_j(i).$$
we denote the associated parabolic type by $\alpha_*^\vee$.
3. Twisting with a line bundle does not affect the parabolic weights.
4. Let $H = \sum_{i=1}^{N} h_i x_i$ be an effective divisor supported on D such that $0 \leqslant H \leqslant (r-1)D$. We define $\mathcal{H}_H(\vec{a})$
$$\mathcal{H}_H(\vec{a}) := \begin{cases} a_{j+h_i}(i) - a_{1+h_i}(i) & j + h_i \leqslant r \\ a_{j+h_i-r}(i) - a_{1+h_i}(i) + 1 & j + h_i > r. \end{cases}$$
we denote the associated parabolic type for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ by $\mathcal{H}_i^j(\alpha_*)$.

So, if $\vec{a}$ is a parabolic system of weights with respect to the divisor D over C, we define for a basic transformation $T = (\tau, s, L, H)$

$$T(\vec{a}) := \begin{cases} \mathcal{H}_H(\vec{a})_j(\tau^{-1}(i)) & s = 1 \\ 1 - \mathcal{H}_H(\vec{a})_{r-j+1}(\tau^{-1}(i)) & s = -1. \end{cases}$$

Chapter 3

Hitchin connection in algebraic geometry

In this chapter we introduce the van Geemen-de Jong approach to the construction of a connection on a direct image of a line bundle by giving a heat operator on the line bundles. We will define connections, heat operators, the relation between them and the van Geemen-de Jong theorem, which is the algebraic geometry analogue of Hitchin's theorem in Kähler geometry. We follow [GdJ98].

Throughout this section we take $\pi : \mathcal{M} \longrightarrow S$, a smooth surjective morphism of regular $\mathbb{C}$-schemes, we have the natural exact sequence on the tangent bundles

$$0 \longrightarrow T_{\mathcal{M}/S} \longrightarrow T_{\mathcal{M}} \xrightarrow{d\pi} \pi^* (T_S) \longrightarrow 0. \tag{3.0.1}$$

We define the sheaf of differential operators and the sheaf of relative operators.

Definition 3.0.1 (Differential operators) *Let E be a locally free sheaf over $\mathcal{M}$. We define the sheaf $\mathcal{D}^{(k)}_{\mathcal{M}}(E)$ of differential operators of order at most k over E by induction on the degree as follow:*

- $\forall k \in \mathbb{N}$, *we have*
  $$\mathcal{D}^{(k)}_{\mathcal{M}}(E) \hookrightarrow \mathrm{End}_{\mathbb{C}}(E).$$
  is a sub-sheaf of $\mathbb{C}$-linear maps of E.
- $\mathcal{D}^{(0)}_{\mathcal{M}}(E) = \mathrm{End}_{\mathcal{O}_{\mathcal{M}}}(E)$.
- *An element $P \in \mathcal{D}^{(k)}_{\mathcal{M}}(E)$ is a $\mathbb{C}$-linear map*
  $$P : E \longrightarrow E$$
  such that for each $f \in \mathcal{O}_{\mathcal{M}}$ we have
  $$[P, f] := Pf - fP \in \mathcal{D}^{(k-1)}_{\mathcal{M}}(E).$$
  The element $[P, Q]$ is called the commutator of the differential operators P and Q.

We define the sheaf $\mathcal{D}^{(k)}_{\mathcal{M}/S}(E)$ of relative differential operators with respect to the map $\pi : \mathcal{M} \longrightarrow S$ as a the sub-sheaf of operators that are $\pi^{-1}(\mathcal{O}_S)$-linear.

Definition 3.0.2 (Symbol map) *We have the natural inclusion for each $k \in \mathbb{N}$*

$$\mathcal{D}^{(k-1)}_{\mathcal{M}}(E) \hookrightarrow \mathcal{D}^{(k)}_{\mathcal{M}}(E).$$

Thus we get a short exact sequence

$$0 \longrightarrow \mathcal{D}^{(k-1)}_{\mathcal{M}}(E) \longrightarrow \mathcal{D}^{(k)}_{\mathcal{M}}(E) \longrightarrow \mathrm{Sym}^k(T_{\mathcal{M}}) \otimes \mathrm{End}(E) \longrightarrow 0.$$

where $\mathrm{Sym}^k(\mathcal{T}_{\mathcal{M}})$ is the k-th symmetric power of the tangent bundle. The natural map

$$\nabla_k : \mathcal{D}^{(k)}_{\mathcal{M}}(E) \longrightarrow \mathrm{Sym}^k(T_{\mathcal{M}}) \otimes \mathrm{End}(E),$$

is what we call the symbol map of order k.

By restriction to the subsheaf $\mathcal{D}^{(k)}_{\mathcal{M}/S}(E)$ of relative differential operators, we get a map

$$\nabla_k : \mathcal{D}^{(k)}_{\mathcal{M}/S}(E) \longrightarrow \mathrm{Sym}^k(T_{\mathcal{M}}) \otimes \mathrm{End}(E)$$

with image in the sub-sheaf $\mathrm{Sym}^k(T_{\mathcal{M}/S})$. Hence

$$\nabla_k : \mathcal{D}^{(k)}_{\mathcal{M}/S}(E) \longrightarrow \mathrm{Sym}^k(T_{\mathcal{M}/S}) \otimes \mathrm{End}(E).$$

we call it the relative symbol map.

3.1 Connections on vector bundles

We follow Atiyah's description of Atiyah algebroids and exact sequences [Ati57] in the context of vector bundles rather than principal bundles.

3.1.1 Atiyah classes

Definition 3.1.1 (Atiyah Class) *Let E be a vector bundle over $\mathcal{M}$. Then the Atiyah exact sequence associated to E is given by the following pull-back*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{A}_{\mathcal{M}}(E) & \xrightarrow{\nabla_1} & T_{\mathcal{M}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle -\otimes id} & & \\
0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{D}^{(1)}_{\mathcal{M}}(E) & \xrightarrow{\nabla_1} & T_{\mathcal{M}} \otimes \mathrm{End}(E) & \longrightarrow & 0
\end{array}$$

The sheaf $\mathcal{A}_{\mathcal{M}}(E)$ is called the Atiyah algebroid of E. We denote its extension class by $at_{\mathcal{M}}(E)$. As an extension

$$at_{\mathcal{M}}(E) \in \mathrm{Ext}^1(T_{\mathcal{M}}, \mathrm{End}(E)) \simeq \mathrm{H}^1(\mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathrm{End}(E))$$

as we deal with locally free sheaves.

For a line bundle $L \in \mathrm{Pic}(\mathcal{M})$ *the Atiyah sequence coincides with the sequence*

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{D}^{(1)}_{\mathcal{M}}(L) \xrightarrow{\nabla_1} T_{\mathcal{M}} \longrightarrow 0 \tag{3.1.1}$$

and its extension class

$$at_{\mathcal{M}}(L) \in \mathrm{H}^1(\mathcal{M}, \Omega^1_{\mathcal{M}}).$$

Note that the Atiyah class can be given as follows: we tensorize the Atiyah class (3.1.1) with the cotangent sheaf $\Omega^1_{\mathcal{M}}$, we get

$$0 \longrightarrow \mathrm{End}(E) \otimes \Omega^1_{\mathcal{M}} \longrightarrow \mathcal{A}_{\mathcal{M}}(E) \otimes \Omega^1_{\mathcal{M}} \xrightarrow{\nabla_1} T_{\mathcal{M}} \otimes \Omega^1_{\mathcal{M}} \longrightarrow 0$$

the connecting morphism in the long exact sequence in cohomology is

$$\delta_1 : \mathrm{H}^0\left(\mathcal{M}, \mathrm{End}(T_{\mathcal{M}})\right) \longrightarrow \mathrm{H}^1\left(\mathcal{M}, \mathrm{End}(E) \otimes \Omega^1_{\mathcal{M}}\right)$$

the class $at_{\mathcal{M}}(E)$ is given by $\delta_1(\mathrm{Id})$. We have the following lemma. [Ati57].

Lemma 3.1.2 *Let* X *be a smooth algebraic variety,* L *a line bundle and* k *a positive integer. Then we have an isomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_X(L^k) & \xrightarrow{\nabla_1} & T_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{1/k} & \mathcal{A}_X(L) & \xrightarrow{\nabla_1} & T_X & \longrightarrow & 0 \end{array}$$

For $\pi : \mathcal{M} \longrightarrow S$ an S-scheme and E a vector bundle over $\mathcal{M}$, there is a relative version of the Atiyah algebroid denoted by $\mathcal{A}_{\mathcal{M}/S}(E)$, given by taking the pull-back

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{A}_{\mathcal{M}/S}(E) & \longrightarrow & T_{\mathcal{M}/S} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \iota & & \\ 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{A}_{\mathcal{M}}(E) & \xrightarrow{\nabla_1} & T_{\mathcal{M}} & \longrightarrow & 0 \end{array}$$

As an extension, we have $at_{\mathcal{M}/S}(E)$ is a global section of the sheaf $R^1\pi_*(\Omega^1_{\mathcal{M}/S} \otimes \mathrm{End}(E))$ over S.i.e.,

$$at_{\mathcal{M}/S}(E) \in \mathrm{H}^0\left(S, \mathcal{E}xt^1(T_{\mathcal{M}/S}, \mathrm{End}(E))\right) \simeq \mathrm{H}^0\left(S, R^1\pi_*(\Omega^1_{\mathcal{M}/S} \otimes \mathrm{End}(E))\right).$$

For a line bundle $L \in \mathrm{Pic}(\mathcal{M})$, we denote its relative Atiyah class by $[L] \in \mathrm{H}^0\left(S, R^1\pi_*(\Omega^1_{\mathcal{M}/S})\right)$.

For our purpose we need the trace-free Atiyah algebroid of vector bundles with fix determinant. We have a direct sum decomposition $\mathrm{End}(E) = \mathrm{End}^0(E) \oplus \mathcal{O}_{\mathcal{M}}$ and let denote by $q : \mathrm{End}(E) \to \mathrm{End}^0(E)$ the first projection map. Then the trace-free Atiyah algebroid is given by the push-out of the standard Atiyah sequence by the map q as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{A}_{\mathcal{M}}(E) & \longrightarrow & T_{\mathcal{M}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{End}^0(E) & \longrightarrow & \mathcal{A}^0_{\mathcal{M}}(E) & \longrightarrow & T_{\mathcal{M}} & \longrightarrow & 0, \end{array}$$

With the same method we define the trace-free relative version Atiyah algebroid $\mathcal{A}^0_{\mathcal{M}/S}(E)$.

Definition 3.1.3 (Lie algebroid structure) *The sheaf $\mathcal{D}^{(1)}_{\mathcal{M}}(E)$ is equipped with a natural Lie brackets given by the commutator. In fact for any $P, Q \in \mathcal{D}^{(1)}_{\mathcal{M}}(E)$ we have*

$$[P, Q] \in \mathcal{D}^{(1)}_{\mathcal{M}}(E).$$

Thus we get a $\mathbb{C}$-bilinear application

$$[\cdot,\cdot] : \mathcal{D}^{(1)}_{\mathcal{M}}(E) \times \mathcal{D}^{(1)}_{\mathcal{M}}(E) \longrightarrow \mathcal{D}^{(1)}_{\mathcal{M}}(E).$$

3.1.2 Connections / Curvature

We will follow Atiyah's approach to define connections on vector bundles as splitting of the associated Atiyah sequence.

Definition 3.1.4 (Connection) *Let E be a vector bundle on $\mathcal{M}$. A (Koszul) connection ∇ on E is a $\mathcal{O}_{\mathcal{M}}$-linear splitting of the Atiyah exact sequence associated to the vector bundle E*

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathcal{A}_{\mathcal{M}}(E) \underset{\nabla}{\overset{\longrightarrow}{\longleftarrow}} T_{\mathcal{M}} \longrightarrow 0.$$

Definition 3.1.5 (Projective connection) *A projective connection is a $\mathcal{O}_{\mathcal{M}}$-linear splitting ∇ of the exact sequence*

$$0 \longrightarrow \mathrm{End}(E)/\mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{A}_{\mathcal{M}}(E)/\mathcal{O}_{\mathcal{M}} \underset{\nabla}{\overset{\longrightarrow}{\longleftarrow}} T_{\mathcal{M}} \longrightarrow 0.$$

Definition 3.1.6 (Flat connection)

A (projective) connection is said to be flat (or integrable) if it preserves the Lie bracket.

3.2 Heat operators

Let $L \in \mathrm{Pic}(\mathcal{M})$ be a line bundle over $\mathcal{M}$ such that $\pi_* L$ is a locally free sheaf over S. We are interested in the subsheaf of differential operators of degree 2 given by

$$\mathcal{W}_{\mathcal{M}/S}(L) := \mathcal{D}^{(1)}_{\mathcal{M}}(L) + \mathcal{D}^{(2)}_{\mathcal{M}/S}(L) \subset \mathcal{D}^{(2)}_{\mathcal{M}}(L).$$

We also denote ∇_2 the restriction of the symbol map to this sub-sheaf

$$\nabla_2 : \mathcal{W}_{\mathcal{M}/S}(L) \longrightarrow \mathrm{Sym}^2(T_{\mathcal{M}/S}),$$

and define the sub-principal symbol

$$\sigma_S : \mathcal{W}_{\mathcal{M}/S}(L) \longrightarrow \pi^* T_S,$$

such that for s a local section of L and f a local section of $\mathcal{O}_S$ we have, for all $D \in \mathcal{W}_{\mathcal{M}/S}(L)$

$$\langle \sigma_S(D), d(\pi^* f) \rangle = D(\pi^* f s) - \pi^* f D(s).$$

The elements of the sheaf $\mathcal{W}_{\mathcal{M}/S}(L)$ satisfy the Leibniz rule (this follow from proprieties of the second order symbol map)

$$D(fgs) = \langle \nabla_2(D), df \otimes dg \rangle s + f D(gs) + g D(fs) - fg D(s).$$

Thus we get a short exact sequence

$$0 \longrightarrow \mathcal{D}^{(1)}_{\mathcal{M}/S}(L) \longrightarrow \mathcal{W}_{\mathcal{M}/S}(L) \xrightarrow{\sigma_S \oplus \nabla_2} \pi^*(T_S) \oplus \mathrm{Sym}^2(T_{\mathcal{M}/S}) \longrightarrow 0. \tag{3.2.1}$$

We now define the heat operators.

Definition 3.2.1 (Heat operator [GdJ98]) *A heat operator H on L is an $\mathcal{O}_S$-linear map of coherent sheaves*

$$H : T_S \longrightarrow \pi_* \mathcal{W}_{\mathcal{M}/S}(L)$$

such that $\sigma_S \circ \tilde{H} = Id$, where $\tilde{H}$ is the $\mathcal{O}_{\mathcal{M}}$-linear map associated to H by adjunction

$$\tilde{H} : \pi^* T_S \longrightarrow \mathcal{W}_{\mathcal{M}/S}(L).$$

Definition 3.2.2 (Projective heat operator) *A projective heat operator H on L is an $\mathcal{O}_S$-linear map of coherent sheaves*

$$H : T_S \longrightarrow \left(\pi_* \mathcal{W}_{\mathcal{M}/S}(L)\right) / \mathcal{O}_S$$

such that $\sigma_S \circ \tilde{H} = Id$. The map $\tilde{H}$ is associated to H by adjunction.

Definition 3.2.3 (Symbol of heat operators) *The symbol map of a (projective) heat operator H is the map*

$$\rho_H := \pi_*(\sigma_2) \circ H : T_S \longrightarrow \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}).$$

3.3 The van Geemen-de Jong approach

Let $\pi : \mathcal{M} \longrightarrow S$ be a smooth surjective morphism of smooth schemes and let L be a line bundle over $\mathcal{M}/S$, such that $\pi_* L$ is a vector bundle over S and let H be a (projective) heat operator over L. We define a (projective) connection on $\pi_* L$ by associating a covariant derivative

$$\nabla_\theta : \pi_* L \longrightarrow \pi_* L,$$

which we define as follows: locally on U an open subset in S, let $\theta \in T_S(U)$ a vector field on S we denote by $\pi^{-1}(\theta)$ the corresponding section of $\pi^{-1}(T_S)(\pi^{-1}(U))$ and for all $s \in \pi_* L(U)$, we define

$$\nabla_\theta(s) := H(\pi^{-1}(\theta))(s).$$

And as the sub-principal symbol of $H(\pi^{-1}(\theta))$ is $\pi^{-1}(\theta)$, the Leibniz rule is satisfied. In fact for any $f \in \mathcal{O}_S(U)$ we have

$$\begin{aligned} \nabla_\theta(fs) &= H(\pi^{-1}(\theta))(\pi^*(f)s) \\ &= \pi^*(\theta(f))s + \pi^*(f)H(\pi^{-1}(\theta))(s) \\ &= \theta(f)s + f\nabla_\theta s. \end{aligned}$$

If the (projective) heat operator preserves the Lie bracket then, the associated (projective) connection is flat.

3.3.1 A heat operator for a candidate symbol

In [GdJ98], van Geemen and de Jong give conditions which imply that a candidate symbol

$$\rho : T_S :\longrightarrow \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}),$$

can be lifted to a (projective) heat operator, i.e. , there exists a (projective) heat operator H such that we have

$$\rho_H := \sigma_S \circ H = \rho.$$

For any line bundle $L \in \mathrm{Pic}(\mathcal{M})$, we have the exact sequence

$$0 \longrightarrow T_{\mathcal{M}/S} \longrightarrow \mathcal{D}^{(2)}_{\mathcal{M}/S}(L)/\mathcal{O}_{\mathcal{M}} \longrightarrow \mathrm{Sym}^2(T_{\mathcal{M}/S}) \longrightarrow 0,$$

The first connecting morphism on cohomology with respect to the map π give rise to a map

$$\mu_L : \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}) \longrightarrow R^1\pi_* \left(T_{\mathcal{M}/S}\right).$$

Proposition 3.3.1 ([Wel83] and [BBMP23]) *For a line bundle $L \in \mathrm{Pic}(\mathcal{M})$. The map μ_L is given by the following formula*

$$\mu_L = \cup [L] - \cup \left(\frac{1}{2}\left[K_{\mathcal{M}/S}\right]\right).$$

where $K_{\mathcal{M}/S}$ is the relative canonical line bundle of $\pi : \mathcal{M} \longrightarrow S$.

The Kodaira-Spencer map is given by the first connecting morphism of the short exact sequence 3.0.1

$$\kappa_{\mathcal{M}/S} : \mathcal{T}_S \longrightarrow R^1\pi_*\mathcal{T}_{\mathcal{M}/S}.$$

Now we can state the following theorem.

Theorem 3.3.2 (van Geemen–de Jong, [GdJ98], §2.3.7)
Let $L \in \mathrm{Pic}(\mathcal{M})$ be a line bundle and $\pi : \mathcal{M} \longrightarrow S$ as before, we have that if, for a given map $\rho : T_S \longrightarrow \pi_*\mathrm{Sym}^2 T_{\mathcal{M}/S}$

1. $\kappa_{\mathcal{M}/S} + \mu_L \circ \rho = 0$,
2. *cupping with the relative Atiyah class*
$$\cup[L] : \pi_* T_{\mathcal{M}/S} \longrightarrow R^1\pi_*\mathcal{O}_{\mathcal{M}}$$
is an isomorphism, and
3. $\pi_*\mathcal{O}_{\mathcal{M}} = \mathcal{O}_S$.

Then there exists a unique projective heat operator H whose symbol is ρ.

Proof. We start by noting that the map $\cup\,[L]$ in the second hypothesis can be seen as the connecting homomorphism in the long exact sequence associated the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{D}^{(1)}_{\mathcal{M}/S}(L) \xrightarrow{\nabla_1} T_{\mathcal{M}/S} \longrightarrow 0$$

So by the second and third hypotheses we get

$$\mathcal{O}_S \cong \pi_*\mathcal{O}_M \cong \pi_*\mathcal{D}^{(1)}_{\mathcal{M}/S}(L).$$

Now, consider the long exact sequence associated to the short exact sequence 3.2.1

$$\begin{array}{ccccccc}
0 \longrightarrow \pi_*\mathcal{D}^{(1)}_{\mathcal{M}/S}(L) & \longrightarrow & \pi_*\mathcal{W}_{\mathcal{M}/S}(L) & \longrightarrow & T_S \oplus \pi^*\mathrm{Sym}^2(T_{\mathcal{M}/S}) \\
 & & & \swarrow^{\delta} & \\
R^1\pi_*\mathcal{D}^{(1)}_{\mathcal{M}/S}(L) & \longrightarrow & R^1\pi_*\mathcal{W}_{\mathcal{M}/S}(L) & \longrightarrow & \cdots
\end{array}$$

we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{M}} & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{M}} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{D}^{(1)}_{\mathcal{M}/S}(L) & \longrightarrow & \pi_*\mathcal{W}_{\mathcal{M}/S}(L) & \longrightarrow & \mathrm{Ker}\delta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & 0 & \longrightarrow & \left(\pi_*\mathcal{W}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_S & \longrightarrow & \mathrm{Ker}\delta & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

therefore an isomorphism
$$\left(\pi_* \mathcal{W}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_S \cong \mathrm{Ker}\delta.$$
Now a projective heat operator with symbol ρ is a lift of ρ relative to the second order map
$$\begin{array}{ccc} & & \left(\pi_* \mathcal{W}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_S \\ & \overset{H}{\nearrow} & \downarrow \nabla_2 \\ T_S & \xrightarrow[\rho]{} & \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}) \end{array}$$
which exists (using the second and third point) if and only if the image of the map
$$T_S \longrightarrow T_S \oplus \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}), \qquad \theta \mapsto (\theta, \rho(\theta)).$$
is in the kernel of δ. It remains to prove that this is equivalent to the first hypothesis. In order to do this , let us decompose $\delta = \delta_1 + \delta_2$ into its two component
$$\delta_1 : T_S \longrightarrow R^1\pi_* \mathcal{D}^{(1)}_{\mathcal{M}/S}(L) \;\; \text{and} \quad \delta_2 : \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}) \longrightarrow R^1\pi_* \mathcal{D}^{(1)}_{\mathcal{M}/S}(L).$$
We can check that
$$R^1\pi\left(\nabla_1\right) \circ \delta_1 = \kappa_{\mathcal{M}/S} \;\; \text{and} \;\; R^1\pi\left(\nabla_1\right) \circ \delta_2 = \mu_L.$$
Take the long exact sequence associated to the Atiyah sequence 3.1.1 of L, we get
$$\cdots \to \pi_* T_{\mathcal{M}/S} \xrightarrow{\cup[L]} \pi_* \mathcal{O}_{\mathcal{M}} \longrightarrow R^1\pi_* \mathcal{D}^{(1)}_{\mathcal{M}/S}(L) \longrightarrow R^1\pi_* T_{\mathcal{M}/S} \to \cdots$$
the natural map
$$R^1\pi\left(\nabla_1\right) : R^1\pi_* \mathcal{D}^{(1)}_{\mathcal{M}/S}(L) \longrightarrow R^1\pi_* T_{\mathcal{M}/S},$$
is injective as by the second hypothesis. Thus
$$\begin{aligned} (\theta, \rho(\theta)) \in \mathrm{Ker}\delta &\Leftrightarrow R^1\pi\left(\nabla_1\right)\delta(\theta, \rho(\theta)) = 0 \\ &\Leftrightarrow \left(\kappa_{\mathcal{M}/S} + \mu_L \circ \rho\right)(\theta) = 0, \end{aligned}$$
for any local section θ. □

3.3.2 Flatness criterion

Theorem 3.3.3 ([Hit90a]; [BBMP23], Theorem 3.5.1) *Under the assumptions of Theorem 3.3.2 the projective connection associated to the symbol ρ is projectively flat if the following conditions holds*

1. *For all local sections θ, θ' of T_S, we have*

$$\{\rho(\theta), \rho(\theta')\}_{T^*_{\mathcal{M}/S}} = 0,$$

   i.e. , the symbol, Poisson-commute with respect to the natural symplectic form over the relative cotangent bundle $T^*_{\mathcal{M}/S}$.

