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Dynamique sur les espaces de modules

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Résumé:

Dans cette thèse, nous nous intéressons à la dynamique de sous-groupes modulaires sur la variété des SU(2)-caractères . Plus précisément, nous étudions des questions d'ergodicité de l'action de sous-groupes Γ du groupe modulaire $\mathrm{Mod}_{\mathrm{g,n}}$ d'une surface compacte $S_{g,n}$ de genre g et n composantes de bord. Ces questions ont été naturellement posées après la preuve de Goldman de l'ergodicité du groupe modulaire sur la variété des caractères. Le premier résultat général dans cette direction est dû à Funar et Marché, en montrant que le premier sous-groupe de Johnson agit de manière ergodique sur la variété des caractères, pour toute surface fermée S_g. D'un autre coté, Brown a montré l'existence de points fixes elliptiques pour tout sous-groupe généré par un homéomorphimse pseudo-Anosov sur le tore épointé $S_{1,1}$. Ceci a permis de démontrer la non-ergodicité de tels sous-groupes par Forni, Goldman, Lawton et Matheus en appliquant la théorie KAM.

Dans la première partie de la thèse, nous étudions une dynamique naturelle sur l'espace des modules des triangles sphériques de $\mathbb{S}^2$ en reliant cette dynamique à la dynamique du groupe modulaire $\mathrm{SL}(2,\mathbb{Z})$ sur la variété des caractères du tore épointé.

La deuxième partie est consacrée à l'étude de l'existence de points fixes elliptiques pour les homéomorphismes pseudo-Anosov sur les variétés de caractères des surfaces épointée $S_{g,n}$, où $g \in \{0,1\}$. On montre que dans le cas de la variété des caractères relative $X_\kappa(\pi_1(S_{1,1}), \mathrm{SU}(2))$ du tore épointé, pour un ensemble de mesure positive et dense de niveaux de la fonction invariante κ , il existe une famille d'élements pseudo-Anosov qui n'agissent pas érgodiquement sur ces niveaux. Un résultat similaire est démontré pour un ensemble de paramètres B sur $X_B(\pi_1(S_{0,4}), \mathrm{SU}(2))$, dans le cas de $S_{0,4}$, la sphère à quatre trous. Ces résultats sont combinés pour construire une famille d'éléments pseudo-Anosov sur le tore à deux trous $S_{1,2}$, qui admettent un point fixe elliptique.

Nous discutons ensuite de l'action d'un groupe Γ généré par des twists de Dehn le long d'une paire de multi-courbes qui remplissent la surface ou plus généralement le long d'une famille des courbes qui remplissent S_g. Nous montrons dans cette partie qu'il existe deux multi-courbes qui remplissent la surface de genre deux S_2 dont les twists de Dehn associées génèrent un groupe Γ agissant de manière non-ergodique sur $\mathrm{Hom}(\pi_1(S_2), \mathrm{SU}(2))$, en trouvant des fonctions rationnelles invariantes explicites. De même, nous montrons l'existence de fonctions rationnelles invariantes par conjugaison et invariantes par un sous-groupe Γ générées par des twists de Dehn le long d'une famille des courbes qui remplissent la surface fermée non-orientable N_4.

Mots Clés: Surfaces de translation et surfaces à petit carreaux, Twists de Dehn, Sous-groupes modulaires, Variétés des caractères et variétés des représentations.

Abstract:

In this thesis, we are interested in the dynamics of the mapping class subgroups on the SU(2)-character variety. More precisely, we deal with ergodicity questions of a subgroup Γ of the mapping class group $\mathsf{Mod}_{\mathsf{g},\mathsf{n}}$ of a compact surface $S_{g,n}$ of genus g and n boundary components. These questions were naturally raised after Goldman's proof of the ergodicity of mapping class groups on the SU(2)-character variety. The first general result in this direction is due to Funar and Marché by showing that the first Johnson subgroups act ergodically on the character variety, for any closed surfaces S_g. On the other hand, Brown showed the existence of an elliptic fixed point (or a double elliptic fixed point) for any subgroup generated by a pseudo-Anosov element on the punctured torus $S_{1,1}$. This led to the proof of the non-ergodicity of such subgroups by Forni, Goldman, Lawton, and Mateus by applying KAM theory.

In the first part of the thesis, we study the natural dynamics of the moduli space of spherical triangles on $\mathbb{S}^2$ relating these dynamics to the dynamics of the mapping class group on the SU(2)-character variety of the punctured torus.

The second part is devoted to the study of the existence of elliptic fixed points for pseudo-Anosov homeomorphisms on the character varieties of punctured surfaces $S_{g,n}$, where $g \in \{0,1\}$. By showing that near any relative character variety $X_\kappa(\pi_1(S_{1,1}), \mathsf{SU}(2))$ of the once punctured torus, for a set of positive measure and dense of levels κ, there exists a family of pseudo-Anosov elements that do not act ergodically on that level, in the case of the punctured torus $S_{1,1}$. A similar result holds for a set of parameters B on $X_B(\pi_1(S_{0,4}), \mathsf{SU}(2))$, in the case of the four-punctured sphere $S_{0,4}$. Then these results can be combined to construct a family of pseudo-Anosov elements on the twice-punctured torus $S_{1,2}$ that admit an elliptic fixed point.

We discuss then the action of a group Γ generated by Dehn-twist along a pair of filling multi-curves or along a family of filling curves on S_g. We show in this part that there exist two filling multi-curves on the surface of genus two S_2 whose associated Dehn twists generate a group Γ acting non-ergodically on $\mathsf{Hom}(\pi_1(S_2), \mathsf{SU}(2))$ by finding explicit invariant rational functions. Similarly, We found invariant rational functions of a subgroup Γ generated by Dehn-twists along a family of filling loops on the character variety of the non-orientable surface N_4.

Key Words: Translation surfaces and square-tiled surfaces, Dehn twists, mapping class subgroups, Character varieties and representation varieties,

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Finally, I take this opportunity to dedicate this work to my parents.

Contents

Chapter 1

Introduction

In this thesis, we are investigating the dynamics of some subgroups of the mapping class group on the SU(2)-character variety. We consider $\pi_1(S)$ to be the fundamental group of some compact surface $S = S_{g,n}$, where g is the genus and n is the number of boundary components $\{b_1, \dots, b_n\}$ of the surface. We define Hom$(\pi_1(S_{g,n}), \mathsf{SU}(2))$ to be the space of representations of $\pi_1(S_{g,n})$ into SU(2). The group SU(2) acts by conjugacy on Hom$(\pi_1(S_{g,n}), \mathsf{SU}(2))$ and the algebraic quotient is called the character variety which is denoted by $X(\pi_1(S_{g,n}), \mathsf{SU}(2))$. The mapping class group $\mathsf{Mod}(\mathsf{S}_{g,n})$ (that is the group of orientation-preserving homeomorphisms of $S_{g,n}$ fixing point-wise the boundary components up to orientation-preserving homeomorphisms which are isotopic to the identity of $S_{g,n}$) has a natural action on $X(\pi_1(S_{g,n}), \mathsf{SU}(2))$ by pre-composition.

Any conjugacy class of a boundary component b_i is fixed by the action of the mapping class group, therefore the function $f_{b_i}([\rho]) := tr(\rho(b_i))$ is invariant under the mapping class group action, for any $i \in \{1, \dots, n\}$. Hence any level B of the evaluation map $(f_{b_1}, \dots, f_{b_n})$ is preserved by the action of $\mathsf{Mod}(\mathsf{S}_{g,n})$. A level of the previous function is called the relative character variety and we denoted it by $X_B(\pi_1(S_{g,n}), \mathsf{SU}(2))$.

For an orientable surface S, Goldman [G1] showed that its character variety/relative character varieties inherit a symplectic form ω and proved that the mapping class group acts ergodically with respect to the induced measure [GX1]. A natural question is then to ask whether a subgroup Γ of the mapping class group acts ergodically or not. The first result in this direction is given by Goldman and Xia [GX2] by proving that on the twice-punctured torus, the Torelli group acts ergodically on its character variety. In [FM], Funar and Marché showed that the first Johnson subgroup, which is the group generated by Dehn-twists along separating curves of S, acts ergodically on the character variety. Recently, Marché and Wolff [MW] proved that any non-central normal subgroup of the mapping class group acts topologically transitively on its SU(2)-character variety. In parallel, the ergodicity of the mapping class group of non-orientable surfaces was proved by Palesi [P] after introducing a measure υ invariant by the mapping class group.