2. *The morphism μ_L is injective.*

3. *There are no vertical vector fields, $\pi_* T_{\mathcal{M}/S} = 0$.*

Proof. We denote by H the projective heat operator, its flatness is equivalent to the vanishing of the operator

$$[H(\theta), H(\theta')] - H([\theta, \theta']) \in \pi_* \left(\mathcal{D}^{(3)}_{\mathcal{M}/S}(L) + \mathcal{D}^{(2)}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_S.$$

its symbol is

$$\nabla_3\left([H(\theta), H(\theta')]\right) \in \pi_* \mathrm{Sym}^3(T_{\mathcal{M}/S})$$

by the isomorphism of Poisson-algebras given by the natural map

$$\pi_*\ \mathrm{Sym}^m(T_{\mathcal{M}/S}) \cong \pi_*(\mathcal{O}_{\mathcal{M}})_m$$

where the right hand side is the weight m part under the action of $\mathbb{G}_m$-action equipped with the natural Poisson-structure. And the Poisson-structure on right hand side is given by the commutators over the sheaf of operators of order at most m. Then by the first hypothesis, we get

$$\{\nabla_2(H(\theta)), \nabla_2(H(\theta'))\}_{T^*_{\mathcal{M}/S}} = \{\rho(\theta), \rho(\theta')\}_{T^*_{\mathcal{M}/S}} = 0.$$

Thus the operator is at most of degree 2 and acts only on the fibres of the map $\pi : \mathcal{M} \longrightarrow S$

$$[H(\theta), H(\theta')] - H([\theta, \theta']) \in \pi_* \left(\mathcal{D}^{(2)}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_S.$$

Now we take the exact sequence

$$0 \longrightarrow T_{\mathcal{M}/S} \longrightarrow \mathcal{D}^{(2)}_{\mathcal{M}/S}(L)/\mathcal{O}_{\mathcal{M}} \longrightarrow \mathrm{Sym}^2(T_{\mathcal{M}/S}) \longrightarrow 0,$$

the associated long exact sequence

$$0 \longrightarrow \pi_* T_{\mathcal{M}/S} \longrightarrow \pi_* \left(\mathcal{D}^{(2)}_{\mathcal{M}/S}(L)/\mathcal{O}_{\mathcal{M}}\right) \longrightarrow \pi_* \mathrm{Sym}^2(T_{\mathcal{M}/S}) \underset{\mu_L}{\longrightarrow} R^1\pi_*\left(T_{\mathcal{M}/S}\right) \to ...,$$

by the second and third hypothesis we get the isomorphisms

$$0 = \pi_* T_{\mathcal{M}/S} \cong \pi_* \left(\mathcal{D}^{(2)}_{\mathcal{M}/S}(L)/\mathcal{O}_{\mathcal{M}}\right) \cong \pi_* \left(\mathcal{D}^{(2)}_{\mathcal{M}/S}(L)\right)/\mathcal{O}_{\mathcal{S}}.$$

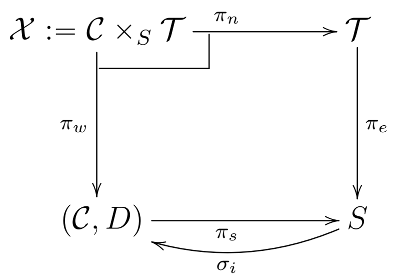
thus concluding the proof. □

Chapter 4

The Hitchin connection for parabolic non-abelian theta functions

In this chapter we prove the main theorem which generalises the algebro-geometric construction of Hitchin's connection given in [BBMP23] over $\mathcal{SU}_{\mathcal{C}/S}(r)$ the relative moduli space of rank-r vector bundles with trivial determinant over a smooth family of complex projective curves of genus $g \geqslant 2$, to $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ the relative moduli space of parabolic rank-r vector bundles of fixed determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ and of fixed parabolic type α_*.

Let S be a smooth complex algebraic variety. We take a smooth family of projective curves $\pi_s : \mathcal{C} \longrightarrow S$ of genus $g \geqslant 2$ and take D a divisor given by N sections of the map π_s such that the relative degree is N, with no non-trivial points (i.e. $\forall i \in I$ such that $\ell_i > 1$). Let $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ a line bundle over the family of curves. Let $\mathcal{E}_*$ be a family of rank-r parabolic vector bundles of fixed parabolic type α_* and fixed determinant δ over $(\mathcal{C}, D)/S$ parameterized by a S-schemes $\mathcal{T}$. We shall denote the fibered product by the diagram:



We set $\mathcal{D} := \pi_w^{-1}(D) = D \times_S \mathcal{T}$.

▸ As a working hypothesis, we suppose that the parabolic system of weights $\alpha_* = (r, \vec{a}, \vec{m})$ is generic [1] in the following sense: α_*-parabolic semi-stability $\Leftrightarrow$ α_*-parabolic stability.

[1] See [BY99] for more details on genericness of parabolic weights

With this working hypothesis, the moduli space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ is smooth over S, and the Picard group is maximal.i.e., all line bundles on the Quot scheme multiplied by the universal flag varieties descend to the moduli space. See [Pau96], [LS97].

4.1 Parabolic Atiyah sequences and algebroids

We define the quasi-parabolic and strongly quasi-parabolic Atiyah sequences and algebroids, that we use to study deformation of marked curves equipped with quasi-parabolic vector bundles and to show existence of the Kodaira-Spencer map in the parabolic case. We recall Yokogawa's isomorphism (Proposition 1.1.7)

$$\mathrm{parEnd}(E)^\vee \cong \mathrm{SparEnd}(E) \otimes \mathcal{O}_C(D).$$

Definition 4.1.1 (Quasi-parabolic Atiyah algebroid (QPA))
We take the push-out of the relative Atiyah exact sequence of the parabolic bundle $\mathcal{E}_*$ by the inclusion $\mathrm{End}^0(\mathcal{E}) \hookrightarrow \mathrm{SparEnd}^0(\mathcal{E})^\vee \cong \mathrm{parEnd}^0(\mathcal{E})(\mathcal{D})$. We get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}^0(\mathcal{E}) & \longrightarrow & \mathcal{A}^0_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) & \longrightarrow & T_{\mathcal{X}/\mathcal{T}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{SparEnd}^0(\mathcal{E})^\vee & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \pi_w^* T_{\mathcal{C}/S} & \longrightarrow & 0
\end{array}$$

Then the QPA sequence is given by tensorizing the exact sequence above by $\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$:

$$0 \longrightarrow \mathrm{parEnd}^0(\mathcal{E}) \longrightarrow \mathcal{A}^{0,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) \longrightarrow \pi_w^*(T_{\mathcal{C}/S}(-D)) \longrightarrow 0,$$

and the QPA algebroid is given by

$$\mathcal{A}^{0,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) := \mathcal{A}_1 \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})\,.$$

Definition 4.1.2 (Strongly quasi-parabolic Atiyah algebroid (SQPA))
We take the push-out of the Atiyah exact sequence of the parabolic bundle $\mathcal{E}_*$ by the inclusion $\mathrm{End}^0(\mathcal{E}) \hookrightarrow \mathrm{parEnd}^0(\mathcal{E})^\vee \cong \mathrm{SparEnd}^0(\mathcal{E})(\mathcal{D})$. We get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}^0(\mathcal{E}) & \longrightarrow & \mathcal{A}^0_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) & \longrightarrow & T_{\mathcal{X}/\mathcal{T}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E})^\vee & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \pi_w^* T_{\mathcal{C}/S} & \longrightarrow & 0
\end{array}$$

Then the SQPA exact sequence is given by tensorizing the exact sequence above by $\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$:

$$0 \longrightarrow \mathrm{SparEnd}^0(\mathcal{E}) \longrightarrow \mathcal{A}^{0,st,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) \longrightarrow \pi_w^*(T_{\mathcal{C}/S}(-D)) \longrightarrow 0,$$

and the SQPA algebroid is given by

$$\mathcal{A}^{0,st,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E}) := \mathcal{A}_2 \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})\,.$$

Remark 4.1.3 *The definitions of $\mathcal{A}^{0,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E})$ and $\mathcal{A}^{0,st,par}_{\mathcal{X}/\mathcal{T}}(\mathcal{E})$ are canonically attached to the family of marked curves $(\mathcal{C}, D)$ and depend only on the quasi-parabolic structure and not on the parabolic weights $\vec{a}$. In particular by construction these Atiyah algebroids are invariant under Hecke modifications.*

4.2 Trace complexes theory

The main ingredient in [BBMP23] is the description the Atiyah class of the relative ample generator $\mathcal{L}$ of the relative Picard of the moduli space $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$ using the theory of complex trace, Sun-Tsai isomorphism (Theorem 4.2.1), Beilinson-Schechtman isomorphism and Bloch-Esnault complex (Theorem 4.2.2). Here we do not need the definition of the complex trace, we use Sun-Tsai characterization of the (-1)- Bloch-Esnault term as definition.
We recall the following fibre product

$$\begin{array}{ccc} \mathcal{X} := \mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta) & \xrightarrow{p_n} & \mathcal{SU}_{\mathcal{C}/S}(r,\delta) \\ \downarrow p_w & & \downarrow p_e \\ \mathcal{C} & \xrightarrow[p_s=\pi_s]{} & S \end{array}$$

Let $\mathcal{U}$ be a universal vector bundle over $\mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta)$. The following theorem give a characterization of the (-1)-Bloch Esnault algebra ${}^0\mathcal{B}^{-1}_{\mathcal{SU}_{\mathcal{C}/S}/S}(\mathcal{U})$, that we will use as a definition.

Theorem 4.2.1 **([ST04])** *There is a canonical isomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^{\vee}_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}} & \longrightarrow & \mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}}}(\mathcal{U})^{\vee} & \longrightarrow & \mathrm{End}(\mathcal{U})^{\vee} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & {\scriptstyle -Tr}\downarrow \cong & & \\ 0 & \longrightarrow & K_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}} & \longrightarrow & {}^0\mathcal{B}^{-1}_{\mathcal{SU}_{\mathcal{C}/S}/S}(\mathcal{U}) & \longrightarrow & \mathrm{End}(\mathcal{U}) & \longrightarrow & 0 \end{array}$$

where $K_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}}$ is the relative canonical bundle with respect to the map p_n.

Theorem 4.2.2 (Beilinson- Schechtman & Bloch-Esnault [BS88], [ET00]) *Let $\mathcal{U}$ be a virtual universal bundle over $\mathcal{X} = \mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta)$. Then we have the following isomorphism of exact sequences over $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1p_{n*}(K_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}/S}}) & \longrightarrow & R^1p_{n*}({}^0\mathcal{B}^{-1}(\mathcal{U})) & \longrightarrow & R^1p_{n*}(\mathrm{End}^0(\mathcal{U})^\vee) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle 2r\cdot\mathrm{id}} \cong & & \downarrow \cong & & \cong \downarrow {\scriptstyle -Tr} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_{\mathcal{C}/S}} & \longrightarrow & \mathcal{A}_{\mathcal{SU}_{\mathcal{C}/S}/S}\left(\lambda(\mathrm{End}^0(\mathcal{U})\right) & \longrightarrow & T_{\mathcal{SU}_{\mathcal{C}/S}/S} & \longrightarrow & 0 \end{array}$$

Combining these two results we get the following theorem, proven in [BBMP23] for $\delta = \mathcal{O}_\mathcal{C}$, but their proof work for any relative line bundle $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$.

Theorem 4.2.3 *We have the following isomorphism of short exact sequences over $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1p_{n*}(K_{\mathcal{X}/\mathcal{SU}}) & \longrightarrow & R^1p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}}(\mathcal{U})^\vee\right) & \longrightarrow & R^1p_{n*}\left(\mathrm{End}^0(\mathcal{U})^\vee\right) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \frac{r}{n}\mathrm{Id}} \cong & & \cong \downarrow & & \cong \downarrow {\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \longrightarrow & \mathcal{A}_{\mathcal{SU}_\mathcal{C}/S}(\mathcal{L}) & \xrightarrow{\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0 \end{array}$$

where $\mathcal{L}$ is the relative ample generator of the group $\mathrm{Pic}\left(\mathcal{SU}_\mathcal{C}(r,\delta)/S\right)$ *and* $n = \gcd(r, \deg(\delta))$.

Proof. By Theorem 4.2.1 and 4.2.2, one has the following isomorphism of short exact sequences $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1p_{n*}\left(K_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}\right) & \longrightarrow & R^1p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}(\mathcal{U})^\vee\right) & \longrightarrow & R^1p_{n*}\left(\mathrm{End}^0(\mathcal{U})^\vee\right) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle 2r\cdot\mathrm{id}} \cong & & \downarrow \cong & & \cong \downarrow {\scriptstyle -Tr} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \longrightarrow & \mathcal{A}_{\mathcal{M}/S}\left(\lambda(\mathrm{End}^0(\mathcal{U}))\right) & \xrightarrow{\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0 \end{array}$$

By Drezet-Narasimhan 1.4.3 theorem and [LS97], we have

$$\lambda(\mathrm{End}^0(\mathcal{U})) = K_{\mathcal{SU}_\mathcal{C}} = \mathcal{L}^{-2n}.$$

Hence we get the following isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1p_{n*}\left(K_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}\right) & \longrightarrow & R^1p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}(\mathcal{U})^\vee\right) & \longrightarrow & R^1p_{n*}\left(\mathrm{End}^0(\mathcal{U})^\vee\right) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle 2r\cdot\mathrm{id}} \cong & & \downarrow \cong & & \cong \downarrow {\scriptstyle -Tr} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \longrightarrow & \mathcal{A}_{\mathcal{M}/S}\left(\mathcal{L}^{-2n}\right) & \xrightarrow{\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0 \end{array}$$

and by applying Lemma 3.1.2 (for $k = -2n$ and $L = \mathcal{L}$), we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^1 p_{n*}\left(K_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}\right) & \longrightarrow & R^1 p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}(\mathcal{U})^\vee\right) & \longrightarrow & R^1 p_{n*}\left(\mathrm{End}^0(\mathcal{U})^\vee\right) & \longrightarrow & 0 \\
 & & {\scriptstyle 2r\cdot\mathrm{id}}\downarrow\cong & & \downarrow\cong & & \cong\downarrow{\scriptstyle -Tr} & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \longrightarrow & \mathcal{A}_{\mathcal{SU}/S}(\mathcal{L}^{-2n}) & \xrightarrow{\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \xrightarrow{\frac{1}{2n}} & \mathcal{A}_{\mathcal{SU}_\mathcal{C}/S}(\mathcal{L}^{-1}) & \xrightarrow{\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SU}_\mathcal{C}} & \xrightarrow{\frac{1}{2n}} & \mathcal{A}_{\mathcal{SU}_\mathcal{C}/S}(\mathcal{L}) & \xrightarrow{-\nabla_1} & T_{\mathcal{SU}_\mathcal{C}/S} & \longrightarrow & 0
\end{array}$$

The right vertical map is $-Tr$, the vertical left map is $2rId$ and the extension class of the last exact map is $-2n\,[\mathcal{L}]$ in $\mathrm{H}^0\left(S, R^1\pi_*\left(\Omega^1_{\mathcal{SU}_{\mathcal{C}/S}(r,\delta)/S}\right)\right)$. Hence we conclude that the extension class of the exact sequence

$$0 \longrightarrow R^1 p_{n*}\left(K_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}\right) \longrightarrow R^1 p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_\mathcal{C}}(\mathcal{U})^\vee\right) \longrightarrow R^1 p_{n*}(\mathrm{End}^0(\mathcal{U})^\vee) \longrightarrow 0$$

equals $\frac{n}{r}\,[\mathcal{L}]$. □

4.3 Parabolic Bloch-Esnault complex

Now, we work over $\mathcal{SM}^{par}_{\mathcal{C}/S} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ the relative moduli space of semi-stable rank-r parabolic vector bundles of fixed parabolic type α_* with determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$ over $\mathcal{C}/S$. We have the following fibre product

$$\begin{array}{ccc}
\mathcal{X}^{par} & \xrightarrow{\pi_n} & \mathcal{SM}^{par}_{\mathcal{C}/S} \\
{\scriptstyle \pi_w}\downarrow & & \downarrow{\scriptstyle \pi_e} \\
(\mathcal{C}, D) & \underset{\sigma_i}{\overset{\pi_s}{\rightleftarrows}} & S
\end{array}$$

We denote by $\mathcal{E}_*$ a virtual universal parabolic bundle over $\mathcal{X}^{par} = \mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}$. For our need we define the (-1)-term of the parabolic Bloch-Esnault complex.

Definition 4.3.1 *We define the (-1)-term of the parabolic Bloch-Esnault* ${}^0\mathcal{P}^{-1}(\mathcal{E})$*, as a pull-back of the (-1)-term of the Bloch-Esnault complex* ${}^0\mathcal{B}^{-1}(\mathcal{E})$*, by the natural inclusion*

$\mathrm{parEnd}^0(\mathcal{E}) \hookrightarrow \mathrm{End}^0(\mathcal{E})$, *as follows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\mathcal{X}^{par}/\mathcal{SM}^{par}} & \longrightarrow & {}^0\mathcal{P}^{-1}(\mathcal{E}) & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{\mathcal{X}^{par}/\mathcal{SM}^{par}} & \longrightarrow & {}^0\mathcal{B}^{-1}(\mathcal{E}) & \longrightarrow & \mathrm{End}^0(\mathcal{E}) & \longrightarrow & 0 \end{array}$$

where $K_{\mathcal{X}^{par}/\mathcal{SM}^{par}}$ is the relative canonical line bundle relatively to the map π_n.

We apply $R^1\pi_{e_*}$ to the (-1)-Bloch-Esnault term exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}}} & \longrightarrow & R^1\pi_{n_*}\left({}^0\mathcal{P}^{-1}(\mathcal{E})\right) & \longrightarrow & R^1\pi_{n_*}\left(\mathrm{parEnd}^0(\mathcal{E})\right) \simeq T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}}p} & \longrightarrow & R^1\pi_{n_*}({}^0\mathcal{B}^{-1}(\mathcal{E})) & \longrightarrow & R^1\pi_{n_*}(\mathrm{End}^0(\mathcal{E})) & \longrightarrow & 0 \end{array}$$

The exact sequence below is the pull-back of the Bloch-Esnault exact sequence of the vector bundle $\mathcal{E}$ seen as a family over the space $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$ by the forgetful morphism map

$$\phi : \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta) \longrightarrow \mathcal{SU}_{\mathcal{C}}(r,\delta)$$

which can be lifted to a map on the fibre product

$$\phi : \mathcal{C} \times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta) \longrightarrow \mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta)$$

we can choose a virtual universal bundle $\mathcal{U}$ over $\mathcal{C} \times_S \mathcal{SU}_{\mathcal{C}/S}(r,\delta)$, such that

$$\phi^*(\mathcal{U}) \cong \mathcal{E}$$

Moreover the differential map

$$d\phi : T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \longrightarrow \phi^*\left(T_{\mathcal{SU}_{\mathcal{C}}/S}\right)$$

is given by applying $R^1\pi_{n_*}$ to the natural inclusion $\mathrm{parEnd}^0(\mathcal{E}) \hookrightarrow \mathrm{End}^0(\phi^*(\mathcal{U}))$.

So we get an identification theorem in the parabolic configuration of Theorem 4.2.1.

Proposition 4.3.2 *For a virtual universal parabolic bundle $\mathcal{E}_*$ over $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$. There is an isomorphism ${}^0\mathcal{P}^{-1}(\mathcal{E}) \simeq \mathcal{A}^{0,par,St}_{\mathcal{X}^{par}/\mathcal{SM}^{par}}(\mathcal{E})(\mathcal{D})^\vee$, such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\mathcal{X}^{par}/\mathcal{SM}^{par}} & \longrightarrow & {}^0\mathcal{P}^{-1}(\mathcal{E}) & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E}) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & K_{\mathcal{X}^{par}/\mathcal{SM}^{par}} & \longrightarrow & \left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}^{par}}(\mathcal{E})(\mathcal{D})\right]^\vee & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E}) & \longrightarrow & 0 \end{array}$$

Hence we get the following parabolic version of [BBMP23] Theorem 4.4.1.