On the other side, Brown [Br] proved that a pseudo-Anosov element (i.e. the iterates of such an element preserve no essential simple closed curve) admits an elliptic fixed point or a double elliptic fixed point for some relative character varieties of the punctured torus $S_{1,1}$. This recently led to fully proving the non-ergodicity of such elements by applying KAM theory in [FGLM] by Forni, Goldman, Lawton, and Matheus.

1.1 Punctured surfaces

We start this thesis by exhibiting in the second Chapter a relationship between natural dynamics on the space of spherical triangles and the dynamics of the mapping class group on SU(2)-character variety of the punctured torus $S_{1,1}$.

In the next Chapter, we generalize in some sense Brown's approach by constructing elliptic fixed points for some pseudo-Anosov maps; For punctured surfaces, we consider a group Γ generated by a single pseudo-Anosov element. The character variety of the punctured is homeomorphic to a ball of dimension 3, foliated by the levels of the function $\kappa := f_b$ where b is the boundary component of $S_{1,1}$ defining the relative character varieties which are homeomorphic to two-dimensional spheres expect at $\kappa = -2$ where it reduces to a point. We found in this context that near each $X_\kappa(\pi_1(S_{1,1}), \mathsf{SU}(2))$, for an open dense set of levels κ, there exists a family of pseudo-Anosov elements which does not act ergodically on that level (See Theorem 4.3.1 in Chapter 4), in the case of the punctured torus $S_{1,1}$. A similar result holds, for a set of parameters B on $X_B(\pi_1(S_{0,4}), \mathsf{SU}(2))$, in the case of the four-punctured sphere $S_{0,4}$ (See Theorem 4.3.2 in Chapter 4). Consider now the two subgroups $\Gamma_n = \langle \tau_\delta^4, \tau_\gamma^n \rangle$ and $\Delta = \langle \tau_\alpha^4, \tau_\beta^4 \rangle$ of the mapping class group of the twice-punctured torus, where α, β, γ, and δ are some closed curves in $S_{1,2}$ as illustrated in Figure 4.4. The degree of freedom in the character variety of punctured surfaces allows us to combine the previous two results to prove in Chapter 4 the following result:

Theorem 1.1.1. *4.4.2 For a homeomorphism $f_2 \in \Delta$ satisfying the conditions of Theorem 4.3.1, there exists $N \in \mathbb{N}$ such that for any $n > N$, for any $f_1 \in \Gamma_n$ satisfying the conditions of Theorem 4.3.2, the homeomorphism $\Phi = f_1 \circ f_2$ is pseudo-Anosov. Moreover, Φ admits a line of elliptic fixed points of irrational frequency vector over the corresponding relative character varieties $X_B(\pi_1(S_{1,2}), \mathsf{SU}(2))$.*

1.2 Closed surfaces

When the surface is closed, we rather consider a group Γ generated by a pair of filling multi-curves of a family of filling curves, such groups contain infinitely many pseudo-Anosov, these elements act by preserving no class of simple closed curves. A consequence of Theorem 1.1 by Charles and Marché in [CM] implies more; Pseudo-Anosov elements admit no invariant polynomial functions on the $\mathsf{SU}(2)$-character varieties. Describing the dynamics of Γ using square-tiled surfaces allows us to predict the existence of rational invariant functions, for a suitable Γ, we establish in Chapter 5, the following:

Theorem 1.2.1. *Let S_2 be the orientable surface of genus two. Then, there exists a pair of filling multi-curves whose associated Dehn twists generate a group Γ that admits an invariant rational function on $\mathsf{Hom}(\pi_1(S_2), \mathsf{SU}(2))$.*

In the non-orientable setting, we have:

Theorem 1.2.2. *On the closed non-orientable surface of genus four N_4, there exists a family of filling curves whose associated Dehn twists generate a group Γ that admits an invariant rational function on the character variety $X(\pi_1(N_4), \mathsf{SU}(2))$.*

The application of Theorem 6.1 by Fathi [F] ensures the existence of pseudo-Anosov elements in such a group. Therefore:

Corollary 1.2.1. There exists a pseudo-Anosov element on N_4 which does not act ergodically on the character variety $X(\pi_1(N_4), \mathsf{SU}(2))$.

We end the thesis by showing the ergodicity of the homeomorphism group fixing a base point p on the $\mathsf{SU}(2)$-representation variety:

Theorem 1.2.3. *The homeomorphism group $\mathsf{Homeo}^+(\mathsf{S_g}, \mathsf{p})$ acts ergodically on the representation variety $\mathsf{Hom}(\pi_1(S_g, p), \mathsf{SU}(2))$ with respect to the class of Lebesgue measure, More precisely, ergodicity is ensured by the action of a group generated by no more than $3g - 1$ Dehn twists on S_g.*

Chapter 2

Background

2.1 Surfaces and the mapping class group

Let $S = S_{g,n}$ be a compact surface of genus g and n boundary, the mapping class group of S that we denote here by Mod(S) is the group of homeomorphisms of S preserving point-wise the boundaries of S and preserving the orientation up to homeomorphisms that are isotopic to the identity. So if we denote the homeomorphism group preserving the orientation by $\mathrm{Homeo}^+(\mathrm{S})$ and by $\mathrm{Homeo}_0(\mathrm{S})$ the group of homeomorphisms isotopic to the identity then the mapping class group of S is the quotient group

$$\mathrm{Mod(S)} := \mathrm{Homeo}^+(\mathrm{S})/\mathrm{Homeo}_0(\mathrm{S})$$

Fixing a point p in the surface, a homeomorphism $h \in \mathrm{Homeo}^+(\mathrm{S})$ induces a homomorphism $h_* \in \mathrm{Aut}(\pi_1(\mathrm{S}))$ from the fundamental group of S to itself, this gives rise to a homomorphism i : $\mathrm{Homeo}^+(\mathrm{S}) \to \mathrm{Aut}(\pi_1(\mathrm{S}))$. In particular, a homeomorphism h that is isotopic to the identity is sent to a homomorphism in the interior group i.e. $h_* \in \mathrm{Inn}(\pi_1(\mathrm{S}))$. Thus i descends to a homomorphism $j : \mathrm{Mod(S)} \to \mathrm{Out}(\pi_1(\mathrm{S}))$. If $[h] \in \mathrm{Mod(S)}$, then by our definition h preserves the orientation, this can be read in h_*; we write $\pi_1(S) = \{a_1, \ldots, a_{2g} \mid R(a_1, \ldots, a_g)\}$, for some relation R, for instance we can take $R(a_1, \ldots, a_{2g}) = \prod_{i=1}^{i=g}[a_i, a_{i+g}]$. The automorphism h_* is lifted to an automorphism $\tilde{h}_*$ that sends the normalizer of R to itself. The group of automorphisms verifying the previous condition is called the group of orientation-preserving automorphisms and it is denoted by $\mathrm{Aut}^+(\pi_1(\mathrm{S}))$. Therefore, the morphism j is defined from Mod(S) into $\mathrm{Out}^+(\pi_1(\mathrm{S}))$. It turns out that

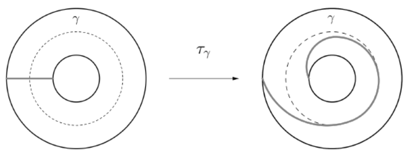
Theorem 2.1.1 (Dehn-Nielsen-Baer)**.** *The homomorphism*

$$j : \mathrm{Mod(S)} \to \mathrm{Out}^+(\pi_1(\mathrm{S}))$$

is an isomorphism.

2.1.1 Generating the mapping class group

The simplest elements of the mapping class group are Dehn twists; Let γ be a simple closed curve in S. The Dehn twist τ_γ along the curve γ is the homeomorphism defined on a tubular neighborhood of γ as follow:



And τ_γ is the identity elsewhere. In fact, we can generate the mapping class group using Dehn-twists:

Theorem 2.1.2 (Dehn)**.** *The mapping class group* Mod(S) *of a compact orientable surface S is generated by the Dehn twists along simple closed curves.*

We only need a finite number of Dehn twists to generate the mapping class group of closed surfaces (See [L] for more details)

Theorem 2.1.3 (Lickorish)**.** *The mapping class group* $\mathrm{Mod(S_g)}$ *of a closed surface of genus g is generated by Dehn twists along the $3g-1$ closed curves $\{\gamma_1, \ldots, \gamma_{3g-1}\}$, as shown in Figure 2.1.*

Non-orientable case

When the surface is a closed non-orientable surface then the mapping class group is generated by Dehn twists along two-sided closed curves together with another type of homeomorphism called the Y-homeomorphisms; Let N be a closed non-orientable surface of genus $g \geq 3$ and $K \subset N$ is a Klein bottle with one boundary component $b = \partial K$ embedded in N, so K is a direct sum of a projective plane with a Möbuis band of boundary ∂K. The Y-homeomorphism associated to K is the homeomorphism defined by sliding a small neighborhood of the projective plane along the Möbuis band once. Notice