Theorem 4.3.3 *Let $\mathcal{E}_\bullet = (\mathcal{E}_\lambda)_{\lambda \in \mathbb{R}}$ be a virtual universal parabolic bundle over $\mathcal{M}_\bullet = \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$. Then for each $\lambda \in \mathbb{R}$, we have the following isomorphism of short exact sequences over $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$*

$$\begin{array}{ccccc} R^1\pi_{n*}\left(K_{\mathcal{X}^{par}/\mathcal{M}_\bullet}\right) & \hookrightarrow & R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{M}_\bullet}(\mathcal{E}_\lambda)(\mathcal{D})\right]^\vee\right) & \twoheadrightarrow & R^1\pi_{n*}\left(\operatorname{parEnd}^0(\mathcal{E}_\lambda)\right) \\ \frac{r}{n(\lambda)}\downarrow \simeq & & \downarrow \simeq & & \downarrow \cong \\ \mathcal{O}_{\mathcal{M}_\bullet} & \hookrightarrow & \mathcal{A}_{\mathcal{M}_\bullet/S}(\Theta(\lambda)) & \xrightarrow{\nabla_1} & T_{\mathcal{M}_\bullet/S} \end{array}$$

where $\Theta(\lambda)$ is the pullback of the ample generator of the group $\operatorname{Pic}\left(\mathcal{SU}_{\mathcal{C}/S}(r, \delta_\lambda)/S\right)$ by the classifying maps

$$\begin{array}{cccc} \phi_\lambda & : & \mathcal{M}_\bullet := \mathcal{M}_\bullet(r, \lambda_\bullet, \delta) & \longrightarrow & \mathcal{SU}_{\mathcal{C}/S}(r, \delta(\lambda)) \\ & & \mathcal{E}_\bullet & \longmapsto & \mathcal{E}_\lambda \end{array}$$

set $d(\lambda) = \deg\ \delta(\lambda)$ and $n(\lambda) = \gcd(r, d(\lambda))$, which is equivalent to the equality

$$\frac{r}{n(\lambda)}\Delta_\lambda = [\Theta(\lambda)] \in \mathrm{H}^0\left(S, R^1\pi_{e*}\left(\Omega^1_{\mathcal{M}_\bullet/S}\right)\right),$$

where we denote by Δ_λ the extension class of the first exact sequence.

The Theorem is equivalent in the parabolic representation to the following theorem using Hecke modification, we recall that Hecke modification acts over the moduli spaces $\mathcal{SM}^{par}_{\mathrm{C}/S}(r, \alpha_*, \delta)$, we get an isomorphism over S

$$\begin{array}{cccc} \mathcal{H}^j_i & : & \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) & \longrightarrow & \mathcal{SM}^{par}_{\mathcal{C}/S}\left(r, \mathcal{H}^j_i(\alpha_*), \mathcal{H}^j_i(\delta)\right) \\ & & E_* & \longmapsto & \mathcal{H}^j_i(E)_* \end{array} \tag{4.3.1}$$

We denote by $\Theta_j(i)$ the pull-backs of the ample generators of the Picard groups $\operatorname{Pic}\left(\mathcal{SM}^{par}_{\mathcal{C}/S}/S\right)$ under the composition of the maps $\mathcal{H}^i_j$ followed by the forgetful maps to the moduli spaces $\mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i))$ where $\mathcal{H}^j_i(\delta) = \delta_j(i)$ and Θ the pull-back of the ample generator of the Picard group of $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$ by the forgetful map. Set $n = \gcd(r, \deg(\delta))$ and $n_j(i) := \gcd(r, \deg(\delta_j(i)))$

Theorem 4.3.4 *Under the same hypothesis. Let $\mathcal{E}_*$ be a virtual universal parabolic bundle, we have the following isomorphism of short exact sequences over $\mathcal{SM}^{par}_{\mathcal{C}}(r, \alpha_*, \delta)$*

$$\begin{array}{ccccc} R^1\pi_{n*}\left(K_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}}}\right) & \hookrightarrow & R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}}}(\mathcal{H}^j_i(\mathcal{E}))(\mathcal{D})\right]^\vee\right) & \twoheadrightarrow & R^1\pi_{n*}\left(\operatorname{parEnd}^0\left(\mathcal{H}^j_i(\mathcal{E})\right)\right) \\ \frac{r}{n_j(i)}\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}}} & \hookrightarrow & \mathcal{A}_{\mathcal{SM}^{par}_{\mathcal{C}}/S}(\Theta_j(i)) & \xrightarrow{\nabla_1} & T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \end{array}$$

We denote the extension class of the first exact sequence by $\Delta_j(i)$ and the Atiyah class of a line bundle L by $[L]$. Then the theorem is equivalent to the equality of global sections

$$\frac{r}{n_j(i)}\Delta_j(i) = [\Theta_j(i)] \in \mathrm{H}^0\left(S, R^1\pi_{e_*}(\Omega^1_{\mathcal{M}^{par}/S})\right).$$

With the same hypothesis we have

$$\begin{array}{ccccc} R^1\pi_{n*}(K_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}}}) & \hookrightarrow & R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}^{par}_{\mathcal{C}}}(\mathcal{E})(\mathcal{D})\right]^{\vee}\right) & \twoheadrightarrow & R^1\pi_{n*}(\mathrm{parEnd}^0(\mathcal{E})) \\ \cong\downarrow \frac{r}{n} & & \downarrow\cong & & \downarrow\cong \\ \mathcal{O}_{\mathcal{SM}^{par}_{\mathcal{C}}} & \hookrightarrow & \mathcal{A}_{\mathcal{SM}^{par}_{\mathcal{C}}/S}(\Theta) & \xrightarrow{\nabla_1} & T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \end{array}$$

which is equivalent to the equality

$$\frac{r}{n}\Delta = [\Theta] \in \mathrm{H}^0\left(S, R^1\pi_{e_*}(\Omega^1_{\mathcal{SM}^{par}_{\mathcal{C}}/S})\right).$$

Proof. Modulo shifting by a rational number λ in the filtered configuration which corresponds to Hecke modifications in the parabolic sitting, it is sufficient to prove the theorem for $\lambda = 0$. Hence take the forgetful map

$$\phi : \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) \longrightarrow \mathcal{SU}_{\mathcal{C}/S}(r, \delta)$$

which can be lifted to the fibre product over S

$$\begin{array}{ccc} \mathcal{C}\times_S \mathcal{SM}^{par}_{\mathcal{C}/S} & \xrightarrow{\phi} & \mathcal{C}\times_S \mathcal{SU}_{\mathcal{C}/S} \\ \pi_n\downarrow & & \downarrow p_n \\ \mathcal{SM}^{par}_{\mathcal{C}/S} & \xrightarrow[\phi]{} & \mathcal{SU}_{\mathcal{C}/S} \end{array}$$

let $\mathcal{E}_*$ be a universal parabolic bundle over $\mathcal{C}\times_S \mathcal{SM}^{par}_{\mathcal{C}/S}$ and let denote by $\mathcal{U}$ a virtual universal bundle over $\mathcal{C}\times_S \mathcal{SU}_{\mathcal{C}/S}$ such that $\phi^*(\mathcal{U}) \cong \mathcal{E}$. Take the pull-back of the exact sequence given in Theorem 4.2.3 by the forgetful map ϕ, we get

$$\begin{array}{ccccccccc} 0 & \to & \phi^*\left(R^1p_{n*}(K_{\mathcal{X}/\mathcal{SU}})\right) & \to & \phi^*\left(R^1p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}}(\mathcal{U})^{\vee}\right)\right) & \to & \phi^*\left(R^1p_{n*}\left(\mathrm{End}^0(\mathcal{U})^{\vee}\right)\right) & \to & 0 \\ & & \frac{r}{n}\mathrm{Id}\downarrow\cong & & \cong\downarrow & & \cong\downarrow\mathrm{Id} & & \\ 0 & \to & \mathcal{O}_{\mathcal{SM}^{par}} & \to & \phi^*\left(\mathcal{A}_{\mathcal{SU}_{\mathcal{C}}/S}(\mathcal{L})\right) & \xrightarrow{\nabla_1} & \phi^*\left(T_{\mathcal{SU}_{\mathcal{C}}/S}\right) & \to & 0 \end{array}$$

take the differential map

$$\mathrm{d}\phi : T_{\mathcal{SM}^{par}/S} \longrightarrow \phi^*\left(T_{\mathcal{SU}_{\mathcal{C}}/S}\right)$$

which correspond to taking the first direct image $R^1\pi_{n_*}$ of the natural inclusion of sheaves

$$\mathrm{parEnd}^0(\mathcal{E}) \hookrightarrow \mathrm{End}^0(\mathcal{E}) = \phi^*\left(\mathrm{End}^0(\mathcal{U})\right)$$

Now, we take the pull-back of this isomorphism of exact sequences by $\mathrm{d}\phi$ by the two realisations as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} & \longrightarrow & R^1\pi_{n*}({}^0\mathcal{P}^{-1}(\mathcal{E})) & \longrightarrow & R^1\pi_{n*}(\mathrm{parEnd}^0(\mathcal{E})) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow d\phi & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} & \longrightarrow & R^1\pi_{n*}({}^0\mathcal{B}^{-1}(\mathcal{E})) & \longrightarrow & R^1\pi_{n*}(\mathrm{End}^0(\mathcal{E})) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \phi^*\left(R^1p_{n*}(K_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}}})\right) & \longrightarrow & \phi^*\left(R^1p_{n*}\left(\mathcal{A}^0_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}}}(\mathcal{U})^\vee\right)\right) & \longrightarrow & \phi^*\left(R^1p_{n*}\left(\mathrm{End}^0(\mathcal{U})^\vee\right)\right) & \longrightarrow & 0 \\
 & & \frac{r}{n}\mathrm{Id}\downarrow\cong & & \cong\downarrow & & \cong\downarrow\mathrm{Id} & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} & \longrightarrow & \phi^*\left(\mathcal{A}_{\mathcal{SU}_{\mathcal{C}}/S}(\mathcal{L})\right) & \xrightarrow{\nabla_1} & \phi^*\left(T_{\mathcal{SU}_{\mathcal{C}}/S}\right) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow d\phi & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} & \longrightarrow & \mathcal{A}_{\mathcal{SM}^{par}/S}(\phi^*(\mathcal{L})) & \longrightarrow & T_{\mathcal{SM}^{par}/S} & \longrightarrow & 0
\end{array}$$

By construction the first and the last exact sequences are isomorphic as they are pull-backs of isomorphic exact sequences by the differential map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^1\pi_{n*}(K_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}) & \longrightarrow & R^1\pi_{n*}({}^0\mathcal{P}^{-1}(\mathcal{E})) & \longrightarrow & R^1\pi_{n*}(\mathrm{parEnd}^0(\mathcal{E})) & \longrightarrow & 0 \\
 & & \frac{r}{n}\mathrm{Id}\,\| & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{SM}^{par}} & \longrightarrow & \mathcal{A}_{\mathcal{SM}^{par}/S}(\phi^*(\mathcal{L})) & \longrightarrow & T_{\mathcal{SM}^{par}/S} & \longrightarrow & 0
\end{array}$$

where $\mathcal{L}$ is the ample generator of the Picard group of the space $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$ that we denote by Θ. Note that we have the equalities

$$K_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}} \cong \pi_w^*\left(K_{\mathcal{C}/S}\right) \cong \phi^*\left(K_{\mathcal{X}/\mathcal{SU}_{\mathcal{C}}}\right).$$

We conclude the proof by applying Proposition 4.3.2, to obtain

$$\begin{array}{ccccc}
R^1\pi_{n*}(K_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}) & \hookrightarrow & R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}(\mathcal{E})(\mathcal{D})\right]^\vee\right) & \twoheadrightarrow & R^1\pi_{n*}(\mathrm{parEnd}(\mathcal{E})) \\
\cong\downarrow\frac{r}{n} & & \downarrow\cong & & \downarrow\cong \\
\mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} & \hookrightarrow & \mathcal{A}_{\mathcal{SM}_{\mathcal{C}}^{par}/S}(\Theta) & \xrightarrow{\nabla_1} & T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}
\end{array}$$

Hence this conclude the proof. □

4.4 Parabolic Hitchin symbol map

Let $\mathcal{E}_* \longrightarrow \mathcal{C}\times_S \mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$ be a virtual universal parabolic vector bundle of fixed parabolic type α_* over $(\mathcal{C},D)/S$ a smooth family of projective marked curves over S. We want a parabolic version of the Hitchin symbol map given in [BBMP23] in section 4.3.

We suppose throughout this section that there is no trivial parabolic points (see section 1.1).i.e. For al $i \in I$ we suppose $\ell_i > 1$. We will use the following notation

$$\mathcal{SM}^{par}_{\mathcal{C}/S} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$$

First approach: We take the trace map which we denote by B as follow

$$\begin{array}{ccc} \pi_{n_*}\left(\mathrm{End}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right) \otimes \pi_{n_*}\left(\mathrm{End}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right) & & (\phi, \psi) \\ \Big\downarrow B & & \Big\downarrow \\ \pi_{n_*}\pi_w^*\left(K^{\otimes 2}_{\mathcal{C}/S}(2D)\right) & & B(\phi,\psi) = \mathrm{Trace}(\phi \circ \psi) \end{array}$$

we take its restriction to the subsheaf $\mathrm{SparEnd}^0(\mathcal{E}) \subset \mathrm{End}^0(\mathcal{E})$ so the left hand side is the cotangent bundle $T^\vee_{\mathcal{SM}^{par}_{\mathcal{C}}/S}$, so we get

$$B : T^\vee_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \otimes T^\vee_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \longrightarrow \pi_{n_*}\pi_w^*\left(K^{\otimes 2}_{\mathcal{C}/S}(2D)\right).$$

A simple calculation gives

$$Image(B) \subset \pi_{n_*}\pi_w^*\left(K^{\otimes 2}_{\mathcal{C}/S}(D)\right).$$

We denote by B the following restriction

$$B : T^\vee_{\mathcal{SM}^{par}/S} \otimes T^\vee_{\mathcal{SM}^{par}/S} \longrightarrow \pi_{n_*}\pi_w^*\left(K^{\otimes 2}_{\mathcal{C}/S}(D)\right)$$

We dualize and by Serre's duality relative to π_n we get

$$B^\vee : \pi_e^*\left(R^1\pi_{s_*}\left(T_{\mathcal{C}/S}(-D)\right)\right) \to T_{\mathcal{SM}^{par}_{\mathcal{C}}/S} \otimes T_{\mathcal{SM}^{par}_{\mathcal{C}}/S}.$$

Definition 4.4.1 (Parabolic Hitchin Symbol map)

The parabolic Hitchin symbol ρ^{par} is the morphism given by applying π_{e_*}

$$\rho_{par} : R^1\pi_{s_*}\left(T_{\mathcal{C}/S}(-D)\right) \longrightarrow \pi_{e_*}\mathrm{Sym}^2\left(T_{\mathcal{SM}^{par}/S}\right).$$

Second approach: Consider the evaluation map of the sheaf: $\mathrm{SparEnd}^0(\mathcal{E})\otimes\pi_w^*\left(K_{\mathcal{C}/S}(D)\right)$, composed with the injection map $\mathrm{SparEnd}^0(\mathcal{E})\subset \mathrm{parEnd}^0(\mathcal{E})$, we get

$$\pi_n^*\,\pi_{n*}\left(\mathrm{SparEnd}^0(\mathcal{E})\otimes\pi_w^* K_{\mathcal{C}/S}(D)\right)\xrightarrow{ev}\mathrm{parEnd}^0(\mathcal{E})\otimes\pi_w^*\left(K_{\mathcal{C}/S}(D)\right)$$

We dualize

$$\mathrm{parEnd}^0(\mathcal{E})^\vee\otimes\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right)\xrightarrow{ev^\vee}\pi_n^*\left(\pi_{n*}\left(\mathrm{SparEnd}^0(\mathcal{E})\otimes\pi_w^*\,K_{\mathcal{C}/S}(D)\right)\right)^\vee$$

This morphism gives a map which we denote by $ev^\vee$

$$\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right)\xrightarrow{ev^\vee}\mathrm{parEnd}^0(\mathcal{E})\otimes\pi_n^*\left(\pi_{n*}\left(\mathrm{SparEnd}^0(\mathcal{E})\otimes\pi_w^*\left(K_{\mathcal{C}/S}(D)\right)\right)\right)^\vee$$

By Serre's duality relatively to π_n

$$\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right)\xrightarrow{ev^\vee}\mathrm{parEnd}^0(\mathcal{E})\otimes\pi_n^*\left(R^1\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E})\right)\right)$$

We apply $\pi_{e*}\circ R^1\pi_{n*}$ and by the projection formula, we get

$$\pi_{e*}\left(R^1\pi_{n*}\left(ev^\vee\right)\right): R^1\pi_{s*}\left(T_{\mathcal{C}/S}\left(-D\right)\right)\longrightarrow\pi_{e*}\left(T_{\mathcal{M}^{par}}\otimes T_{\mathcal{M}^{par}}\right)$$

Finally we get the morphism:

$$\pi_{e*}\left(R^1\pi_{n*}\left(ev^\vee\right)\right): R^1\pi_{s*}\left(T_{\mathcal{C}/S}(-D)\right)\longrightarrow\pi_{e*}\mathrm{Sym}^2\left(T_{\mathcal{SM}^{par}/S}\right)$$

Lemma 4.4.2 *This application coincide with the parabolic Hitchin symbol ρ_{par}.i.e.*

$$\rho_{par}:=\pi_{e*}\left(R^1\pi_{n*}\left(ev^\vee\right)\right): R^1\pi_{s*}\left(T_{\mathcal{C}}\left(-D\right)\right)\longrightarrow\pi_{e*}\mathrm{Sym}^2\left(T_{\mathcal{M}^{par}}\right).$$

Proof. The lemma follows from commutativity of the diagram

$$\begin{array}{ccc} & & R^1\pi_{n*}(\mathrm{parEnd}^0(\mathcal{E}))\otimes R^1\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E})\right) \\ & \overset{B^\vee}{\nearrow} & \\ R^1\pi_{n*}\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right) & & \Big\downarrow Id\otimes\left(R^1\pi_{n*}Tr^{-1}\right)^\vee \\ & \underset{R^1\pi_{n*}(ev^\vee)}{\searrow} & \\ & & T_{\mathcal{M}^{par}/S}\otimes T_{\mathcal{M}^{par}/S} \end{array}$$

This follows if we in turns dualize, apply Serre duality, for which

$$\left(R^1\pi_{n*}(ev^\vee)\right)^\vee=\pi_{n*}(ev\otimes\mathrm{Id}),$$

□

Proposition 4.4.3 *The symbol map ρ_{par} is invariant under Hecke modifications.*

The proposition is a consequence of the following: Take E_* a parabolic vector bundle over a curve C of parabolic type α_* with respect to a divisor D. Let $g \in \text{parEnd}(E)$ be a parabolic endomorphism. Then

Lemma 4.4.4 *The trace is invariant under Hecke modifications.i.e.*

$$tr(\mathcal{H}_i^j(g)) = tr(g) \text{ for all } i \in I \text{ and } j \in \{1, 2, ..., \ell_i\}.$$

Proof. For $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ take the Hecke modification of E with respect to the subspace $F_i^{j+1}(E)$ so we get a sub sheaf

$$f : \mathcal{H}_i^j(E) \hookrightarrow E$$

which is an isomorphism over $C \backslash \{x_i\}$, thus

$$tr(\mathcal{H}_i^j(g)) = tr(g) \quad \text{over} \quad C\backslash\{x_i\}.$$

The vector bundle $\mathcal{H}_i^j(E)$ inherits a parabolic structure, see Section 2.4, and $\mathcal{H}_i^j(g)$ is a parabolic endomorphism with respect to this parabolic structure.

$$\begin{array}{ccc} E & \xrightarrow{\ g\ } & E \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ \mathcal{H}_i^j(E) & \xrightarrow[\mathcal{H}_i^j(g)]{} & \mathcal{H}_i^j(E) \end{array}$$

Now we describe the map $\mathcal{H}_i^j(g)_{x_i} : \mathcal{H}_i^j(E)_{x_i} \to \mathcal{H}_i^j(E)_{x_i}$. We have the decomposition of the map g with respect to the quotient exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_i^{j+1}(E) & \longrightarrow & E_{x_i} & \longrightarrow & Q_i^j(E) := E_{x_i}/F_i^{j+1}(E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g|_{F_i^{j+1}(E)}} & & \downarrow{\scriptstyle g_{x_i}} & & \downarrow{\scriptstyle \overline{g}} & & \\ 0 & \longrightarrow & F_i^{j+1}(E) & \longrightarrow & E_{x_i} & \longrightarrow & Q_i^j(E) := E_{x_i}/F_i^{j+1}(E) & \longrightarrow & 0 \end{array}$$

thus we have

$$g_{x_i} = \begin{pmatrix} g|_{F_i^{j+1}(E)} & * \\ 0 & \overline{g} \end{pmatrix} \Longrightarrow tr(g_{x_i}) = tr(g|_{F_i^{j+1}(E)}) + tr(\overline{g}).$$

the Heck modification $\mathcal{H}_i^j(E)$ fit in the same diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_i^j(E) & \longrightarrow & \mathcal{H}_i^j(E) & \longrightarrow & F_i^{j+1}(E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \mathcal{H}_i^j(g)_{x_i}} & & \downarrow{\scriptstyle g|_{F_i^{j+1}(E)}} & & \\ 0 & \longrightarrow & Q_i^j(E) & \longrightarrow & \mathcal{H}_i^j(E)_{x_i} & \longrightarrow & F_i^{j+1}(E) & \longrightarrow & 0 \end{array}$$

hence we get

$$\mathcal{H}_i^j(g)_{x_i} = \begin{pmatrix} \overline{g} & 0 \\ * & g|_{F_i^{j+1}(E)} \end{pmatrix} \Longrightarrow tr(\mathcal{H}_i^j(g)_{x_i}) = tr(\overline{g}) + tr(g|_{F_i^{j+1}(E)}) = tr(g_{x_i}),$$

There for one has globally the equality

$$tr(g) = tr(\mathcal{H}_i^j(g)) \in \mathcal{O}_C.$$

This ends the proof. □

Proposition 4.4.5 *The parabolic Hitchin symbol map ρ_{par} is an isomorphism.*

Proof. Take the relative cotangent bundle over $\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}$

$$q : T^{\vee}_{\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}/S} \longrightarrow \mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}$$

one gets the following isomorphism

$$(\pi_e \circ q)_* \ \mathcal{O}_{T^{\vee}_{\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}/S}} \cong \bigoplus_{q \geqslant 0} \pi_{e*} \mathrm{Sym}^q \left(T_{\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}/S}\right)$$

and take the $\mathbb{G}_m$-action over the moduli space of the parabolic Higgs bundles $\mathcal{H}iggs^P(\alpha_*)$ that contain the cotangent space $T^{\vee}_{\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}/S}$ as a big open space. Thus elements of $\pi_{e*}\mathrm{Sym}^q\left(T_{\mathcal{SM}_{\mathcal{C}/\mathcal{S}}^{par}/S}\right)$ can be seen as regular functions over $T^{\vee}_{\mathcal{SM}^{par}/S}$ of degree 2 with respect to the action of $\mathbb{G}_m$ that can be extend by Hartog's theorem to all the space $\mathcal{H}iggs^P(\alpha_*)$. As the parabolic Hitchin system is equivariant, they are obtained from the quadratic part of the parabolic Hitchin base given by the space

$$\pi_{s*} K_{\mathcal{C}/S}^{\otimes 2}(D) \cong R^1\pi_{s*}\left(T_{\mathcal{C}/S}(-D)\right)$$

□

4.5 Kodaira-Spencer map

4.5.1 Infinitesimal deformations

We study the infinitesimal deformations of a triple $\mathbb{E} := (C, D, E_*)$ given by a smooth marked projective curve C of genus $g \geqslant 2$ and D a reduced divisor of degree N equipped with a quasi-parabolic rank-r vector bundle E_* of fixed quasi-parabolic type $\vec{m}$. We follow [Wel83] to prove the following theorem. See also [Mar09].