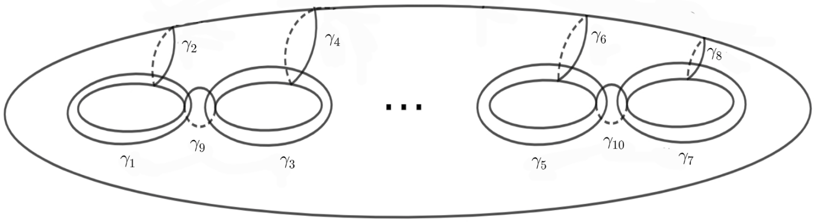


Figure 2.1: Generators

that the square of this homeomorphism is just sliding twice the projective plane, thus the square is isotopic to the Dehn twist along the boundary of the Möbuis band $b = \partial K$. (Following these terms, the Dehn twist along a curve γ can be defined as sliding a point in γ along the curve itself).

2.1.2 Relations on the mapping class group

Previously, we saw that Mod(S) is finitely generated, it is natural to ask then about a presentation of the group Mod(S). If γ is sent to γ' by a homeomorphism h then $[\tau_{\gamma'}] = [h \circ \tau_\gamma \circ h^{-1}]$. In particular, if $h = \tau_\lambda$ and λ does not intersect γ then $h(\gamma)$ is isotopic to γ. Hence the two Dehn-twists τ_λ and τ_γ commutes with each others. This relation i.e $\tau_\gamma.\tau_\lambda = \tau_\lambda.\tau_\gamma$ is called the disjointness relation.

If $i(\lambda, \gamma) = 1$ i.e. the geometric intersection between γ and λ is 1, then we can check on a small neighborhood of $\gamma \cup \lambda$, which is homeomorphic to a punctured torus, that $\tau_\gamma(\lambda) = \tau_\lambda^{-1}(\gamma)$, therefore $\tau_{\tau_\gamma(\lambda)} = \tau_{\tau_\lambda^{-1}(\gamma)}$, hence

$$\tau_\gamma.\tau_\lambda.\tau_\gamma^{-1} = \tau_\lambda^{-1}.\tau_\gamma.\tau_\lambda$$

Thus, we obtain what is called the Braid relation:

$$\tau_\gamma.\tau_\lambda.\tau_\gamma = \tau_\lambda.\tau_\gamma.\tau_\lambda$$

We can deduce another relation called the k-chain relation (See Chapter 09 in [MF]) on the surface $S = S_{g,b}$, whenever $g + 2b \geq 2$:

Proposition (k-chain relation)**.** Let $\gamma_1, \ldots, \gamma_k$ be a chain of simple closed curves in S i.e. $i(\gamma_i, \gamma_j) = 1$, if $j = i+1$ and $i(\gamma_i, \gamma_j) = 0$, otherwise. Let K be a closed regular neighborhood of $\gamma_1, \ldots, \gamma_n$. Then, we have:

- For k even, $(\tau_{\gamma_1} \ldots \tau_{\gamma_k})^{2k+2} = \tau_\lambda$, where $\lambda := \partial K$.

- For k odd, $(\tau_{\gamma_1} \ldots \tau_{\gamma_k})^{k+1} = \tau_{\lambda_1}.\tau_{\lambda_2}$, where $\lambda_1 \cup \lambda_2 = \partial K$.

The last three relations, together with relations called the hyperelliptic relations are enough for representing the mapping class group of the closed surfaces of genus 2:

Theorem 2.1.4 (Birman-Hilden). *Setting $A := \tau_{\alpha_1}$, $B := \tau_{\beta_1}$, $C := \tau_{\gamma_1}$, $D := \tau_{\beta_2}$ and $E := \tau_{\alpha_2}$, as shown if Figure 2.2. We have that:*

$$\mathrm{Mod(S_2)} = \{\mathrm{A, B, C, D, E} \mid \mathrm{disjointness, braid, (ABC)^4 = E^2, [H, A] = 1, H^2 = 1}\}$$

where $H := EDCBA^2BCDE$.