Theorem 4.5.1 *The infinitesimal deformations of $\mathbb{E}$ are parameterized by $\mathrm{H}^1\left(C, \mathcal{A}_C^{par}(E)\right)$.*

Proof. Let $\mathcal{U} = \{U_\lambda\}_\lambda$ be an affine cover of the curve C such that any open affine set contains at most one point of the divisor D, we set $U_{\lambda,\mu} = \mathrm{Spec}(A_{\lambda,\mu})$.

1. First we recall the infinitesimal deformations of a marked curve (C, D): As the infinitesimal deformations of an affine scheme are trivial, the infinitesimal deformation C_ε of the curve C is locally trivial and globally given by the transition maps

$$\theta_{\lambda,\mu} : U_{\lambda,\mu} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \xrightarrow{\theta_\mu} \mathcal{C}\mid_{U_{\lambda,\mu}} \xrightarrow{\theta_\lambda^{-1}} U_{\lambda,\mu} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon])$$

which is equivalent to an isomorphism of rings

$$\begin{array}{cccc} \theta_{\lambda,\mu} & : & A_{\lambda,\mu}[\varepsilon] & \longrightarrow & A_{\lambda,\mu}[\varepsilon] \\ & & a + \varepsilon b & \longmapsto & a + \varepsilon\left(\vartheta_{\lambda,\mu}(a) + b\right). \end{array}$$

where $A_{\lambda,\mu}[\varepsilon] := A_{\lambda,\mu} \otimes \mathbb{C}[\varepsilon] = A_{\lambda,\mu} + \varepsilon A_{\lambda,\mu}$, and $\vartheta_{\lambda,\mu} : A_{\lambda,\mu} \longrightarrow A_{\lambda,\mu}$ are $\mathbb{C}$-derivations, hence $\{\vartheta_{\lambda,\mu}\}$ is a 1-cocycle with values in T_C. The latter gives the Kodaira-Spencer class of the deformation C_ε in $\mathrm{H}^1(C, T_C)$. Let denote $(C_\varepsilon, D_\varepsilon)$ an infinitesimal deformation of the couple (C, D). On an open set $U_{\lambda,\mu}$ a point $x \in D$ is given by a maximal ideal $I_{\lambda,\mu} \subset A_{\lambda,\mu}$, hence the 1-cocycle $\{\vartheta_{\lambda,\mu}\}$ must preserve the ideal $I_{\lambda,\mu}[\varepsilon]$ which means $\vartheta_{\lambda,\mu}(I_{\lambda,\mu}) \subset I_{\lambda,\mu}$. Hence a derivation over $I_{\lambda,\mu}$. Therefore the 1-cocycle $\{\vartheta_{\lambda,\mu}\}$ has values in $T_C(-D)$ and gives the Kodaira-Spencer class of the deformation $(C_\varepsilon, D_\varepsilon)$ in $\mathrm{H}^1(C, T_C(-D))$.

2. Now, let us study the infinitesimal deformation of the triple (C, D, E), without the quasi-parabolic structure: we choose the affine covering such that we have $E|_{U_{\lambda,\mu}} = \mathcal{O}_C|_{U_{\lambda,\mu}}^{\oplus r}$, so over the affine open subset $U_{\lambda,\mu}$ the vector bundle E is given by an $A_{\lambda,\mu}$-module $M_{\lambda,\mu}$. Let $(C_\varepsilon, D_\varepsilon, E_\varepsilon)$ be an infinitesimal deformation of the triple, where the deformation $(C_\varepsilon, D_\varepsilon)$ is given by the cocycle $\{\vartheta_{\lambda,\mu}\}$ with values in $T_C(-D)$ and the vector bundle E_ε in given by the gluing isomorphisms that induce the identity over $M_{\lambda,\mu}$

$$\begin{array}{cccc} \tau_{\lambda,\mu} & : & M_{\lambda,\mu}[\varepsilon] & \longrightarrow & M_{\lambda,\mu}[\varepsilon] \\ & & m + \varepsilon n & \longmapsto & m + \varepsilon\left(\xi_{\lambda,\mu}(m) + n\right). \end{array}$$

which is an $A_{\lambda,\mu}[\varepsilon]$-linear map via the isomorphism $\theta_{\lambda,\mu}$. This is equivalent to the following equality for all $a \in A_{\lambda,\mu}$ and $m \in M_{\lambda,\mu}$

$$\xi_{\lambda,\mu}(am) - a\ \xi_{\lambda,\mu}(m) = \vartheta_{\lambda,\mu}(a)\ m,$$

this can be written for all $a \in A_{\lambda,\mu}$ as follows

$$[\xi_{\lambda,\mu}, a] = \vartheta_{\lambda,\mu}(a)\mathrm{Id}_{M_{\lambda,\mu}}.$$

Hence $\{\xi_{\lambda,\mu}\}$ yields a 1-cocycle with values in the sheaf $\widehat{\mathcal{A}_C(E)}$ given by the pull-back

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathcal{A}_C(E) & \longrightarrow & T_C & \longrightarrow & 0 \\ & & \downarrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \widehat{\mathcal{A}_C(E)} & \longrightarrow & T_C(-D) & \longrightarrow & 0 \end{array} \tag{4.5.1}$$

3. Finally, we study the infinitesimal deformations of $\mathbb{E}$: Let$(C_\varepsilon, D_\varepsilon, E_{*\varepsilon})$ be such a deformation of $\mathbb{E}$, where the deformation $(C_\varepsilon, D_\varepsilon)$ is given by the 1-cocycle $\{\vartheta_{\lambda,\mu}\}$ and the vector bundle E_ε is given by the 1-cocycle $\{\xi_{\lambda,\mu}\}$. By definition, a parabolic bundle is given for all $i \in I$ by the Hecke filtration (see Proposition 1.3.2)

$$\mathcal{H}_i^{\ell_i}(E) \subset \mathcal{H}_i^{\ell_i-1}(E) \subset \cdots \subset \mathcal{H}_i^2(E) \subset \mathcal{H}_i^1(E) \subset \mathcal{H}_i^0(E) = E. \tag{4.5.2}$$

Hence the parabolic vector bundle $E_{*\varepsilon}$ is given also for all $i \in I$ by filtrations of locally free sheaves

$$\mathcal{H}_i^{\ell_i}(E_\varepsilon) \subset \mathcal{H}_i^{\ell_i-1}(E_\varepsilon) \subset \cdots \subset \mathcal{H}_i^2(E_\varepsilon) \subset \mathcal{H}_i^1(E_\varepsilon) \subset \mathcal{H}_i^0(E_\varepsilon) = E_\varepsilon,$$

and the 1-cocycle $\{\tau_{\lambda,\mu}\}$ must preserve this filtrations, locally the sheaf $\mathcal{H}_i^j(E)$ is identified with a $A_{\lambda,\mu}[\varepsilon]$-submodule denoted by $M_{\lambda,\mu}^{i,j}[\varepsilon] \subset M_{\lambda,\mu}^0[\varepsilon] = M_{\lambda,\mu}[\varepsilon]$,

$$\begin{array}{ccc} M_{\lambda,\mu}[\varepsilon] & \xrightarrow{\tau_{\lambda,\mu}} & M_{\lambda,\mu}[\varepsilon] \\ \bigcup & & \bigcup \\ M_{\lambda,\mu}^{i,j}[\varepsilon] & \longrightarrow & M_{\lambda,\mu}^{i,j}[\varepsilon] \end{array}$$

The fact that the diagram commutes is equivalent to the fact that the 1-cocycle $\{\xi_{\lambda,\mu}\}$ preserve the filtration given by the $A_{\lambda,\mu}$-modules $\{M_{\lambda,\mu}^{i,j}\}$ associated to the filtration (4.5.2). Hence the 1-cocycle $\{\xi_{\lambda,\mu}\}$ has values in the sheaf $\widehat{\mathcal{A}_C^{par}(E)}$ defined as the subsheaf of $\widehat{\mathcal{A}_C(E)}$ given locally by differential operators preserving the subsheaves $\mathcal{H}_i^j(E)$. Hence the infinitesimal deformations of $\mathbb{E} = (C, D, E_*)$ are given by the cohomology group $\mathrm{H}^1(C, \widehat{\mathcal{A}_C^{par}(E)})$. Note that the sheaf $\widehat{\mathcal{A}_C^{par}(E)}$ can be included in an exact sequence

$$0 \longrightarrow \mathrm{parEnd}(E) \longrightarrow \widehat{\mathcal{A}_C^{par}(E)} \xrightarrow{\nabla_1} T_C(-D)$$

where the map ∇_1 is the restriction of the natural map : $\widehat{\mathcal{A}_C(E)} \to T_C(-D)$ given in the exact sequence (4.5.1).

To conclude the proof we need to show the following isomorphism $\widehat{\mathcal{A}_C^{par}(E)} \cong \mathcal{A}_C^{par}(E)$. Note that by definition of $\mathcal{A}_C^{par}(E)$ as push-out we have

$$\mathcal{A}_C^{par}(E) := \{(f, \partial) \,/\, f \in \mathrm{parEnd}(E)\ \partial \in \mathcal{A}_C(E)(-D) \text{ and } (f, 0) \sim (0, f) \text{ if } f \in \mathrm{End}(E)(-D)\}.$$

Thus we can define an $\mathcal{O}_C$-linear map ϱ as follows

$$\begin{array}{cccc} \varrho & : & \mathcal{A}_C^{par}(E) & \longrightarrow & \widehat{\mathcal{A}_C^{par}(E)} \\ & & (f, \partial) & \longmapsto & f + \partial. \end{array}$$

Clearly the map ϱ induces identity map on $\mathrm{parEnd}(E)$. Let us prove that ϱ is an isomorphism:

1. Injectivity: Let $(f,\partial) \in \mathcal{A}_C^{par}(E)$ such that $\varrho(f,\partial) = 0 \Leftrightarrow f + \partial = 0 \Leftrightarrow \partial = -f$, hence $\partial, f \in \mathrm{End}(E)(-D)$ by definition of $\mathcal{A}_C^{par}(E)$, we have $(f,\partial) = (f,-f) \sim (f-f,0) = 0 \in \mathcal{A}_C^{par}(E)$.

2. Surjectivity: Let $\partial \in \widehat{\mathcal{A}_C^{par}(E)}$ we associate its symbol $\nabla_1(\partial) \in T_C(-D)$. Take a lifting $\widehat{\nabla_1(\partial)} \in \mathcal{A}_C^{par}(E)$ (modulo $\mathrm{parEnd}(E)$), which can be written $\widehat{\nabla_1(\partial)} = (f,\widehat{\partial})$, where $\widehat{\partial} \in \mathcal{A}_C(E)(-D)$ with $\nabla_1(\widehat{\partial}) = \nabla_1(\partial)$ and f any element in $\mathrm{parEnd}(E)$. Note that $\partial, \widehat{\partial} \in \mathcal{A}_C(E)(-D) \Rightarrow \partial - \widehat{\partial} \in \mathrm{parEnd}(E)$. For $f = \partial - \widehat{\partial}$, one has $q(\partial - \widehat{\partial}, \widehat{\partial}) = \partial$.

Hence we get an isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{parEnd}(E) & \longrightarrow & \mathcal{A}_C^{par}(E) & \longrightarrow & T_C(-D) & \longrightarrow & 0 \\ & & \| \,\mathrm{Id} & & \varrho \downarrow \cong & & \| \,\mathrm{Id} & & \\ 0 & \longrightarrow & \mathrm{parEnd}(E) & \longrightarrow & \widehat{\mathcal{A}_C^{par}(E)} & \longrightarrow & T_C(-D) & \longrightarrow & 0 \end{array}$$

This concludes the proof. □

Remark 4.5.2 *Note that [BDHP22] studied the infinitesimal deformations of* $\mathbb{E} = (C, D, E_*)$ *a marked curve equipped with a quasi-parabolic vector bundle of type* $\vec{m}$*. Where they defined* $At(E_*)$ *a parabolic Atiyah algebroid as following: Let*

$$\mathcal{D}_{par}^{(1)}(E,E) \subset \mathcal{D}_C^{(1)}(E),$$

be the coherent sheaf of all differentiable operators $\partial_U : E|_U \to E|_U$*, where* $U \subset C$ *is any open subset, satisfying the condition that for any section* $s \in \Gamma(U\mathcal{H}_j^i(E))$ *of a Hecke modification of* E*, we get* $\partial_U(s) \in \Gamma(U\mathcal{H}_j^i(E))$*. Now, we define the sheaf* $At(E_*)$ *by*

$$At(E_*) := \{\partial \in \mathcal{D}_{par}^{(1)}(E,E) \quad / \quad \nabla_1(\partial) \in T_C(-D)\},$$

where $\nabla_1 : \mathcal{D}_C^1(E) \longrightarrow T_C$ *is the first symbol map. And they prove the following result*

Theorem 4.5.3 (Lemma 3.1 [BDHP22])

The infinitesimal deformations of $\mathbb{E}$ *are parameterized by* $\mathrm{H}^1(C, At(E_*))$*.*

For the proof they use Seshadri's identification (see introduction Theorem 0.0.5) hence for $\pi : X \to C$ *a Galois covering with Galois group* Γ*, they identify*

1. *Infinitesimal deformations of* (C, D, E_*)*, and*

2. *Infinitesimal deformations of* $(X, F := \pi^*(E))$ *equipped with the natural* Γ*-linearisation.*

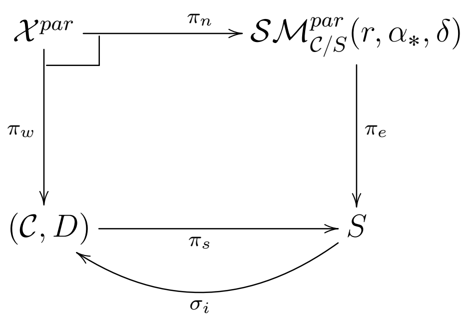
The second point is given by $\mathrm{H}^1(X, At_X(F))^\Gamma$ *the* Γ*-invariant part of the cohomology group* $\mathrm{H}^1(X, At_X(F))$*, that parametrises the infinitesimal deformations of the couple* (Y, F)*, where* $At_X(F)$ *is the Atiyah algebroid of the vector bundle* F *over* X*. They conclude the proof by the following remark: the sheaf* $At(E_*)$ *is the vector bundle underlying the parabolic bundle corresponding the* Γ*-equivariant bundle* $At_X(F)$ *over* X*. i.e.* $At(E_*) = \pi_*^\Gamma(At_X(F))$*. Hence we get*

$$\mathcal{T}_{\mathrm{Def}(X,F,\Gamma)} \cong \mathcal{T}_{\mathrm{Def}_{\mathbb{E}}} \cong \mathrm{H}^1(X, At_X(F))^\Gamma \cong \mathrm{H}^1(C, \pi_*^\Gamma(At_X(F)) \cong \mathrm{H}^1(C, At(E_*)).$$

where $\mathcal{T}_{\mathrm{Def}(X,F,\Gamma)}$ *is the space of infinitesimal deformations of* (Y, F, Γ) *a curve and a* Γ*-linearised bundle* F *on* X*. Note that the last isomorphism depends on the chosen Galois cover. This concludes the proof. Note that the sheaf* $At(E_*)$ *given in* [BDHP22] *is by definition the sheaf* $\widehat{\mathcal{A}_C^{par}}(E)$ *given in the proof of theorem* 4.5.1*, hence isomorphic to the parabolic Atiyah algebroid* $\mathcal{A}_C^{par}(E)$*.*

4.5.2 Parabolic Kodaira-Spencer map

Let $\pi_s : (\mathcal{C}, D) \longrightarrow S$ be a smooth family of projective marked curves parametrized by an algebraic variety S and let $\pi_e : \mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta) \longrightarrow S$ the relative moduli spaces of parabolic rank-r vector bundles of fixed parabolic type α_* and determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$. Let $\mathcal{E}_*$ be a virtual universal parabolic vector bundle over $\mathcal{X}^{par} := \mathcal{C} \times_S \mathcal{SM}_{\mathcal{C}}^{par}(r, \alpha_*, \delta)$. We shall denote the fiber product by the diagram



We use the following notation $\mathcal{SM}_{\mathcal{C}}^{par} := \mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$. We have two fundamental maps

- The Kodaira-Spencer of the family of marked curves:

  $$\kappa_{\mathcal{C}/S} : T_S \longrightarrow R^1\pi_{s_*}\left(T_{\mathcal{C}/S}(-D)\right),$$

  given as the first connecting morphism on cohomology of the short exact sequence

  $$0 \longrightarrow T_{\mathcal{C}/S}(-D) \longrightarrow \mathcal{T} \longrightarrow \pi_s^* T_S \longrightarrow S.$$

  where the sheaf $\mathcal{T}$ is given as follow:

  $$\mathcal{T} := \{v \in T_{\mathcal{C}} \mid v(I_D) \subset I_D\} \subset T_{\mathcal{C}}$$

  where I_D is the ideal sheaf of the divisor D.

- The Kodaira-Spencer of the family $\pi_e : \mathcal{SM}_{\mathcal{C}}^{par} \longrightarrow S$:

$$\kappa_{\mathcal{SM}_{\mathcal{C}}^{par}/S} : T_S \longrightarrow R^1\pi_{e*}\left(T_{\mathcal{SM}^{par}/S}\right)$$

  where

$$T_{\mathcal{SM}^{par}/S} \cong R^1\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E})\right)$$

  given as the first connecting morphism on cohomology of the short exact sequence

$$0 \longrightarrow T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \longrightarrow T_{\mathcal{SM}^{par}_{\mathcal{C}/S}} \longrightarrow \pi_e^* T_S \longrightarrow S.$$

Take the QPA sequence of the bundle $\mathcal{E}_*$ over $\mathcal{X}^{par}$

$$0 \longrightarrow \mathrm{parEnd}^0(\mathcal{E}) \longrightarrow \mathcal{A}^{0,par}_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}(\mathcal{E}) \longrightarrow \pi_w^*\left(T_{\mathcal{C}/S}\left(-D\right)\right) \longrightarrow 0$$

As $\pi_{e*}\left(\pi_w^*\left(T_{\mathcal{C}}\left(-D\right)\right)\right) = 0$ and $R^2\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E})\right) = 0$ (the relative dimension of π_n is 1), we apply $R^1\pi_{n*}$ we get an exact sequence on $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$

$$0 \longrightarrow T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \longrightarrow R^1\pi_{n*}\left(\mathcal{A}^{0,par}_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}(\mathcal{E})\right) \longrightarrow R^1\pi_{n*}\left(\pi_w^*\left(T_{\mathcal{C}/S}\left(-D\right)\right)\right) \longrightarrow 0$$

Proposition 4.5.4 *The first connecting homomorphism with respect to π_{e*} denoted Φ^{par} commutes with the Kodaira-Spencer maps of the two families*

$$\Phi_{par} \circ \kappa_{\mathcal{C}/S} = \kappa_{\mathcal{SM}_{\mathcal{C}}^{par}/S}.$$

We call the map Φ^{par} the Kodaira-Spencer map.

Let ∂ is the first connecting homomorphism of the long exact sequence for π_e of the sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} \hookrightarrow R^1\pi_{n*}\left(\left[\mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}(\mathcal{E})\left(\mathcal{D}\right)\right]^{\vee}\right) \twoheadrightarrow T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \longrightarrow 0$$

given by applying $R^1\pi_{n*}$ to the dual of the SQPA sequence tensorized by $\mathcal{O}_{\mathcal{X}^{par}}(\mathcal{D})$

$$0 \longrightarrow \Omega^1_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}} \hookrightarrow \mathcal{A}^{0,par,st}_{\mathcal{X}^{par}/\mathcal{SM}_{\mathcal{C}}^{par}}(\mathcal{E})\left(\mathcal{D}\right)^{\vee} \twoheadrightarrow \mathrm{parEnd}^0(\mathcal{E}) \longrightarrow 0$$

We prove a parabolic version of proposition 4.7.1 in [BBMP23]

Proposition 4.5.5 *The following diagram commute*

$$\begin{array}{ccc} R^1\pi_{s*}\left(T_{\mathcal{C}/S}(-D)\right) & \xrightarrow{\quad -\Phi_{par} \quad} & R^1\pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right) \\ & {\searrow}^{\rho_{par}} \qquad {\nearrow}^{\partial} & \\ & \pi_{e*}\mathrm{Sym}^2\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right) & \end{array}$$

i.e.: $\Phi_{par} + \partial \circ \rho_{par} = 0$.