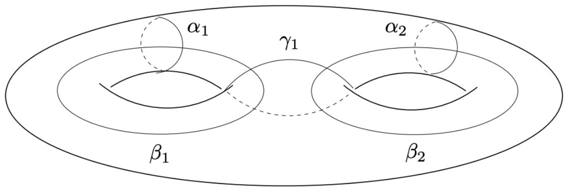


Figure 2.2: Presentation of Mod($\mathrm{S_2}$)

2.1.3 Classification of elements of the mapping class group

Elements of the mapping class group are sorted into three different categories:

Theorem 2.1.5 (Nielsen-Thurston). *Let h be a homeomorphism of S_g for $g > 1$, then up to isotopy, at least one of the following holds:*

- *h is periodic.*

- *h is reductive i,e h fixes a simple closed curve in S_g.*

- *h is pseudo-Anosov.*

One can define pseudo-Anosov using this theorem as the homeomorphisms h such that no power of h preserves an isotopy class of an essential simple closed curve. A more descriptive definition would be the following (See next section for more details):

Definition 2.1.1. $[h] \in$ Mod(S) is said to be pseudo-Anosov, if there exists a half-translation S' and a diagonal matrix

$$D = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix} \in SL(2, \mathbb{R})$$

with $|K| > 1$, such that $h_*(S') = D.S'$

2.2 half-translation surfaces

2.2.1 Geometric definition

A half-translation structure on a closed surface S is the data of an atlas of charts $(\phi_i, U_i)_{i \in I}$ on $S \setminus \Sigma$ for a finite set Σ such that the transition maps $\phi_{i,j} : \phi_i(U_i) \longrightarrow \phi_j(U_j)$ are half-translations on the complex plane $\mathbb{C}$ i.e. of the form $z \mapsto \pm z + c$ for some $c \in \mathbb{C}$. The metric dz^2 is invariant by the transition maps. Hence the surface $S \setminus \Sigma$ is endowed with the flat Riemannian metric dz^2. We also require that the previous metric admits an extension to the surface S. If we look at the circle γ_r centered at a point in Σ of radius r in S then the charts around the circle can be chosen so that the image of the developing map along γ_r is again a Euclidean circle of radius r. Hence the Disk $D_r \subset S$ that consists of points of distance less than r from the singular point is isometric to a flat cone of angle $k\pi$ for $k \in \mathbb{N}$.

Remark 2.2.1. We shall require here that $k \geq 2$, otherwise, for instance, Definition 2.1.1 would not agree with Theorem 2.1.5.

From a half-translation surface, we can construct a translation surface; for a chart (ϕ_i, U_i) we can derive two new charts $(-\phi_i, U_i)$ and (ϕ_i, U_i). Consider the double cover $\pi : S' \longrightarrow S \setminus \Sigma$ endowed with charts $(-\phi_i, U_i')_{i \in I} \cup (\phi_i, U_i'')_{i \in I}$, such that $\pi(U_i') = \pi(U_i'') = U_i$. On these new charts, the transition maps are translations in the plane i.e. of the form $z \mapsto z + c$, for some $c \in \mathbb{C}$. Around a singular point of Σ , we have two possibilities:

- The flat cone D_r of angle $k\pi$ is lifted to a cone of angle $2k\pi$, in this case, k is odd.
- The lift of D_r consists of two cones each of them is a copy of D_r, in this case, k is even.

Therefore the cover on $S \setminus \Sigma$ is in fact a double ramified cover over S and the lift is a translation surface.

2.2.2 Analytic definition

A half-translation surface S is the data of a Riemann surface S and a quadratic differential q on S. Denote by Σ the zeros of q. On a small neighborhood of any point, we can write $q_z = z^k dz^2$, if q does not vanish at z then we can consider a neighborhood U_i where we can simply write $q_z = dz^2$ around a neighborhood of z, notice that the transition maps between these charts must be of the form $z \mapsto \pm z + c$. We can endow $S \setminus \Sigma$ with the flat metric $\mid dz^2 \mid$. At a point in Σ, the quadratic differential vanishes, hence around this singular point we can write $q = z^n.dz^2$. Consider the pull-back of q via $\phi_2 : z \mapsto \frac{1}{2}z^2$ to get a new quadratic differential $q' = z^{2n+2}.dz^2$. The pull-back of dz^2 via $\phi_n : z \mapsto \frac{1}{n+2}.z^{n+2}$ gives also q'. This means that around this singular point, we have a flat cone of angle $(n+1)\pi$. Conversely, If $(\phi_i, U_i)_{i\in I}$ is an atlas for half-translation surface, then we can consider the quadratic differential $q = dz^2$ on S. The lift defined above is exactly an attempt to define the square root of q which yields a 1-holomorphic form.