Proof. We need the following lemma

Lemma 4.5.6 ([BBMP23] lemma 4.5.1 page 23.) *Let X a scheme, V and L respectively a vector and a line bundle on X. Moreover let $F \in Ext^1(L, V)$*

$$0 \longrightarrow V \xrightarrow{\ \mathrm{i}\ } F \xrightarrow{\ \pi\ } L \longrightarrow 0$$

by taking the dual and tensorizing with $V \otimes L$, we get

$$0 \longrightarrow V \longrightarrow F^* \otimes V \otimes L \longrightarrow V^* \otimes V \otimes L \longrightarrow 0$$

consider the injection

$$\begin{array}{cccc} \psi : & L & \longrightarrow & V^* \otimes V \otimes L \\ & t & \longmapsto & Id_V \otimes t \end{array}$$

then there exist a canonical injection $\phi : F \longrightarrow F^* \otimes V \otimes L$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & F & \xrightarrow{-\pi} & L & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & V & \longrightarrow & F^* \otimes V \otimes L & \longrightarrow & V^* \otimes V \otimes L & \longrightarrow & 0 \end{array}$$

Now, we prove the proposition. Take the parabolic Atiyah sequence on $\mathcal{X}^{par}$ of the universal bundle $\mathcal{E}$ relative to π_n. We note : $\mathcal{A}^{par} := \mathcal{A}^{0,par}_{\mathcal{X}^{par}/\mathcal{M}^{par}}(\mathcal{E})$ and $\mathcal{A}^{str} := \mathcal{A}^{0,par,str}_{\mathcal{X}^{par}/\mathcal{M}^{par}}(\mathcal{E})$, and take the evaluation map composed with the inclusion $\mathrm{SparEnd}(\mathcal{E}) \hookrightarrow \mathrm{parEnd}(\mathcal{E})$

$$\pi_n^* \ \pi_{n*} \left(\mathrm{SparEnd}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right) \xrightarrow{ev} \mathrm{parEnd}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)$$

We dualize

$$\mathrm{parEnd}^0(\mathcal{E})^\vee \otimes \pi_w^* \ T_{\mathcal{C}/S}(-D) \xrightarrow{ev^*} \pi_n^* \ \pi_{n*} \left(\mathrm{SparEnd}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right)^\vee$$

We get the following morphism of exact sequences

$$\begin{array}{ccc} 0 & & 0 \\ \big\downarrow & & \big\downarrow \\ \mathrm{parEnd}^0(\mathcal{E}) & = & \mathrm{parEnd}^0(\mathcal{E}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{parEnd}^0(\mathcal{E}) \otimes \mathcal{A}^{par\vee} \otimes \pi_w^* T_{\mathcal{C}/S}(-D) & \xrightarrow{q} & \mathrm{parEnd}^0(\mathcal{E}) \otimes \pi_n^* \pi_{n*} \left(\mathcal{A}^{str} \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right)^\vee \\ \big\downarrow & & \big\downarrow \\ \mathrm{parEnd}^0(\mathcal{E}) \otimes \mathrm{parEnd}^0(\mathcal{E})^\vee \otimes \pi_w^* T_{\mathcal{C}/S}(-D) & \xrightarrow{ev^\vee} & \mathrm{parEnd}^0(\mathcal{E}) \otimes \pi_n^* \ \pi_{n*} \left(\mathrm{SparEnd}^0(\mathcal{E}) \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right)^\vee \\ \big\downarrow & & \big\downarrow \\ 0 & & 0 \end{array}$$

The map q is given by taking the dual of the evaluation map

$$ev : \pi_n^* \pi_{n*} \left(\mathcal{A}^{par} \otimes \pi_w^* K_{\mathcal{C}/S}(D)\right) \longrightarrow \mathcal{A}^{par} \otimes \pi_w^* K_{\mathcal{C}/S}(D)$$

and composite with the natural inclusion $\mathcal{A}^{str} \hookrightarrow \mathcal{A}^{par}$. We apply the lemma 4.5.6 to the left exact sequence (for $V = \mathrm{parEnd}^0(\mathcal{E})$, $L = \pi_w^* T_{\mathcal{C}/S}(-D)$ and $F = \mathcal{A}^{par}$), and we apply the Serre duality relative to π_n for the right exact sequence we get the morphism of exact sequences

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{parEnd}^0(\mathcal{E}) & = & \mathrm{parEnd}^0(\mathcal{E}) \\
\downarrow & & \downarrow \\
\mathcal{A}^{par} & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E}) \otimes \pi_n^* R^1\pi_{n*}\left(\mathcal{A}^{str}(\mathcal{D})^\vee\right) \\
\downarrow & & \downarrow \\
\pi_w^* T_{\mathcal{C}/S}(-D) & \longrightarrow & \mathrm{parEnd}^0(\mathcal{E}) \otimes \pi_n^* R^1\pi_{n*}\left(\mathrm{parEnd}^0(\mathcal{E})\right) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The left exact sequence is the parabolic Atiyah sequence where we multiply the map $\mathcal{A}^{par} \longrightarrow \pi_w^*(T_{\mathcal{C}/S}(-D))$ by -1, see Lemma 4.5.6 .

We apply $R^1\pi_{n*}$ to get

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} & = & T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \\
\downarrow & & \downarrow \\
R^1\pi_{n*}\left(\mathcal{A}^{par}\right) & \longrightarrow & T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \otimes R^1\pi_{n*}\left(\mathcal{A}^{str}(\mathcal{D})^\vee\right) \\
\downarrow & & \downarrow \\
R^1\pi_{n*}\left(\pi_w^*\left(T_{\mathcal{C}/S}(-D)\right)\right) & \longrightarrow & T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \otimes T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The right exact sequence is the $R^1\pi_{e*}$ applied to the dual of strongly parabolic Atiyah sequence tensorized by $T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}$, and the first connecting homomorphism in cohomology with respect to π_e is given by cup product with the class Δ.

We take the first connecting homomorphism of the long exact sequence with respect to the map π_e, we get

$$\begin{CD}
\pi_{e*}R^1\pi_{n*}\left(\pi_w^*\left(T_{\mathcal{C}/S}\left(-D\right)\right)\right) @>>> \pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\otimes T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right)\\
@VVV @VV{\cup\Delta}V\\
R^1\pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right) @= R^1\pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right)
\end{CD}$$

we have

$$\pi_{e*}R^1\pi_{n*}\left(\pi_w^*\left(T_{\mathcal{C}/S}\left(-D\right)\right)\right)\simeq R^1\pi_{s*}\left(T_{\mathcal{C}/S}\left(-D\right)\right)$$

we get the following commutative diagram

$$\begin{CD}
R^1\pi_{s*}\left(\mathcal{T}_{\mathcal{C}/S}\left(-D\right)\right) @>{\rho_{par}}>> \pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\otimes T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right)\\
@V{-\Phi^{par}}VV @VV{\partial}V\\
R^1\pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right) @= R^1\pi_{e*}\left(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}\right)
\end{CD}$$

Thus conclude the proof. □

4.5.3 Some equalities and consequences

We recall the equalities given in Theorem 4.3.4. For all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ one has

1. $\frac{r}{n}\Delta = [\Theta] \in \mathrm{H}^0\left(S, R^1\pi_{e*}(\Omega^1_{\mathcal{SM}^{par}/S})\right)$. We denote the associated application given by the contraction with this class by

$$\partial := \cup\Delta : \pi_{e*}\left(\mathrm{Sym}^2\left(T_{\mathcal{SM}^{par}/S}\right)\right) \longrightarrow R^1\pi_{e*}\left(T_{\mathcal{SM}^{par}/S}\right),$$

2. $\frac{r}{n_j(i)}\Delta_j(i) = [\Theta_j(i)] \in \mathrm{H}^0\left(S, R^1\pi_{e*}(\Omega^1_{\mathcal{SM}^{par}/S})\right)$. We denote the associated application given by the contraction with this class, as follows

$$\partial_j(i) := \cup\Delta_j(i) : \pi_{e*}\left(\mathrm{Sym}^2\left(T_{\mathcal{M}^{par}/S}\right)\right) \longrightarrow R^1\pi_{e*}\left(T_{\mathcal{M}^{par}/S}\right).$$

Combining the above equalities, we get the following result.

Theorem 4.5.7 *Assume that the family $\pi_s : (\mathcal{C}, D) \to S$ is versal* [2]*. Then for all $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$ we have the equalities over the moduli space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$*

1. $\cup[\Theta] \circ \rho_{par} = -\frac{r}{n} \cdot \Phi_{par}$.
2. $\cup[\Theta_j(i)] \circ \rho_{par} = -\frac{r}{n_j(i)} \cdot \Phi_{par}$.

Proof. The first equality is a direct consequence of Proposition 4.5.5, where we have

$$\partial \circ \rho_{par} = -\Phi_{par},$$

we multiply the equality by $\frac{r}{n}$ and use the first equality above

$$\frac{r}{n}\partial \circ \rho_{par} = \cup[\Theta] \circ \rho_{par} = -\frac{r}{n}\Phi_{par}.$$

Fix $i \in I$ and $j \in \{1, 2, ..., \ell_i\}$. We take the Hecke isomorphism over S

$$\begin{array}{ccc} \mathcal{H}^j_i : \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta) & \xrightarrow{\cong} & \mathcal{SM}^{par}_{\mathcal{C}/S}\left(r, \mathcal{H}^j_i(\alpha_*), \mathcal{H}^j_i(\delta)\right) \\ {\scriptstyle \pi_e} \searrow & \circlearrowleft & \swarrow {\scriptstyle \pi_e^{i,j}} \\ & S & \end{array}$$

where the map $\mathcal{H}^j_i$ is given in (4.3.1). Then by Proposition 4.5.4 applied over $\mathcal{SM}^{par} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ and $\mathcal{SM}^{par}_{i,j} := \mathcal{SM}^{par}_{\mathcal{C}/S}\left(r, \mathcal{H}^j_i(\alpha_*), \mathcal{H}^j_i(\delta)\right)$, the following diagram commute under the assumption that the map $\kappa_{\mathcal{C}/S}$ is an isomorphism

$$\begin{array}{ccccc} & & & & R^1\pi_{e*}\left(T_{\mathcal{SM}^{par}/S}\right) \\ & \nearrow^{\kappa_{\pi_e}} & & \nearrow^{\Phi_{par}} & \\ T_S & \xrightarrow[\cong]{\kappa_{\mathcal{C}/S}} & R^1\pi_{s*}\left(T_{\mathcal{C}/S}(-D)\right) & & \cong \downarrow \mathcal{H}^j_i \\ & \searrow_{\kappa_{\pi_e^{i,j}}} & & \searrow_{\Phi^{i,j}_{par}} & \\ & & & & R^1\pi^{i,j}_{e*}\left(T_{\mathcal{SM}^{par}_{i,j}/S}\right) \end{array}$$

In fact by Proposition 4.5.4 applied over $\mathcal{SM}^{par}$ and $\mathcal{SM}^{par}_{i,j}$, one has

$$\left.\begin{array}{r} \Phi_{par} \circ \kappa_{\mathcal{C}/S} = \kappa_{\pi_e} \\ \Phi^{i,j}_{par} \circ \kappa_{\mathcal{C}/S} = \kappa_{\pi_e^{i,j}} \\ \mathcal{H}^j_i \circ \kappa_{\pi_e} = \kappa_{\pi_e^{i,j}} \\ \kappa_{\mathcal{C}/S} \quad \text{isomorphism} \end{array}\right\} \Longrightarrow \mathcal{H}^j_i \circ \Phi_{par} = \Phi^{i,j}_{par}. \tag{4.5.3}$$

[2]The Kodaira-Spencer of the family of marked curves is an isomorphism

Now we define the parabolic Hitchin symbol map $\rho_{par}^{i,j}$ over the moduli space $\mathcal{SM}_{i,j}^{par}$ (see definition 4.4.1). We have the following commutative diagram

$$\begin{array}{ccccc} & \overset{\rho_{par}}{\nearrow} & \pi_{e_*}\left(\mathrm{Sym}^2 T_{\mathcal{SM}^{par}/S}\right) & \xrightarrow{\cup[\Theta_j(i)]} & R^1\pi_{e_*}\left(T_{\mathcal{SM}^{par}/S}\right) \\ R^1\pi_{s_*}\left(T_{\mathcal{C}/S}(-D)\right) & \circlearrowleft & \cong \downarrow \mathcal{H}_i^j & \circlearrowleft & \cong \downarrow \mathcal{H}_i^j \\ & \underset{\rho_{par}^{i,j}}{\searrow} & \pi_{e_*}\left(\mathrm{Sym}^2 T_{\mathcal{SM}_{i,j}^{par}/S}\right) & \xrightarrow[\cup[\Theta_j(i)]]{} & R^1\pi_{e_*}^{i,j}\left(T_{\mathcal{SM}_{i,j}^{par}/S}\right) \end{array}$$

The first diagram commute by Proposition 4.4.3. Hence by the above diagram one has

$$\begin{aligned} \cup[\Theta_j(i)] \circ \rho_{par} &= \left((\mathcal{H}_i^j)^{-1} \circ \cup[\Theta_j(i)] \circ \mathcal{H}_i^j\right) \circ \rho_{par} \\ &= (\mathcal{H}_i^j)^{-1} \circ \left(\cup[\Theta_j(i)] \circ \rho_{par}^{i,j}\right) \end{aligned}$$

We apply Proposition 4.5.5, to get

$$\cup[\Theta_j(i)] \circ \rho_{par} = (\mathcal{H}_i^j)^{-1} \circ \left(-\frac{r}{n_j(i)}\Phi_{par}^{i,j}\right),$$

and by equation (4.5.3)

$$\cup[\Theta_j(i)] \circ \rho_{par} = -\frac{r}{n_j(i)}\Phi_{par}.$$

This concludes the proof. □

4.6 Line bundles over $\mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$

4.6.1 Parabolic determinant bundle

Let $\mathcal{E}_*$ be a family of parabolic rank r vector bundles of fixed parabolic type α_* over a smooth family of curves $\mathcal{C}/S$ parametrized by a S-variety $\mathcal{T}$. Let $p : \mathcal{C} \times_S \mathcal{T} \longrightarrow \mathcal{T}$ the projection map. We recall the definition of the parabolic determinant line bundle under the hypothesis (1.6)

$$\lambda_{par}(\mathcal{E}_*) := \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i} \left\{\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right)^{-a_j(i)}\right\} \otimes \det(\mathcal{E}_\sigma)^{\frac{k}{r}\chi_{par}}$$

which is a line bundle over $\mathcal{T}$, where

- $\mathcal{E}_\sigma = \mathcal{E}|_{\sigma \times_S T}$, for some section $\sigma : S \longrightarrow \mathcal{C}$.

- $\chi_{par} = d + r(1-g) + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i} m_j(i)a_j(i)$..

Pauly in [Pau96] gives another definition as following

Definition 4.6.1 (Parabolic determinant bundle) *Let $\mathcal{E}_*$ be a family of parabolic rank-r vector bundles of parabolic type α_* over a smooth family of curves $\pi_s : \mathcal{C} \longrightarrow S$ parameterized by a S-variety $\mathcal{T}$, then we have*

$$\Theta_{par}(\mathcal{E}_*) := \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1}\left\{\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)^{p_j(i)}\right\} \otimes \det(\mathcal{E}_\sigma)^e$$

where the determinant is with respect to the projection $\mathcal{C}\times_S\mathcal{T}\longrightarrow\mathcal{T}$ and for all $i\in I$ and $j\in\{1,2,...,\ell_i-1\}$

- $\mathcal{E}_{\sigma_i} := \mathcal{E}|_{\sigma_i(S)\times_S\mathcal{T}}$*, where $\sigma_i : S\longrightarrow\mathcal{C}$ the parabolic section of π_s.*
- $p_j(i) = a_{j+1}(i) - a_j(i)$.
- $r_j(i) := \sum_{i=1}^{q} m_i(q) = \dim_{\mathbb{C}}(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E}))$.
- $re = k\chi - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)r_j(i)$*, where $\chi = d + r(1-g)$.*

We prove in the following Proposition that the two definition are the same.

Proposition 4.6.2 *Let $\mathcal{E}_*$ be a family of parabolic rank-r vector bundles of parabolic type α_* over a smooth family of curves $\pi_s : \mathcal{C}\longrightarrow S$ parameterized by a S-variety $\mathcal{T}$, then*

$$\Theta_{par}(\mathcal{E}_*) \cong \lambda_{par}(\mathcal{E}_*).$$

Proof. To prove the equality of the line bundles over $\mathcal{T}$, we begin by replacing $\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right)$ by $\det\left(\mathcal{E}_{x_i}/F_i^{j+1}(\mathcal{E})\right)$. In fact we have for all $i\in I$ and $j\in\{1,2,...,\ell_i\}$ the equality

$$\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right) = \left(\det\left(\mathcal{E}_{x_i}/F_i^j(\mathcal{E})\right)\right)^{-1}\otimes\det\left(\mathcal{E}_{x_i}/F_i^{j+1}(\mathcal{E})\right). \quad (4.6.1)$$

for the proof, we take for all $i\in I$ and $j\in\{1,2,...,\ell_i\}$, the quotient exact sequences

$$0\longrightarrow F_i^j(\mathcal{E})\hookrightarrow\mathcal{E}_{\sigma_i}\twoheadrightarrow Q_i^{j-1}(\mathcal{E}) := \mathcal{E}_{\sigma_i}/F_i^j(\mathcal{E})\longrightarrow 0,$$

$$0\longrightarrow F_i^{j+1}(\mathcal{E})\hookrightarrow\mathcal{E}_{\sigma_i}\twoheadrightarrow Q_i^{j}(\mathcal{E}) := \mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\longrightarrow 0,$$

We calculate the determinants line bundles

$$\det F_i^j(\mathcal{E}) = \det(\mathcal{E}_{\sigma_i})\otimes\left(\det Q_i^{j-1}(\mathcal{E})\right)^{-1}$$
$$\det F_i^{j+1}(\mathcal{E}) = \det(\mathcal{E}_{\sigma_i})\otimes\left(\det Q_i^{j}(\mathcal{E})\right)^{-1}.$$

Now we calculate the determinant using the above equalities

$$\begin{aligned}\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right) &= \det F_i^j(\mathcal{E})\otimes\left(\det F_i^{j+1}(\mathcal{E})\right)^{-1}\\ &= \det(\mathcal{E}_{\sigma_i})\otimes\left(\det Q_i^{j-1}(\mathcal{E})\right)^{-1}\otimes\left(\det(\mathcal{E}_{\sigma_i})\otimes\left(\det Q_i^j(\mathcal{E})\right)^{-1}\right)^{-1}\\ &= \left(\det Q_i^{j-1}(\mathcal{E})\right)^{-1}\otimes\det Q_i^j(\mathcal{E})\\ &= \left(\det\left(\mathcal{E}_{\sigma_i}/F_i^j(\mathcal{E})\right)\right)^{-1}\otimes\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right).\end{aligned}$$

Now we can proof the proposition

$$\begin{aligned}\lambda_{par}(\mathcal{E}_*) &:= \lambda(\mathcal{E})^k\otimes\bigotimes_{i=1}^N\bigotimes_{j=1}^{\ell_i}\left\{\det\left(F_i^j(\mathcal{E})/F_i^{j+1}(\mathcal{E})\right)\right\}^{-a_j(i)}\otimes\det(\mathcal{E}_\sigma)^{\frac{k}{r}\chi_{par}}\\ &= \lambda(\mathcal{E})^k\otimes\bigotimes_{i=1}^N\bigotimes_{j=1}^{\ell_i}\left\{\det\left(\mathcal{E}_{\sigma_i}/F_i^j(\mathcal{E})\right)^{-1}\otimes\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)\right\}^{-a_j(i)}\otimes\det(\mathcal{E}_\sigma)^{\frac{k}{r}\chi_{par}}\end{aligned}$$

By rearranging the terms, we get

$$\lambda_{par}(\mathcal{E}_*) = \lambda(\mathcal{E})^k\otimes\bigotimes_{i=1}^N\left\{\det(\mathcal{E}_{\sigma_i})^{a_{\ell_i}(i)}\otimes\bigotimes_{j=1}^{\ell_i-1}\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)^{p_j(i)}\right\}\otimes\det(\mathcal{E}_\sigma)^{\frac{k}{r}\chi_{par}}$$

As $\det(\mathcal{E}_\sigma)$ is independent of the section σ, we get

$$\lambda_{par}(\mathcal{E}_*) = \lambda(\mathcal{E})^k\otimes\bigotimes_{i=1}^N\bigotimes_{j=1}^{\ell_i-1}\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)^{p_j(i)}\otimes\det(\mathcal{E}_\sigma)^{\left(\frac{k}{r}\chi_{par}-\sum_{i=1}^N a_{\ell_i}(i)\right)}$$

Now we observe the following equality:

$$\sum_{i=1}^N\sum_{j=1}^{\ell_i}a_j(i)m_j(i) = -\sum_{i=1}^N\sum_{j=1}^{\ell_i-1}d_j(i)r_j(i)+r\sum_{i=1}^N a_{\ell_i}(i) \tag{4.6.2}$$

So the exponent

$$\begin{aligned}\frac{k}{r}\chi_{par}-\sum_{i=1}^N a_{\ell_i}(i) &= \frac{k}{r}\left(\chi+\frac{1}{k}\sum_{i=1}^N\sum_{j=1}^{\ell_i}a_j(i)m_j(i)\right)-\sum_{i=1}^N a_{\ell_i}(i)\\ &= \frac{k}{r}\chi+\frac{1}{r}\left(\sum_{i=1}^N\sum_{j=1}^{\ell_i}a_j(i)m_j(i)-r\sum_{i=1}^N a_{\ell_i}(i)\right)\end{aligned}$$

By (4.6.2) we get

$$\frac{k}{r}\chi_{par}-\sum_{i=1}^N a_{\ell_i}(i) = \frac{k}{r}\chi-\frac{1}{r}\sum_{i=1}^N\sum_{j=1}^{\ell_i-1}d_j(i)r_j(i).$$

which is equivalent to the equality:

$$k\chi_{par} - r\sum_{i=1}^{N} a_{\ell_i}(i) = re.$$

This concludes the proof. □

We give another description of the parabolic determinant line bundle.

Proposition 4.6.3 (Parabolic determinant bundle and Hecke modifications) *Let $\mathcal{E}_*$ be a family of parabolic rank-r vector bundles and determinant $\delta \in \mathrm{Pic}(\mathcal{C}/S)$ of parabolic type α_* over a smooth family of curves $\pi_s : \mathcal{C} \longrightarrow S$ parameterized by a S-variety $\mathcal{T}$. Then*

$$\lambda_{par}(\mathcal{E}_*)^r = \Theta^a \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \Theta_j(i)^{q_j(i)} \tag{4.6.3}$$

where, for all $i \in I$ and $j \in \{1, 2, ..., \ell_i - 1\}$

- *Θ is the pull-back of the ample generator of $\mathrm{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r, \delta)/S)$ by the classifying map $\phi_{\mathcal{T}}$ and $n = \gcd(r, d)$.*
- *$\Theta_{i,j}$ is the pull-back of the ample generators of $\mathrm{Pic}(\mathcal{SU}_{\mathcal{C}/S}(r, \delta_j(i))/S)$ by the classifying maps $\phi_{i,j}^{\mathcal{T}}$ and $n_j(i) = \gcd(r, d_j(i))$, where $d_j(i) = \deg(\delta_j(i))$.*
- *$p_j(i) = a_{j+1}(i) - a_j(i)$ and $q_j(i) = n_j(i)p_j(i)$.*
- *$a = n\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i - 1} p_j(i)\right)$.*

Proof. By the Proposition 4.6.1, we prove the equality

$$\Theta_{par}(\mathcal{E}_*)^r = \Theta^a \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \Theta_{i,j}^{n_j(i)p_j(i)}.$$

we have by definition

$$\Theta_{par}(\mathcal{E}_*) = \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \left\{\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)^{p_j(i)}\right\} \otimes \det(\mathcal{E}_\sigma)^e$$

Take the Hecke exact sequences

$$0 \longrightarrow \mathcal{H}_i^j(\mathcal{E}) \hookrightarrow \mathcal{E} \twoheadrightarrow Q_i^j(\mathcal{E}) := \mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E}) \longrightarrow 0,$$

By Lemma 1.4.5, we get

$$\lambda(\mathcal{E}) = \lambda(\mathcal{H}_i^j(\mathcal{E})) \otimes \left(\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)\right)^{-1}.$$

We rearrange the terms and by the above equality and take the r-th power

$$\begin{aligned}
\Theta_{par}(\mathcal{E}_*)^r &= \lambda(\mathcal{E})^{rk} \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \left\{\lambda\left(\mathcal{H}_i^j(\mathcal{E})\right) \otimes \lambda(\mathcal{E})^{-1}\right\}^{rp_j(i)} \otimes \det(\mathcal{E}_\sigma)^{re} \\
&= \left\{\lambda(\mathcal{E})^k \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \lambda(\mathcal{E})^{-p_j(i)}\right\}^r \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \lambda\left(\mathcal{H}_i^j(\mathcal{E})\right)^{rp_j(i)} \otimes \det(\mathcal{E}_\sigma)^{re} \\
&= \left\{\lambda(\mathcal{E})^{\frac{r}{n}} \otimes \det(\mathcal{E}_\sigma)^{\aleph}\right\}^a \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \left\{\lambda(\mathcal{H}_i^j(\mathcal{E}))^{\frac{r}{n_j(i)}} \otimes \det(\mathcal{E}_\sigma)^{\aleph_j(i)}\right\}^{n_j(i)p_j(i)} \otimes \det(\mathcal{E}_\sigma)^q
\end{aligned}$$

where

$$(*)\begin{cases}
a = n\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\right) \quad \text{and} \quad re = k\chi - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)r_j(i). \\
\chi = d + r(1-g) \quad \text{and} \quad n\,\aleph = \chi \quad \text{for} \quad n = \gcd(r,d) \\
\chi_j(i) = d_j(i) + r(1-g) \quad \text{and} \quad n_j(i)\,\aleph_j(i) = \chi_j(i) \quad \text{for} \quad n_j(i) = \gcd(r, d_j(i)) \\
d_j(i) = d - r_j(i).
\end{cases}$$

Thus

$$q = re - \aleph a - \left(\sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} \aleph_j(i)p_j(i)n_j(i)\right),$$

By $(*)$, we get

$$\begin{aligned}
q &= \left(k\chi - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)r_j(i)\right) - \chi\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\right) - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\chi_j(i) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\left(-r_j(i) + \chi - \chi_j(i)\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\left(-r_j(i) + \underbrace{d - d_j(i)}_{r_j(i)}\right) \\
q &= 0.
\end{aligned}$$

Hence for $q_j(i) = n_j(i)p_j(i)$ one has: $\Theta_{par}(\mathcal{E}_*)^r = \Theta^a \otimes \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{q_j(i)}$. □

Remark 4.6.4 *We apply this proposition for* $\mathcal{T} = \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ *and* $\mathcal{E}_*$ *a virtual universal parabolic bundle, we get*

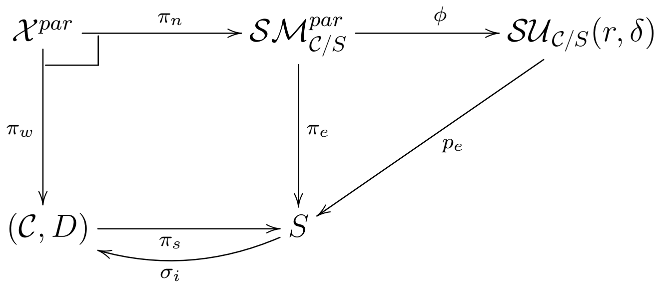
$$\Theta^r_{par} = \Theta_{par}(\mathcal{E}_*)^r = \Theta^{n\left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i - 1} p_j(i)\right)} \otimes \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{n_j(i)p_j(i)}$$

4.6.2 Canonical bundle

In this section we calculate the canonical bundle of the relative moduli space of semi-stable parabolic bundles for a fixed parabolic type $\mathcal{SM}^{par}_{\mathcal{C}} := \mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)/S$ over a smooth family of marked projective curves parameterized by a scheme S.

Canonical bundle in the Grassmannian case

We suppose that the divisor is of degree one and the flag type is of length one. Let $\mathcal{E}_*$ be a virtual universal parabolic vector bundle



where $D = \sigma(S)$. In this case the map ϕ is a Grassmannian bundle over the stable locus of $\mathcal{SU}_{\mathcal{C}/S}$ (the relative moduli space of semi-stable rank-r vector bundles of determinant δ) and we set $\mathcal{D} := \pi_w^*(D) = D \times_S \mathcal{SM}^{par}_{\mathcal{C}}$. Then we have the Hecke exact sequence

$$0 \longrightarrow \mathcal{H}(\mathcal{E}) \hookrightarrow \mathcal{E} \longrightarrow Q(\mathcal{E}) = \mathcal{E}|_{\mathcal{D}}/F(\mathcal{E}) \longrightarrow 0 \tag{4.6.4}$$

and the natural exact sequence supported over $\mathcal{D}$

$$0 \longrightarrow F(\mathcal{E}) \hookrightarrow \mathcal{E}|_{\mathcal{D}} \longrightarrow Q(\mathcal{E}) \longrightarrow 0 \tag{4.6.5}$$

The relative tangent bundle of the fibration ϕ is given as follow

$$T_\phi = Hom\left(F(\mathcal{E}), Q(\mathcal{E})\right) = F(\mathcal{E})^{-1} \otimes Q(\mathcal{E})$$

So the relative canonical bundle is

$$K_\phi = \det(T_\phi^{-1}) = \det\left(F(\mathcal{E})\right) \otimes Q(\mathcal{E})^{-1}\right)$$

We put $r' := \mathrm{rank}(Q(\mathcal{E})) = r - \mathrm{rank}(F(\mathcal{E}))$, we get

$$K_\phi = \det(T_\phi)^{-1} = \det(F(\mathcal{E}))^{r'} \otimes \det(Q(\mathcal{E}))^{-(r-r')}$$

The short exact sequence (4.6.5)

$$\det(\mathcal{E}_D) = \det(F(\mathcal{E})) \otimes \det(Q(\mathcal{E}))$$

which is equivalent to

$$\det F(\mathcal{E}) = \det(\mathcal{E}_D) \otimes \det(Q(\mathcal{E}))^{-1}$$

We replace in the previous equation

$$K_\phi = \det(\mathcal{E}_D)^{r'} \otimes \det(Q(\mathcal{E}))^{-r}$$

so Lemma 1.4.5 applied to the Hecke modification sequence (4.6.4) gives the equality

$$\lambda(\mathcal{E}) = \lambda(\mathcal{H}(\mathcal{E})) \otimes \det(Q(\mathcal{E}))^{-1}$$

which implies that

$$K_\phi = \det(\mathcal{E}_D)^{r'} \otimes \lambda(\mathcal{H}(\mathcal{E}))^{-r} \otimes \lambda(\mathcal{E})^{r}$$

$$K_{\phi'} = \left[\lambda(\mathcal{H}(\mathcal{E}))^{\frac{r}{n'}} \otimes \det(\mathcal{E}_y)^{\frac{\chi'}{n'}}\right]^{-n'} \otimes \left[\lambda(\mathcal{E})^{\frac{r}{n}} \times \det(\mathcal{E}_D)^{\frac{\chi}{n}}\right]^{n} \otimes \det(\mathcal{E}_D)^{r'-\chi+\chi'}$$

where: $n = \gcd(r, \deg(\mathcal{E}))$, $n' = gcd(r, \deg(G_D(\mathcal{E}))$, $\chi = \chi(\mathcal{E})$ and $\chi' = \chi(G_D(\mathcal{E}))$.

$$\chi' - \chi = \deg(G_D(\mathcal{E})) - \deg(\mathcal{E}) = -r' \Longrightarrow r' - \chi + \chi' = 0$$

If we denote Θ_D the pull-back of the ample generator of $\mathrm{Pic}(\mathcal{SU}_\mathcal{C}(r, \delta')/S)$, we get:

$$\omega_{\phi'} = \Theta^n \otimes \Theta_D^{-n'}$$

$$K_{\mathcal{SM}^{par}_{\mathcal{C}/S}} = K_{\mathcal{SU}_\mathcal{C}(r,\delta')/S} \otimes K_{\phi'} = \Theta^{-2n} \otimes \left(\Theta^n \otimes \Theta_D^{-n'}\right).$$

Hence

$$K_{\mathcal{SM}^{par}_\mathcal{C}/S} = \Theta^{-n} \otimes \Theta_D^{-n'}. \tag{4.6.6}$$

General case: Now we can calculate the relative canonical bundle $K_{\mathcal{SM}^{par}_\mathcal{C}/S}$ of the moduli space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ of parabolic bundles.

Proposition 4.6.5 *Let* $b = -n\left(2 + \deg(D) - \sum_{i=1}^{N} \ell_i\right)$. *Then the canonical bundle is given by the formula:*

$$K_{\mathcal{SM}^{par}_\mathcal{C}/S} = \Theta^b \otimes \left(\bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i - 1} \Theta_j(i)^{-n_j(i)}\right).$$

Proof. Let $\phi : \mathcal{SM}^{par}_\mathcal{C} \to \mathcal{SU}_\mathcal{C}(r, \delta)$ be the forgetful map and denote its relative canonical bundle by K_ϕ, then we have $K_{\mathcal{SM}^{par}_\mathcal{C}/S} = K_{\mathcal{SU}_\mathcal{C}(r,\delta)/S} \otimes K_\phi$ and by Drezet-Narasimhan 1.4.3 we get $K_{\mathcal{SM}^{par}_\mathcal{C}/S} = \Theta^{-2n} \otimes K_\phi$, where Θ is the pull-back of the relative ample generator of $\mathrm{Pic}(\mathcal{SU}_\mathcal{C}(r, \delta)/S)$ by ϕ and $n = \gcd(r, \deg(\delta))$. Now, as the map ϕ is generically a product of a flag varieties we can decompose the relative canonical bundle $K_\phi = \bigotimes_{i=1}^{N} K_{\phi(i)}$, where for all $i \in I$ the bundle $K_{\phi(i)}$ is the canonical bundle of a flag variety. Hence as the flag variety is embedded canonically in a product of Grassmanians and that its canonical bundle is given by the product of the canonical bundle over the Grassmanians, then by the equality (4.6.6), we have

$$K_{\phi(i)} = \bigotimes_{j=1}^{\ell_i - 1} \left(\Theta^n \otimes \Theta_j(i)^{-n_j(i)}\right).$$

We replace and rearrange the terms, to get

$$\begin{aligned} K_{\mathcal{SM}^{par}_{\mathcal{C}}/S} &= K_{\mathcal{SU}_{\mathcal{C}}(r,\delta)/S} \otimes K_\phi = \Theta^{-2n} \otimes \bigotimes_{i=1}^{N} K_{\phi(i)} \\ &= \Theta^{-2n} \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \left(\Theta^n \otimes \Theta_j(i)^{-n_j(i)}\right) \\ &= \Theta^{-n\left(2+\deg(D)-\sum_{i=1}^{N}\ell_i\right)} \otimes \left(\bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{-n_j(i)}\right) \end{aligned}$$

This proves the formula. □

Example: Rank 2 Parabolic bundles

Let $\mathcal{E}_* = (\mathcal{E}, F_i(\mathcal{E})_{i\in I})$ be a relative family of rank-2 parabolic vector bundles and degree-d of fixed parabolic type

$$(0 \leqslant a_1(i) < a_2(i) < k)_{i\in I}$$

over $(\mathcal{C}/S, D)$ parameterized by an S-variety $\mathcal{T}/S$, we set

- $\chi = d + 2(1-g)$, $n = \gcd(2,d)$ and $n' = \gcd(2, d-1)$.
- $p(i) = a_2(i) - a_1(i)$ and $2e = k\chi - \sum_{i=1}^{N} p(i)$.
- $a = n\left(k - \sum_{i=1}^{N} p(i)\right)$.

Then the parabolic determinant bundle

$$\lambda_{par}(\mathcal{E}) = \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} (\mathcal{E}_i/F_i(\mathcal{E}))^{p_i} \otimes \det(\mathcal{E}_\sigma)^e.$$

We get the following description in rank 2 case.

Proposition 4.6.6 *With the above hypothesis, we get*

1. *The parabolic determinant bundle is given by:* $\Theta_{par}^2 = \Theta^a \otimes \bigotimes_{i=1}^{N} \Theta(i)^{n' p(i)}$.
2. *The canonical bundle is given by:* $K_{\mathcal{SM}_{\mathcal{C}}^{par}/S} = \Theta^{-n(2-N)} \otimes \bigotimes_{i=1}^{N} \Theta(i)^{-n'}$.

4.7 Existence and flatness of the connection

With the same hypothesis.

Theorem 4.7.1 *Take the parabolic symbol map* ρ_{par}*, then the parabolic determinant line bundle* Θ_{par} *satisfies the van Geemen-de Jong equation. i.e.*

$$\mu_{\Theta_{par}} \circ \rho_{par} = -(k+r)\Phi_{par}$$

Proof. By Proposition 3.3.1 we have that

$$\mu_{\Theta_{par}} = \cup[\Theta_{par}] - \cup\left(\frac{1}{2}[K_{\mathcal{SM}_{\mathcal{C}}^{par}/S}]\right)$$

Thus the Theorem is equivalent to the following points

1. We prove the equality: $\cup[\Theta_{par}] \circ \rho_{par} = -k\ \Phi^{par}$, so called the metaplectic case (or correction)

   By Proposition 4.6.3, Theorem 4.5.7 and linearity with respect to the the tensor product, we get

$$\begin{aligned}
\cup[\Theta_{par}^r] \circ \rho_{par} &= \cup\left[\Theta^a \otimes \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{n_j(i)p_j(i)}\right] \circ \rho_{par} \\
&= a\left(\cup\left[\Theta\right] \circ \rho_{par}\right) + \sum_{i=1}^{N} \sum_{j=1}^{\ell_i-1} n_j(i)p_j(i)\left(\cup\left[\Theta_j(i)\right] \circ \rho_{par}\right) \\
&= a\left(-\frac{r}{n}\Phi^{par}\right) + \sum_{i=1}^{N} \sum_{j=1}^{\ell_i-1} n_j(i)p_j(i)\left(-\frac{r}{n_j(i)}\Phi_{par}\right) \\
&= -\left(\frac{r}{n}a + \sum_{i=1}^{N} \sum_{j=1}^{\ell_i-1} n_j(i)p_j(i)\left(\frac{r}{n_j(i)}\right)\right)\Phi_{par}
\end{aligned}$$

   Thus

$$\cup[\Theta_{par}] \circ \rho_{par} = -\left(\frac{a}{n} + \sum_{i=1}^{N} \sum_{j=1}^{\ell_i-1} p_j(i)\right)\Phi_{par}$$

and by the following identity

$$\frac{a}{n} + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i) = \left(k - \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i)\right) + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} p_j(i) = k$$

we get the equality

$$\cup\,[\Theta_{par}] \circ \rho_{par} = -k\ \Phi_{par} \tag{4.7.1}$$

2. We prove the equality: $\cup[K_{\mathcal{SM}^{par}_{\mathcal{C}}/S}] \circ \rho_{par} = 2r\ \Phi_{par}$

   By Proposition 4.6.5, Theorem 4.5.7 and linearity with respect to the the tensor product, we have

$$\begin{aligned}
\cup[K_{\mathcal{SM}^{par}_{\mathcal{C}}/S}] \circ \rho_{par} &= \cup\left[\Theta^{-b} \otimes \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{-n_j(i)}\right] \circ \rho_{par} \\
&= -b\,(\cup[\Theta]\circ\rho_{par}) + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} -n_j(i)\,(\cup[\Theta_j(i)]\circ\rho_{par}) \\
&= -n\left(2+\deg(D) - \sum_{i=1}^{N}\ell_i\right)\left(-\frac{r}{n}\Phi^{par}\right) + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} -n_j(i)\left(-\frac{r}{n_j(i)}\Phi^{par}\right) \\
&= r\left(2+\deg(D) - \sum_{i=1}^{N}\ell_i + \sum_{i=1}^{N}\sum_{j=1}^{\ell_i-1} 1\right)\ \Phi_{par}
\end{aligned}$$

   Thus

$$\cup[K_{\mathcal{SM}^{par}_{\mathcal{C}}/S}] \circ \rho_{par} = 2r\ \Phi_{par}$$

Adding the two equations we get

$$\left(\cup[\Theta_{par}] - \cup\frac{1}{2}[K_{\mathcal{SM}^{par}_{\mathcal{C}}/S}]\right) \circ \rho_{par} = -k\ \Phi_{par} - r\ \Phi_{par} = -(k+r)\Phi_{par}$$

thus

$$\mu_{\Theta_{par}} \circ \rho_{par} = -(k+r)\cdot\Phi_{par}. \tag{4.7.2}$$

□

We observe that the composition $\mu_{\Theta_{par}} \circ \rho_{par}$ does not depend on the parabolic weights but depends on the level-k, in some sense what contributes in the decomposition (4.6.3) is the term Θ^k, we rearrange the terms as follow

$$\Theta^r_{par} = \Theta^a \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1} \Theta_j(i)^{n_j(i)p_j(i)} = \Theta^{nk} \bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1} \left(\Theta^{-n}\otimes\Theta_j(i)^{n_j(i)}\right)^{p_j(i)}.$$

By the Definition 4.6.1 and Propositions 4.6.2, 4.6.3 the identification we get

$$\bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1}\left\{\det\left(\mathcal{E}_{\sigma_i}/F_i^{j+1}(\mathcal{E})\right)^{rp_j(i)}\right\}\otimes\det(\mathcal{E}_\sigma)^{(re-k\chi)}=\bigotimes_{i=1}^{N}\bigotimes_{j=1}^{\ell_i-1}\left(\Theta^{-n}\otimes\Theta_j(i)^{n_j(i)}\right)^{p_j(i)}.$$

which we call the flag part of the determinant line bundle and we denote by $\mathcal{F}(\alpha_*)$. By Corollary 4.5.7 for all $i\in I$ and $j\in\{1,2,...,\ell_i\}$ we have

$$\left(\cup\left[\Theta^{-n}\right]+\cup\left[\Theta_j(i)^{n_j(i)}\right]\right)\circ\rho_{par}=0$$

Thus we get

$$\cup\left[\mathcal{F}(\alpha_*)\right]\circ\rho_{par}=0. \tag{4.7.3}$$

By Proposition 4.4.5 one has

$$\cup\left[\mathcal{F}(\alpha_*)\right]=0. \tag{4.7.4}$$

Remark 4.7.2

1. *In general case (see. [Sin21]) : If X is a Hitchin variety and L line bundle over X, then we have a map*

$$\cup[L]:\mathrm{H}^0(X,\mathrm{Sym}^q\,T_X)\longrightarrow\mathrm{H}^1(X,\mathrm{Sym}^{q-1}T_X)$$

which can be seen as the first connecting map in cohomology of the short exact sequence

$$0\longrightarrow\mathrm{Sym}^{q-1}\left(\mathcal{D}_X^{(1)}(L)\right)\longrightarrow\mathrm{Sym}^q(\mathcal{D}_X^{(1)}(L))\longrightarrow\mathrm{Sym}^q(T_X)\longrightarrow0$$

which is the q-th symmetric power of the the Atiyah sequence (3.1.1), and we have the following theorem

Theorem 4.7.3 ([Sin21], Theorem 2.2)

If L is an ample line bundle then the map above is an isomorphism.

2. *The varieties $\mathcal{SU}_{\mathcal{C}/S}(r,\delta)$ and $\mathcal{SM}^{par}_{\mathcal{C}/S}(r,\alpha_*,\delta)$ are Hitchin varieties in the sense of [Sin21], and by Theorem 1.6.3 the parabolic determinant line bundle Θ_{par} is ample thus the map*

$$\cup[\Theta_{par}]:\pi_{e_*}\mathrm{Sym}^2\left(T_{\mathcal{SM}^{par}_{\mathcal{C}}/S}\right)\longrightarrow R^1\pi_{e_*}\left(T_{\mathcal{SM}^{par}_{\mathcal{C}}/S}\right),$$

is an isomorphism.

By equalities (4.7.2) and (4.7.4), one has

$$\mu_{\Theta_{par}} = \frac{n\,(k+r)}{r} \cdot \cup [\Theta] = \frac{k+r}{k} \cdot \cup [\Theta_{par}]$$

and for all positive integer ν, one has

$$\mu_{\Theta_{par}^{\nu}} = \frac{n\,(\nu k+r)}{r} \cdot \cup [\Theta] = \left(\frac{\nu k+r}{k}\right) \cup [\Theta_{par}] \,.$$

Thus by the previous remark, on has

Proposition 4.7.4 *For ν a positive integer the map $\mu_{\Theta_{par}^{\nu}}$ is an isomorphism.*

We get the van Geemen and de Jong equation Theorem 4.7.1 for any positive power of the theta line bundle

$$\mu_{\Theta_{par}^{\nu}} \circ \rho_{par} = -(\nu k + r)\Phi_{par}$$

Theorem 4.7.5 *Let $\nu \in \mathbb{N}^*$ be a positive integer. Suppose $\pi_s : (\mathcal{C}, D) \longrightarrow S$ a smooth family of complex projective marked curves of genus $g \geqslant 2$ and D a reduced divisor of relative degree N, take $\alpha_* = (k, \vec{a}, \vec{m})$ a fixed rank-r parabolic type with respect to the divisor D. We denote by $\pi_e : \mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta) \longrightarrow S$ the relative moduli space of parabolic rank-r vector bundles over $(\mathcal{C}, D)/S$ with determinant $\delta \in \mathrm{Pic}^d(\mathcal{C}/S)$, equipped with the parabolic determinant bundle Θ_{par}. Then there exists a unique projective flat connection on the vector bundle $\pi_{e*}(\Theta_{par}^{\nu})$ of non-abelian parabolic theta functions, induced by a heat operator with symbol*

$$\rho_{par}^{Hit}(\nu) := \frac{1}{(\nu k + r)} \left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right) .$$

Proof. Let prove the theorem for $\nu = 1$, we denote $\rho_{par}^{Hit} := \rho_{par}^{Hit}(1)$.