2.2.3 Constructive definition

Let $\mathcal{P} = \cup_{i\in I}\mathcal{P}_i$ be a finite collection of polygons in the complex plane $\mathbb{C}$. Consider E to be the collection of all edges in $\mathcal{P}$. A partition of the set E two by two, such that the two curves forming each pair have the same lengths, gives a half-translation surface by identifying the edges via half-translation in such a way that the interiors of polygons do not overlap once we apply the half-translation, Considering the quadratic differential dz^2 on each polygon yields a half-translation surface in the analytic sense.

2.2.4 Strata of the Teichmüller space of translation surfaces

For each translation surface, we can list the conical singularities with their angles. Denote by $(p_1, p_2, \ldots, p_n)$ the list of singularities of S and by $(2\pi k_1, \ldots, 2\pi k_n)$ their angles in the same order. For instance, the genus of the surface can be computed using Poincaré-Hopf formula; let X be the horizontal vector field defined on the translation surface S, then the sum of the indexes of X at the singular points gives the relation:

$$\sum_{i=1}^{i=n} k_i - 1 = -\chi(S) \tag{2.1}$$

The Teichmüller space of translation surfaces which is the space of all translation surfaces on a fixed surface S up to isotopies of S, denoted here by $\mathcal{TH}(S)$ is stratified.

Let us denote $\mathcal{H}(k_1 - 1, k_2 - 1, \ldots, k_n - 1)$ the space of all translation surfaces up to isotopies on S

such that $S \in \mathcal{H}(k_1 - 1, k_2 - 1, \ldots, k_n - 1)$, if S has n conical singularities of angles $(2\pi k_1, \ldots, 2\pi k_n)$, then we have a stratification of $\mathcal{TH}(S)$ via $\mathcal{H}(k_1 - 1, k_2 - 1, \ldots, k_n - 1)$, where $(k_1, k_2, \ldots, k_n)$ span the possible combinations such that the topological condition above holds, in other words:

$$\mathcal{TH}(S) = \bigcup_{\sum_{i=1}^{i=n}(k_i - 1) = -\chi(S)} \mathcal{H}(k_1 - 1, k_2 - 1, \ldots, k_n - 1)$$

The deformation space of quadratic differentials is stratified similarly; if S is a half-translation surface with singular points of angles $(\pi k_1, \ldots, \pi k_n)$, then the list of the singularities yields a stratification of the deformation space.

Remark 2.2.2. In what follows, we normalize the volume of (half-)translation surfaces, so we are considering only those that have volume 1.

2.2.5 The period map

Each stratum $\mathcal{H}(\sigma)$ on the Teichmüller space of translation surfaces has a natural affine structure, in fact, the evaluation of the 1-holomorphic form over the relative homology $H_1(S, \Sigma, \mathbb{Z})$ determines locally the translation structure.

Theorem 2.2.1. *The period map Θ defined below is a local homeomorphism.*

$$\begin{aligned} \Theta : \mathcal{H}(\sigma) &\longrightarrow H^1(S, \Sigma, \mathbb{Z}) \\ w &\longmapsto (\gamma \mapsto \int_\gamma w) \end{aligned}$$

Example 2.2.1. Let v_1, v_2, v_3 and v_4 be vectors in the plane. Consider the polygon $\mathcal{P}$ with eight edges $\{v_1, v_2, v_3, v_4, v_1, v_2, v_3, v_4\}$, identified via translations we get a translation surface in the stratum $\mathcal{H}(2)$.