- First we prove existence of the connection: We apply van Geemen-de Jong Theorem 3.3.2 for $L = \Theta_{par}$ over $\mathcal{SM}_{\mathcal{C}}^{par} = \mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$. Thus by Theorem 4.7.1 and Proposition 4.3.3, we get the first condition of Theorem 3.3.2

  $$\begin{aligned} \mu_{\Theta_{par}} \circ \left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right) &= -(k+r)\Phi_{par} \circ \kappa_{\mathcal{C}/S} \\ &= -(k+r)\ \kappa_{\mathcal{SM}_{\mathcal{C}}^{par}/S}. \end{aligned}$$

  The second condition follows by Theorem 4.7.3 for $q = 1$.i.e.

  $$\cup [\Theta_{par}] : \pi_{e*} T_{\mathcal{SM}_{\mathcal{C}}^{par}/S} \longrightarrow R^1 \pi_{e_*} \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}/S}$$

  is an isomorphism, as the relative Picard group $\mathrm{Pic}(\mathcal{SM}_{\mathcal{C}}^{par}/S)$ is discrete then the infinitesimal deformations of any line bundle L over $\mathcal{SM}_{\mathcal{C}}^{par}$ are trivial and parameterized by the sheaf $R^1 \pi_{e_*} \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}}$ over S, thus

  $$R^1 \pi_{e_*} \mathcal{O}_{\mathcal{SM}_{\mathcal{C}}^{par}} \cong 0,$$

as consequence, we get that there are no global vector fields

$$\pi_{e*}(T_{\mathcal{SM}_{\mathcal{C}}^{par}/S}) \cong 0,$$

The third condition follows from the algebraic Hartogs's Theorem and the fact that the space $\mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$ is normal variety and proper over S. Hence the smooth locus is a big open subset of $\mathcal{SM}_{\mathcal{C}/S}^{par}(r, \alpha_*, \delta)$.

- Flatness of the connection: We apply the flatness criterion Theorem 3.3.3 to the parabolic symbol map ρ_{par}. The first condition holds since by definition of the parabolic Hitchin map corresponds to homogeneous functions on $T^{\vee}_{\mathcal{SM}_{\mathcal{C}}^{par}/S}$ of degree two under the action of $\mathbb{G}_m$ in the parabolic Hitchin system, hence Poisson-commute. The second point is given in Proposition 4.7.4 and the third point is given in the first part of the proof.

□

For $D = \varnothing$ and $\alpha_* = k \in \mathbb{N}^*$ the trivial parabolic type, we have the identification $\mathcal{SM}_{\mathcal{C}/S}^{par}(r, 0_*, \delta)$ with $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$ the moduli space of semi-stable rank-r vector bundles with determinant δ, hence $\Theta_{par}^{r/n} = \mathcal{L}^k$ for $n := \gcd(r, \deg(\delta))$ and $\rho_{par} = \rho^{Hit}$.

We obtain the following special case for non-parabolic vector bundles.

Theorem 4.7.6 *Let k a positive integer. Suppose a smooth family $p_s : \mathcal{C} \longrightarrow S$ of complex projective curves of genus $g \geqslant 2$ (and $g \geqslant 3$ if $r = 2$ and $\deg(\delta)$ even), set $n := \gcd(r, \deg(\delta))$. Let $\mathcal{L}$ be the ample generator of the Picard group of $\mathcal{SU}_{\mathcal{C}/S}(r, \delta)$. Then there exists a unique projective flat connection on the vector bundle $p_{e*}(\mathcal{L}^k)$ of non-abelian theta functions, induced by a heat operator with symbol*

$$\rho(k) := \frac{n}{r(k+n)} \left(\rho^{Hit} \circ \kappa_{\mathcal{C}/S}\right).$$

In fact for $k \in \mathbb{N}^*$, we have $\Theta_{par}^{r/n} = \mathcal{L}^k$ hence $\rho(k) = \rho_{par}^{Hit}(\frac{r}{n})$.

Remark 4.7.7 *The van Geemen-de Jong criterion cannot be applied in following cases:*

1. *Genus zero: not interesting case as we have*

$$\mathcal{SU}_{\mathbb{P}^1}(r, \delta) = \begin{cases} \{pt\} & \text{if} \quad \deg(\delta) \quad \text{divides} \quad r. \\ \varnothing & \text{otherwise.} \end{cases}$$

2. *Elliptic curves: we have the following description of the moduli space [Tu93]*

$$\mathcal{SU}_C(r, \delta) \cong \mathbb{P}^{m-1},$$

where $m = \gcd(r, \deg(\delta))$, for any line bundle $\delta \in \mathrm{Pic}(C)$. Note that the case $m = 1$ is not interesting.

Hence the second condition in Theorem 3.3.2 does not hold, as on the one hand the projective space admits global holomorphic vector fields $\mathrm{H}^0(\mathbb{P}^m, T_{\mathbb{P}^m}) \neq 0$ *by Euler exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^m} \longrightarrow \mathcal{O}_{\mathbb{P}^m}(1)^{m+1} \longrightarrow T_{\mathbb{P}^m} \longrightarrow 0$$

on the other hand $\mathrm{H}^1(\mathbb{P}^m, \mathcal{O}) = 0$ *as the Picard group is isomorphic to* $\mathbb{Z}$ *hence no infinitesimal deformations.*

Some comments We apply Theorem 4.7.6 for $\delta = \mathcal{O}_{\mathcal{C}}$ thus $n = r$ and $\Theta_{par} = \mathcal{L}^k$, we get

$$\rho(k) := \frac{1}{(k+r)} \left(\rho_{Hit} \circ \kappa_{\mathcal{C}/S}\right)$$

Hence we recover Theorems 4.8.1 and 4.8.2 in [BBMP23], which was generalized in [BMW21a] and [BMW21b] to the space $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*)$ the space of rank-r parabolic bundle with trivial determinant and parabolic type α_*. The symbol map is given by

$$\rho^{Hit}_{par,\Gamma}(\nu) := |\Gamma| \; \mu^{-1}_{\Theta^\nu} \circ \left(\cup[\Theta] \circ \rho_{par} \circ \kappa_{\mathcal{C}/S}\right).$$

for a positive integer ν. By 4.7.4 and (4.7.4), we get

$$|\Gamma|(\mu^{-1}_{\Theta^\nu} \circ \cup[\Theta]) = \frac{|\Gamma|}{(\nu k + r)} \mathrm{Id},$$

hence

$$\rho^{Hit}_{par,\Gamma}(\nu) := \frac{|\Gamma|}{(\nu k + r)} \left(\rho_{par} \circ \kappa_{\mathcal{C}/S}\right) = |\Gamma| \rho^{Hit}_{par}(\nu).$$

The factor $|\Gamma|$ is because they work over $\mathcal{SU}^{\Gamma}_{\widehat{\mathcal{C}}}(r)$ the moduli space of Γ-linearised bundles for family of Galois coverings $h : \widehat{\mathcal{C}} \longrightarrow (\mathcal{C}, D)$ parameterized by the variety S.

Remark 4.7.8 *If the system of weights* α_* *is not generic in the sense of Yokogawa then the moduli space* $\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)$ *is not smooth, and its Picard group is not maximal. In other words, note all line bundles on the Quot scheme descends to the moduli space. In fact we can choose the weights* α_* *in such a way that we have the following isomorphism:*

$$\mathrm{Pic}\left(\mathcal{SM}^{par}_{\mathcal{C}/S}(r, \alpha_*, \delta)/S\right) \simeq \mathbb{Z}\Theta_{par}$$

In this case, we must work over the stack of parabolic vector bundles, where the Picard group is maximal, and the Hecke maps $\mathcal{H}^j_i$ *and the forgetful map are maps between stacks (no stability conditions). The decompositions of the parabolic determinant line bundle 4.6.3 and the canonical line bundle 4.6.5 still holds.*

Example 4.7.9 (Non-generic weights. See [Pau98] for the details) *Let's consider the rank two case with* D *being a parabolic divisor of degree* $N = 2m \geqslant 4$*. For all* $i \in I$*, we choose the following system of weights:*

$$a_1(i) = 0, \; a_2(i) = 1 \textit{ and } k = 2.$$

In this case the Picard group of the moduli space $\mathcal{SM}^{par}_{\mathcal{C}/S}(2, \alpha_*, \mathcal{O}_{\mathcal{C}})$ *is generated by the parabolic determinant line bundle* Θ_{par}*.*

Chapter A

Deformation functors

In this appendix we recall definitions of deformation theory over Artin $\mathbb{C}$-algebras and their properties. We give some examples of geometric deformations. We follow [Mar09].

A.1 Formal deformation theory

Definition A.1.1 (Deformation over Artin rings) *A functor of Artin rings is a covariant functor* $\mathcal{F} : \mathrm{Art}_{\mathbb{C}} \longrightarrow$ *Set from the category of local Artin* $\mathbb{C}$*-algebras with residue field* $\mathbb{C}$ *to the category of sets, such that* $\mathcal{F}(\mathbb{C}) = \{$*fixed one point set*$\}$.

Definition A.1.2 (Tangent space) *Let* $\mathcal{F}$ *a functor of Artin ring. The tangent space to* $\mathcal{F}$ *is the set* $\mathcal{T}_{\mathcal{F}} := \mathcal{F}(\mathbb{C}[\varepsilon])$*, where* $\mathbb{C}[\varepsilon] := \mathbb{C}[x]/(x^2)$*, the dual number* $\mathbb{C}$*-algebra.*

We want to have some control over such functors, so we define a special class of functors. Let $\mathcal{F}$ a functor of Artin ring. Let $B \longrightarrow A$ and $C \longrightarrow A$ be morphisms of Artin algebras and let

$$\zeta : \mathcal{F}(B \times_A C) \longrightarrow \mathcal{F}(B) \times_{\mathcal{F}(A)} \mathcal{F}(C)$$

be the induced morphism. The Schlessinger conditions are the following:

A) If $C \longrightarrow A$ is surjective, then ζ is surjective,

B) If $C = \mathbb{C}[\varepsilon]$ and $A = \mathbb{C}$, then ζ is bijective,

C) $\dim_{\mathbb{C}} \mathcal{F}(\mathbb{C}[\varepsilon])$ is finite,

D) If $B = C \longrightarrow A$ is surjective, then ζ is bijective.

Definition A.1.3 *Let* $\mathcal{F}$ *be a functor of Artin rings.* $\mathcal{F}$ *is a functor with good theory of deformation if conditions (A) and (B) holds.* $\mathcal{F}$ *is homogeneous, if* ζ *is bijective, whenever* $C \longrightarrow A$ *is surjective.*

Proposition A.1.4 *Let $\mathcal{F}$ be a functor of Artin rings. The set $\mathcal{T}_{\mathcal{F}}$ has the structure of a $\mathbb{C}$-vector space. Let $\upsilon : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of functors of Artin rings. Then the induced map $\upsilon : \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{T}_{\mathcal{G}}$ is a $\mathbb{C}$-linear map.*

A.2 Examples of deformation functors

A.2.1 Deformation of schemes

Definition A.2.1 *Let X be an algebraic $\mathbb{C}$-scheme. An infinitesimal deformation of X over $A \in \mathrm{Art}_{\mathbb{C}}$ is a Cartesian diagram of morphisms of schemes*

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X}_A \\ \downarrow & & \downarrow \pi \\ Spec(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

where π is flat.

An isomorphism of infinitesimal deformations $\mathcal{X}_A$ and $\mathcal{X}'_A$ of X over A, is an isomorphism $\nu : \mathcal{X}_A \longrightarrow \mathcal{X}'_A$ that makes the following diagram commutative

$$\begin{array}{ccccc} & \swarrow & X & \searrow & \\ \mathcal{X}_A & & \xrightarrow{\nu} & & \mathcal{X}'_A \\ & {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\pi'} & \\ & & \mathrm{Spec}(A) & & \end{array}$$

Definition A.2.2 (Locally trivial deformations) *Let X be an algebraic $\mathbb{C}$-scheme. An infinitesimal deformation $\mathcal{X}_A$ of X over A is locally trivial, if every $x \in X$ has a neighbourhood $U_x \subset X$, such that*

$$\begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}_A|_{U_x} \\ \downarrow & & \downarrow \pi \\ Spec(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

is a trivial deformation of U_x.

We define isomorphism of locally trivial deformation as isomorphism of deformations.

Proposition A.2.3 *Let X an affine scheme, then we have*

1. *Every infinitesimal deformation of X is affine.*

2. If X is smooth, then it is rigid, i.e. all its infinitesimal deformations are trivial.

Definition A.2.4 (Functor of deformation of schemes) *Let X be an algebraic scheme. We define the functor of infinitesimal deformations of X:*

$$\mathrm{Def}_X : \mathrm{Art}_{\mathbb{C}} \longrightarrow Set$$

which associates to every local Artinian $\mathbb{C}$-algebra A, the set of isomorphism classes of infinitesimal deformations of X over A.

The functor of locally trivial infinitesimal deformations of X:

$$\mathrm{Def}'_X : \mathrm{Art}_{\mathbb{C}} \longrightarrow Set$$

which associates to every local Artinian $\mathbb{C}$-algebra A, the set of isomorphism classes of locally trivial infinitesimal deformations of X over A.

Remark A.2.5 *By Proposition A.2.3 if X is a non singular scheme, all its infinitesimal deformations are locally trivial. Thus $\mathrm{Def}_X \cong \mathrm{Def}'_X$.*

Now we recall without proof, the most important results and properties of the functors Def_X and Def'_X.

Theorem A.2.6 *The functors Def_X and Def'_X are with a good deformation theory, and we have the following*

- $\mathcal{T}_{\mathrm{Def}'_X} = H^1(X, \mathcal{T}_X)$ *and* $\mathcal{T}_{\mathrm{Def}_X} = Ext^1(X, \mathcal{T}_X)$.
- *If X is smooth,* $\mathrm{Def}_X \cong \mathrm{Def}'_X$ *and we have* $\mathcal{T}_{\mathrm{Def}_X} \cong \mathcal{T}_{\mathrm{Def}'_X} \cong H^1(X, \mathcal{T}_X)$.
- *The obstruction space of the functor* Def_X *is the space* $H^2(X, \mathcal{T}_X)$.
- *If X is smooth, then* $\mathrm{Def}_X \cong \mathrm{Def}'_X$ *and the obstruction space is* $H^2(X, \mathcal{T}_X)$.

A.2.2 Deformation of sheaves

Definition A.2.7 *Let X be an algebraic scheme over $\mathbb{C}$. Let $\mathcal{E}$ a locally free sheaf over X. An infinitesimal deformation of $\mathcal{E}$ over $A \in \mathrm{Art}_{\mathbb{C}}$ is a locally free sheaf of $\mathcal{O}_X \otimes A$-module $\mathcal{E}_A$ over $X \times \mathrm{Spec}(A)$ with a morphism of sheaves $\pi_A : \mathcal{E}_A \longrightarrow \mathcal{E}$, such that $\pi_A : \mathcal{E}_A \otimes \mathbb{C}\mathcal{E}$ is an isomorphism.*

Definition A.2.8 *Two infinitesimal deformation of $\mathcal{E}$ on X, $\mathcal{E}_A$ and $\mathcal{E}'_A$ are isomorphic, if there exist an isomorphism of sheaves $\varphi : \mathcal{E}_A \longrightarrow \mathcal{E}'_a$, that commutes with the maps $\pi_A : \mathcal{E}_A \longrightarrow \mathcal{E}$ and $\pi'_A : \mathcal{E}'_A \longrightarrow \mathcal{E}$, i.e. $\pi'_A \circ \varphi = \pi_A$.*

Definition A.2.9 *Let $\mathcal{E}$ a locally free over the scheme X. We define the Artin functor of infinitesimal deformations of $\mathcal{E}$: $\mathrm{Def}_{\mathcal{E}} : \mathrm{Art}_{\mathbb{C}} \longrightarrow Set$. Which associates to every local Artinian $\mathbb{C}$-algebra A, the set of isomorphism classes of infinitesimal deformation of $\mathcal{E}$ over A.*

Theorem A.2.10 *The functor* $\mathrm{Def}_{\mathcal{E}}$ *is a functor with good deformation theory.*

Theorem A.2.11 *The tangent space to the functor* $\mathrm{Def}_{\mathcal{E}}$ *is given by* $\mathcal{T}_{\mathrm{Def}_{\mathcal{E}}} \cong H^1(X, \mathrm{End}(\mathcal{E}))$ *and* $H^2(X, \mathrm{End}(\mathcal{E}))$ *is the obstruction space.*

A.2.3 Deformation of a pair (scheme, sheaf)

Definition A.2.12 *Let* X *a scheme over* $\mathbb{C}$ *and* $\mathcal{E}$ *a locally free sheaf. An infinitesimal deformation of the pair* $(X, \mathcal{E})$ *over* $A \in \mathrm{Art}_{\mathbb{C}}$ *is the following*

- *A deformation* $\mathcal{X}_A$ *of the scheme* X *over* A*, i.e.*

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X}_A \\ \downarrow & & \downarrow \pi \\ Spec(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

- *A locally free sheaf* $\mathcal{E}_A$ *over* $\mathcal{X}_A$*, with a morphism* $\pi_A : \mathcal{E}_A \longrightarrow \mathcal{E}$*, such that* $\pi_A : \mathcal{E}_A \otimes \mathbb{C} \longrightarrow \mathcal{E}$ *is an isomorphism.*

two such deformation $(\mathcal{X}_A, \mathcal{E})$ *and* $(\mathcal{X}'_A, \mathcal{E}')$ *over* A *are isomorphic, if*

- *there is an isomorphism of deformations* $\nu : \mathcal{X}_A \longrightarrow \mathcal{X}'_A$.

- *we have an isomorphism of vector bundles over* $\mathcal{X}_A$*,* $\mu : \mathcal{E} \longrightarrow \nu^*(\mathcal{E}')$*, such that* $\pi_A = \pi'_A \circ \mu$.

Definition A.2.13 *Let* X *be a* $\mathbb{C}$*-scheme and let* $\mathcal{E}$ *a locally free sheaf on* X*. The functor of infinitesimal deformation of the pair* $(X, \mathcal{E})$ *is the following:*

$$\mathrm{Def}_{(X,\mathcal{E})} : \mathrm{Art}_{\mathbb{C}} \longrightarrow Set$$

which associates to every local Artinian $\mathbb{C}$*-algebra* A*, the set of isomorphism classes of infinitesimal deformations of the pair* $(X, \mathcal{E})$ *over* $\mathrm{Spec}(A)$.

Theorem A.2.14 *Let* X *be a* $\mathbb{C}$*-scheme and* $\mathcal{E}$ *a locally free sheaf on* X*. Then*

1. *The functor* $\mathrm{Def}_{(X,\mathcal{E})}$ *is a functor with a good deformation theory.*

2. *If* X *is a non singular projective variety, the tangent space of the functor* $\mathrm{Def}_{(X,\mathcal{E})}$ *is isomorphic to* $\mathrm{H}^1(X, \mathcal{A}_X(\mathcal{E}))$ *and* $\mathrm{H}^2(X, \mathcal{A}_X(\mathcal{E}))$ *is an obstruction space for it, where* $\mathcal{A}_X(\mathcal{E})$ *is the Atiyah algebroid 3.1.1 associated to the locally free sheaf* $\mathcal{E}$.

Remark A.2.15 *We have a natural transformation* $\mu : \mathrm{Def}_{(X,\mathcal{E})} \longrightarrow \mathrm{Def}_X$*, which corresponds to the forgetful map, for a deformation of a pair* $(X, \mathcal{E})$ *over* $\mathrm{Spec}(A)$

$$\begin{array}{ccc} & & \mathcal{E}_A \\ & & \downarrow \\ X & \hookrightarrow & \mathcal{X}_A \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

We associate the deformation $\mathcal{X}_A$ *of* X *over* $\mathrm{Spec}(A)$*. Hence we get a* $\mathbb{C}$*-linear map*

$$\mathrm{H}^1(X, \mathcal{A}_X(\mathcal{E})) \longrightarrow \mathrm{H}^1(X, T_X)$$

Example A.2.16 *Let* X *be a projective smooth complex curve of genus* $g \geqslant 2$ *and let* E *be a rank-*r *vector bundle. Take the Atiyah exact sequence* (3.1.1)

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathcal{A}_X(E) \longrightarrow T_X \longrightarrow 0$$

we associate the short-exact sequence in cohomology as dimension is one $\mathrm{H}^2(X, F) = 0$ *for any locally free sheaf* F *and there is no global sections*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(X, \mathrm{End}(E)) & \longrightarrow & \mathrm{H}^1(X, \mathcal{A}_X(E)) & \longrightarrow & \mathrm{H}^1(X, T_X) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathrm{Def}_E(\mathbb{C}[\varepsilon]) & \longrightarrow & \mathrm{Def}_{(X,E)}(\mathbb{C}[\varepsilon]) & \longrightarrow & \mathrm{Def}_X(\mathbb{C}[\varepsilon]) & \longrightarrow & 0 \end{array}$$

A.3 Deformations of a quasi-parabolic tuple

Let $\mathbb{E} = (C, D, E_*)$ where $\left(C, D = \sum_{i=1}^{N} x_i\right)$ a marked complex projective smooth curve and E_* a quasi-parabolic vector bundle of rank-r and type $\vec{m}$. Following the same pattern we define for an Artinian $\mathbb{C}$-algebra A a deformation of $\mathbb{E}$ over $\mathrm{Spec}(A)$, as follow

Definition A.3.1 *A deformation of* $\mathbb{E}$ *over* $T = \mathrm{Spec(A)}$*, is given by the following data: a deformation of marked curves* $\pi : (\mathcal{C}, \mathcal{D}) \longrightarrow T$ *where* π *is a flat morphism,* $\mathcal{D} = \Sigma\sigma_i(S)$ *and* $\mathcal{E}_*$ *a quasi-parabolic vector bundle of rank-*r *over* $(\mathcal{C}, \mathcal{D})$ *of quasi-parabolic type* $\vec{m}$*, we denote such data by* $\chi = (\mathcal{C}, \mathcal{D}, \mathcal{E}_*)$*. We have the following fibre product:*

$$\begin{array}{ccc} & & \mathcal{E}_* \\ & & \downarrow \\ (C, D) & \xrightarrow{\varphi} & (\mathcal{C}, \mathcal{D}) \\ \downarrow & & \downarrow \pi \\ Spec(\mathbb{C}) & \hookrightarrow & T \end{array}$$

and

$$\varphi^*(\mathcal{E}_*) \simeq E_*$$

isomorphic as quasi-parabolic vector bundles over (C, D).

In the following we define a deformation functor for $X = (C, D, E_*)$.

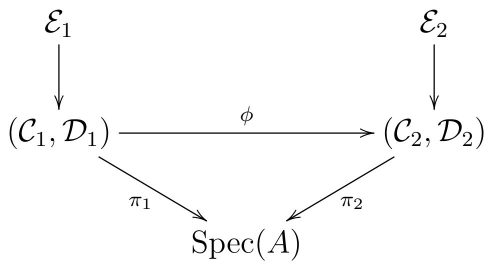
Definition A.3.2 (Deformation functor)
For $\mathbb{E} = (C, D, E_*)$ *as before, we define a deformation functor as follows:*

$$\begin{array}{rcl} Def_{\mathbb{E}} : \mathrm{Art}_{\mathbb{C}} & \rightarrow & Set \\ A & \mapsto & Def_{\mathbb{E}}(A), \end{array}$$

which associates to every local Artinian $\mathbb{C}$*-algebra* A*, the set of isomorphism classes of infinitesimal deformation of* $\mathbb{E}$ *over* $\mathrm{Spec}(A)$*. And two deformations* χ_1 *and* χ_2 *of* $\mathbb{E}$ *over* A *are isomorphic if:*

1. $\exists\ \phi : (\mathcal{C}_1, \mathcal{D}_1) \longrightarrow (\mathcal{C}_2, \mathcal{D}_2)$ *an isomorphism of marked curves over* $T = \mathrm{Spec}(A)$.

2. $\exists\ \psi : \mathcal{E}_1 \longrightarrow \phi^*(\mathcal{E}_2)$ *an isomorphism of quasi-parabolic vector bundles.*



Proposition A.3.3 *The deformation functor* $Def_{\mathbb{E}} : Art_{\mathbb{C}} \longrightarrow Set$*, is a functor with a good theory of deformation.*

Proof. The proof is just an adaptation to the quasi-parabolic case of the proof given in [Mar09] for the deformation functor for (X, E) where X is a complex variety and E a vector bundle over X. □

Remark A.3.4 *The result remains true in higher dimensions for a compact variety* X *and a normal crossing divisor. In this case, the space of obstructions is given by* $H^2\left(X, \mathcal{A}_X^{par}(E)\right)$ *which is trivial in dimension one.*

Chapter B

Rank 2 parabolic vector bundles

In this appendix we present Bertram's work on rank 2 parabolic bundles [Ber93].

Let C be a smooth projective complex curve of genus g and $I = \{x_1, x_2, \cdots, x_N\}$ and let α_* a full flag rank two parabolic type over I, so to each $i \in I$ we have

$$0 \leqslant a_1(i) < a_2(i) < k,$$

we associate the positive integer: $p_i := a_2(i) - a_1(i) \in \mathbb{N}$.

We get a new data (C, D, k), where

1. C a smooth projective complex curve of genus g.
2. $D = \sum_{i=1}^{N} p_i \, x_i$ an effective divisor of even degree.
3. An integer $k \geqslant \max\{p_i\}$.

To such data and $\delta \in \operatorname{Pic}^d(C)$, we get the space

$$\mathcal{SM}_C^{par}(D, k, \delta) := \mathcal{SM}_C^{par}(2, \alpha_*, \delta)$$

the moduli space of rank two full flag semi-stable parabolic vector bundles of determinant δ and parabolic type α_*, of dimension

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{SM}_C^{par}(D, k, \delta) &= 3(g-1) + \deg(D_{red}) \\ &= 3(g-1) + N.\end{aligned}$$

For any line bundle L over C, we get an isomorphism

$$\begin{array}{rccc} - \otimes L & : & \mathcal{SM}_C^{par}(D, k, \delta) & \longrightarrow & \mathcal{SM}_C^{par}(D, k, \delta \otimes L^2) \\ & & E_* & \longmapsto & (E \otimes L)_* \end{array}$$

Hence we have two cases:

1. even degree and we take $\delta = \mathcal{O}_C$.

2. odd degree and take $\deg\, \delta = \pm 1$.

Lemma B.0.1 ([Ber93], Lemma 2.3) *If $k > \frac{1}{2}\deg(D)$, then there is a morphism*

$$\phi : \mathcal{SM}_C^{par}(D,k,\delta) \longrightarrow \mathcal{SU}_\mathcal{C}(r,\delta)$$

such that over $\mathcal{SU}_\mathcal{C}^s(r,\delta)$ the stable locus is a $(\mathbb{P}^1_\mathbb{C})^N$-bundle. where $N = \deg(D_{red})$.

In fact if degree of δ is odd the moduli space $\mathcal{SU}_\mathcal{C}(r,\delta)$ is smooth. Thus it has a universal bundle $\mathcal{U}$ and we get the following isomorphism of moduli spaces

$$\mathcal{SM}_C^{par}(D,k,\delta) \simeq \mathbb{P}(\mathcal{U}_{x_1}^\vee) \times_{\mathcal{SU}_\mathcal{C}(r,\delta)} \mathbb{P}(\mathcal{U}_{x_2}^\vee) \times_{\mathcal{SU}_\mathcal{C}(r,\delta)} \cdots \times_{\mathcal{SU}_\mathcal{C}(r,\delta)} \mathbb{P}(\mathcal{U}_{x_N}^\vee)$$

where $\mathcal{U}_x = \mathcal{U}|_{\{x\}\times \mathcal{SU}_\mathcal{C}(r,\delta)}$, and the map ϕ is the natural map.

Example B.0.2 *If $D = \varnothing$, then we get*

$$\mathcal{SM}_C^{par}(D,k,\delta) = \mathcal{SU}_\mathcal{C}(2,\delta)$$

and the parabolic determinant bundle $\Theta_{par} = \mathcal{L}^k$, where $\mathcal{L}$ is the ample generator of $\mathrm{Pic}(\mathcal{SU}_\mathcal{C}(2,\delta))$.

Example B.0.3 *If $D = px$ and $p < k$, then by lemma B.0.1, $\mathcal{SM}_C^{par}(D,k,\mathcal{O}_C)$ maps to $\mathcal{SU}_\mathcal{C}(2,\mathcal{O}_C)$ with $\mathbb{P}^1$-fibres over $\mathcal{SU}_\mathcal{C}^s(2,\mathcal{O}_C)$. But by Hecke modification it maps to $\mathcal{SU}_\mathcal{C}^s(2,\mathcal{O}_C(-x))$ as follow*

$$\begin{array}{rcccc} \mathcal{H}_x & : & \mathcal{SM}_C^{par}(D,k,\mathcal{O}_C) & \longrightarrow & \mathcal{SU}\,(2,\mathcal{O}_C(-x)) \\ & & E_* = (E, D_x \subset E_x) & \longmapsto & \mathcal{H}_x(E) = \mathrm{Ker}\{E \twoheadrightarrow E_x/D_x\}. \end{array}$$

as a $\mathbb{P}^1$-bundle.

In fact, If $\mathcal{U}$ is a universal bundle over $C \times \mathcal{SU}\,(2,\mathcal{O}_C(-x))$, then

$$\mathcal{SM}_C^{par}(D,k,\mathcal{O}_C) \cong \mathbb{P}(\mathcal{U}_x^\vee)$$

We get the following diagram called the Hecke correspondence

$$\begin{array}{ccccc} & & \mathcal{SM}_C^{par}(D,k,\mathcal{O}_C) & & \\ & \swarrow^{\phi} & & \searrow^{\mathcal{H}_x} & \\ \mathcal{SU}\,(2,\mathcal{O}_C) & & & & \mathcal{SU}\,(2,\mathcal{O}_C(-x)) \end{array}$$

Furthermore, let $\mathcal{L}$ and $\mathcal{L}_\mathcal{H}$ the ample generator of the groups $\mathrm{Pic}(\mathcal{SU}\,(2,\mathcal{O}_C))$ *and* $\mathrm{Pic}(\mathcal{SU}\,(2,\mathcal{O}_C(-x)))$ *respectively. If we denote their pull backs to $\mathcal{SM}_C^{par}(D,k,\mathcal{O}_C)$ by Θ and $\Theta_\mathcal{H}$ respectively. Then*

$$\Theta_{par} = \Theta^{(k-p)} \otimes \Theta_\mathcal{H}^{p/2}$$

$$K_{\mathcal{SM}_C^{par}} = \Theta^{-2} \otimes \Theta_\mathcal{H}^{-1}.$$

Proposition B.0.4 (Generalized Hecke correspondence. Lemma.2.4, [Ber93]) *Suppose there is $j \in I$, such that $k > p_j > \frac{1}{2}\deg D$ in the data (C, D, k). Let $\mathcal{U}$ be a universal bundle on $C \times \mathcal{SM}_C^{par}(D, k, \mathcal{O}_C(-x_j))$, Then*

1. *The moduli space $\mathcal{SM}_C^{par}(D, k, \mathcal{O}_C)$ is isomorphic to the fibre product*

$$\mathbb{P}(\mathcal{U}_{x_1}^\vee) \times_{\mathcal{SU}_\mathcal{C}(r,\mathcal{O}_C(-x_j))} \mathbb{P}(\mathcal{U}_{x_2}^\vee) \times_{\mathcal{SU}_\mathcal{C}(r,\mathcal{O}_C(-x_j))} \cdots \times_{\mathcal{SU}_\mathcal{C}(r,\mathcal{O}_C(-x_j))} \mathbb{P}(\mathcal{U}_{x_N}^\vee)$$

   of $\mathbb{P}^1$-bundles over $\mathcal{SU}_\mathcal{C}(r, \mathcal{O}_C(-x_j))$.

2. *For $i \neq j$, there are morphisms*

$$\begin{aligned} \phi_{j,i}: \quad \mathcal{SM}_C^{par}(D, k, \mathcal{O}_C) &\longrightarrow \mathcal{SU}_\mathcal{C}(r, \mathcal{O}_C(-x_j - x_i)) \\ E_* = (E, D_x \subset E_x) &\longmapsto \mathcal{H}_i\left(\mathcal{H}_j(E)\right). \end{aligned}$$

   in addition to the projections $\mathcal{H}_j$ and ϕ given in example B.0.3. We get the following diagram

$$\begin{array}{ccccc} & & \mathcal{SM}_C^{par}(D,k,\mathcal{O}_C) & & \\ & \swarrow^{\phi} & \downarrow^{\mathcal{H}_j} & \searrow^{\phi_{j,i}} & \\ \mathcal{SU}(2,\mathcal{O}_C) & & \mathcal{SU}(2,\mathcal{O}_C(-x_j)) & & \mathcal{SU}_\mathcal{C}(r,\mathcal{O}_C(-x_j-x_i)) \end{array}$$

   If we denote Θ, $\Theta_{\mathcal{H}_j}$ and $\Theta_{j,i}$ the pull backs of the ample generators of the groups $\mathrm{Pic}(\mathcal{SU}(2, \mathcal{O}_C))$, $\mathrm{Pic}(\mathcal{SU}(2, \mathcal{O}_C(-x_j)))$ and $\mathrm{Pic}(\mathcal{SU}(2, \mathcal{O}_C(-x_j - x_i)))$ respectively. Then

$$\Theta_{par} = \Theta^{(k-p_j)} \otimes \Theta_{\mathcal{H}_j}^{\left(p_j - \frac{1}{2}\deg(D)\right)} \otimes \bigotimes_{i \neq j}^{N} \Theta_{j,i}^{p_j}$$

$$K_{\mathcal{SM}_C^{par}} = \Theta^{-2} \otimes \Theta_{\mathcal{H}_j}^{(N-2)} \otimes \bigotimes_{i \neq j}^{N} \Theta_{j,i}^{-2}$$

Proof. We give an explicit proof. Note that because of the condition $k > p_j > \frac{1}{2}\deg D$, every semistable parabolic bundle is stable. Thus the space $\mathcal{SM}_C^{par}(D, k, \mathcal{O}_C)$ is smooth. And we get

$$E_* \text{ is stable} \Longrightarrow E,\ \mathcal{H}_j(E) \text{ and } \phi_{j,i}(E_*) = \mathcal{H}_i(\mathcal{H}_j(E)) \text{ for } i \neq j \text{ are semistable.}$$

We recall the Hecke modification over x_j with respect to the line D_{x_j}

$$0 \longrightarrow \mathcal{H}_j(E) \longrightarrow E \longrightarrow E_{x_j}/D_{x_j} \longrightarrow 0$$

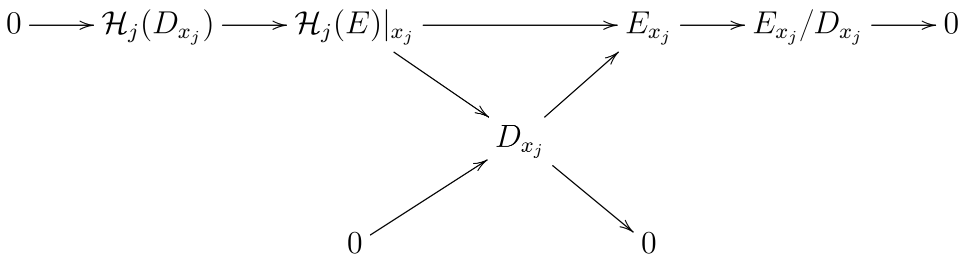
so we have an isomorphism of rank-2 vector bundle over $C\backslash\{x_j\}$

$$\mathcal{H}_j(E)|_{C\backslash\{x_j\}} \cong E|_{C\backslash\{x_j\}}$$

and by Lemma 1.4.5 we get

$$E_{x_i}/D_{x_i} = \lambda(E)^{-1} \otimes \lambda(\mathcal{H}_j(E)). \tag{B.0.1}$$

We pull-back the parabolic structure over $I\backslash\{x_j\}$, and at the point x_j we get



In the above diagram the line $\mathcal{H}_j(D_{x_j})$ is the Hecke transform of the line D_{x_j}. Thus we get a parabolic structure over $\mathcal{H}_j(E)$ with respect to the parabolic divisor I. To this parabolic bundle we can apply Hecke modification over any element of the divisor I, thus we can define $\mathcal{H}_i(\mathcal{H}_j(E))$, the associated exact sequence is

$$0 \longrightarrow \mathcal{H}_i(\mathcal{H}_j(E)) \longrightarrow \mathcal{H}_j(E) \longrightarrow \mathcal{H}_j(E)|_{x_i}/\mathcal{H}_j(D_{x_i}) \cong E_{x_i}/D_{x_i} \longrightarrow 0$$

By Lemma 1.4.5 we have

$$E_{x_i}/D_{x_i} = \lambda(\mathcal{H}_j(E))^{-1} \otimes \lambda(\mathcal{H}_i(\mathcal{H}_j(E))). \tag{B.0.2}$$

We recall the definition of the parabolic determinant bundle. Let $\mathcal{E}_* = (\mathcal{E}, (D_i)_{i\in I})$ be a virtual universal bundle over $C \times \mathcal{SM}_C^{par}(D, k, \mathcal{O}_C)$. Then

$$\Theta_{par} := \lambda(\mathcal{E})^k \otimes \bigotimes_{i=1}^{N} \{\mathcal{E}_{\sigma_i}/D_i\}^{p_i} \otimes \det(\mathcal{E}_\sigma)^e$$

and $2e = k\chi - \deg(D)$. We use equations (B.0.1) and (B.0.2), to get

$$\begin{aligned}
\Theta_{par} &:= \lambda(\mathcal{E})^k \otimes \{\mathcal{E}_{\sigma_j}/D_j\}^{p_j} \bigotimes_{i\neq j}^{N} \{\mathcal{E}_{\sigma_i}/D_i\}^{p_i} \otimes \det(\mathcal{E}_\sigma)^e \\
&= \lambda(\mathcal{E})^k \otimes \left\{\lambda(\mathcal{E})^{-1} \otimes \lambda(\mathcal{H}_j(\mathcal{E}))\right\}^{p_j} \bigotimes_{i\neq j}^{N} \{\lambda(\mathcal{H}_j(\mathcal{E}))^{-1} \otimes \lambda(\mathcal{H}_i(\mathcal{H}_j(\mathcal{E})))\}^{p_i} \otimes \det(\mathcal{E}_\sigma)^e \\
&= \left\{\lambda(\mathcal{E}) \otimes \det(\mathcal{E}_\sigma)^{\frac{\chi}{2}}\right\}^{(k-p_j)} \otimes \left\{\lambda(\mathcal{H}_j(\mathcal{E}))^2 \otimes \det(\mathcal{E}_\sigma)^{\chi_j}\right\}^{\left(p_j - \frac{1}{2}\deg(D)\right)} \\
&\quad \bigotimes_{i\neq j}^{N} \left\{\lambda(\mathcal{H}_i(\mathcal{H}_j(\mathcal{E}))) \otimes \det(\mathcal{E}_\sigma)^{\frac{\chi_{j,i}}{2}}\right\}^{p_i} \otimes \det(\mathcal{E}_\sigma)^q
\end{aligned}$$

where $\chi = 2(1-g)$, $\chi_j = \chi - 1$, $\chi_{j,i} = \chi - 2$ and

$$\begin{aligned}
q &= e - \left(\frac{k-p_j}{2}\right)\chi - \left(p_j - \frac{1}{2}\deg(D)\right)\chi_j - \sum_{i\neq j}^{N}\left(\frac{\chi_{j,i}}{2}\right)p_i \\
&= \frac{k}{2}\chi - \frac{1}{2}\deg(D) - \left(\frac{k-p_j}{2}\right)\chi - \left(p_j - \frac{1}{2}\deg(D)\right)(\chi-1) - \left(\frac{\chi}{2}-1\right)\sum_{i\neq j}^{N} p_i \\
&= -\underbrace{\left(-\frac{k}{2} + \frac{k-p_j}{2} + p_j - \frac{1}{2}\deg(D) + \frac{1}{2}\sum_{i\neq j}^{N} p_i\right)}_{0}\chi + \underbrace{\left(-\frac{1}{2}\deg(D) + p_j - \frac{1}{2}\deg(D) + \sum_{i\neq j}^{N} p_i\right)}_{0}
\end{aligned}$$

$q = 0.$

and if we denote $\mathcal{L}$, $\mathcal{L}_j$ and $\mathcal{L}_{j,i}$ the ample generators of the groups $\mathrm{Pic}(\mathcal{SU}(2,\mathcal{O}_C))$, $\mathrm{Pic}(\mathcal{SU}(2,\mathcal{O}_C(-x_j)))$ and $\mathrm{Pic}(\mathcal{SU}_{\mathcal{C}}(r,\mathcal{O}_C(-x_j-x_i)))$, respectively then by Theorem 1.5.2, we get

$$\begin{aligned}
&\Theta := \phi^*(\mathcal{L}) := \lambda(\mathcal{E}) \otimes \det(\mathcal{E}_\sigma)^{\frac{\chi}{2}} \\
&\Theta_{\mathcal{H}_j} := \mathcal{H}_j^*(\mathcal{L}_j) = \lambda(\mathcal{H}_j(\mathcal{E}))^2 \otimes \det(\mathcal{E}_\sigma)^{\chi_j} \\
&\Theta_{j,i} := \phi_{j,i}^*(\mathcal{L}_{j,i} = \lambda\left(\mathcal{H}_i(\mathcal{H}_j(\mathcal{E}))\right) \otimes \det(\mathcal{E}_\sigma)^{\frac{\chi_{j,i}}{2}}
\end{aligned}$$

Hence

$$\Theta_{par} = \Theta^{(k-p_j)} \otimes \Theta_{\mathcal{H}_j}^{\left(p_j - \frac{1}{2}\deg(D)\right)} \otimes \bigotimes_{i\neq j}^{N} \Theta_{j,i}^{p_j}.$$

Let calculate the canonical bundle of the space $\mathcal{SM}_C^{par}(D,k,\mathcal{O}_C)$. First we may choose the universal bundle $\mathcal{U}$ over $C \times \mathcal{SU}_C(r,\mathcal{O}_C(-x_j))$, such that for every point $x \in C$, $\Lambda^2\mathcal{U}_x = \mathcal{L}_j$ where $\mathcal{U}_x := \mathcal{U}|_{\{x\}\times\mathcal{SU}_C(r,\mathcal{O}_C(-x_j))}$. Next we take the Euler exact sequence over

$$\mathbb{P}(\mathcal{U}_{x_j}^\vee) \longrightarrow \mathcal{SU}_C(r,\mathcal{O}_C(-x_j))$$

This implies that the relative canonical sheaf is

$$K_{\mathbb{P}(\mathcal{U}_{x_j}^\vee)/\mathcal{SU}_C(r,\mathcal{O}_C(-x_j))} \cong \mathcal{O}_{\mathbb{P}(\mathcal{U}_{x_j}^\vee)}(-2) \otimes \mathcal{H}_j^*(\mathcal{L}_j).$$

For this choice of universal bundle, $\mathcal{O}_{\mathbb{P}(\mathcal{U}_{x_j}^\vee)}(1) = \phi_{j,i}^*(\mathcal{L}_{j,i})$ and $\mathcal{O}_{\mathbb{P}(\mathcal{U}_{x_j}^\vee)}(-1) = \phi^*(\mathcal{L})$

$$\begin{aligned}
K_{\mathcal{SM}_C^{par}(D,k,\mathcal{O}_C)} &= \mathcal{H}_j^*\left(\omega_{\mathcal{SU}_C(r,\mathcal{O}_C(-x_j))}\right) \otimes \left(\Theta^{-2} \otimes \Theta_{\mathcal{H}_j}\right) \otimes \bigotimes_{i\neq j}^{N}\left(\Theta_{j,i}^{-2} \otimes \Theta_{\mathcal{H}_j}\right) \\
&= \Theta^{-2} \otimes \Theta_{\mathcal{H}_j}^{(N-2)} \otimes \bigotimes_{i\neq j}^{N} \Theta_{j,i}^{-2}.
\end{aligned}$$

□

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