2.2.6 Left and Right actions

The deformation space $\mathcal{TH}(S)$ has a natural projection to The Teichmüller space $\mathcal{T}(S)$ or equivalently the deformation space of the complex structures by simply considering the underlying complex structure. The mapping class group Mod(S) acts on both spaces by pre-composition and the quotient spaces are the moduli spaces of the prescribed structures. The group $\mathsf{SL}(2, \mathbb{R})$ acts on $\mathcal{TH}(S)$ by post-composition; if $A \in \mathsf{SL}(2, \mathbb{R})$ and S is a (half-)translation structure then $A.S$ is the surface obtained by composing

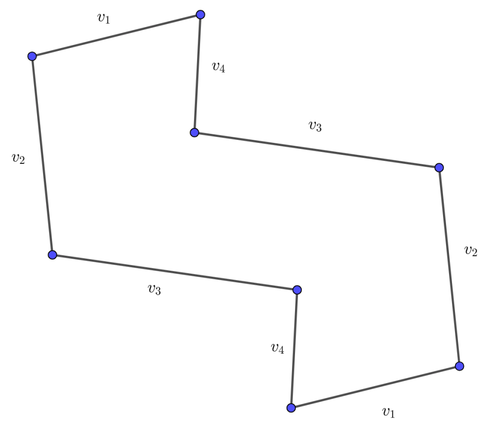


Figure 2.3: Translation surface

the charts of S by A, this produces a new surface with (half-)translations as transition maps. Since Mod(S) and SL$(2,\mathbb{R})$ act by left and right composition, respectively, then their actions commute.

The action of the subgroup SO(2) preserves the underlying complex structure, therefore the orbits of SL$(2,\mathbb{R})$ on $\mathcal{H}(S)$ project to the Teichmüller space as copies of an embedded hyperbolic plane $\mathbb{H}^2$ called the Teichmüller disks. Let S be a translation surface, the subgroup of the mapping class group that preserves the Teichmüller disk of S is called the Veech group of S. Equivalently, we can define it as the group of linear parts of the affine transformations on the translation surface S. The Figure 2.4 below illustrate the action of the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and the pre-composition by $\phi \in$ Mod(S) on the surface $S \in \mathcal{H}(2)$.

2.2.7 Square-tiled surfaces, decomposition into cylinders

Definition 2.2.1. An origami of a square-tiled surface of genus g and degree d is a ramified cover $O : S \mapsto \mathbb{T}^2$, where S is a surface of genus g and O is of degree d with only one singular point p at the basis $\mathbb{T}^2$. We say that two origamis $O_1 : S_1 \mapsto \mathbb{T}^2$ and $O_2 : S_2 \mapsto \mathbb{T}^2$ are equivalents if there exists a homeomorphism $f : S_1 \mapsto S_2$ s.t $O_1 \circ f = O_2$ i.e. they are in the same class of equivalence in the moduli space of translation surfaces

An origami of degree d can be seen as a representation of $\mathbb{F}_2$ into the permutation group S_d of a

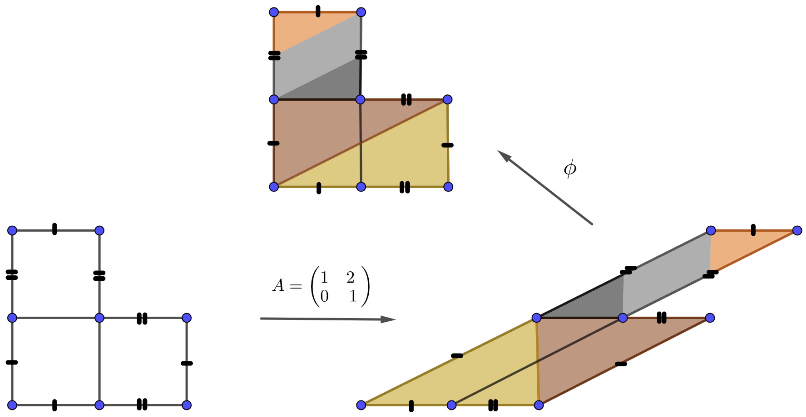


Figure 2.4: Element in the Veech group

set of d elements. Consider $E = \mathbb{T}^2 \setminus p$, $S' = S \setminus O^{-1}(p)$ and let $\tilde{S}$ be the universal cover of E. Then $\pi_1(S')$ is a subgroup of the Deck transformation of $\tilde{S} \mapsto E$ which is isomorphic to a free group with two generators $\langle \alpha_r, \alpha_u \rangle = \mathbb{F}_2$, where α_r(resp. α_u) corresponds to the operation of going right (resp. going up). We have that $S' = \tilde{S}/\pi_1(S')$ which is made of d squares glued together along their edges. Now $\mathbb{F}_2$ acts on these squares in the following way, each square is identified to a class of $\mathbb{F}_2/\pi_1(S')$ and $\mathbb{F}_2$ acts on these classes by left multiplication. This implies that the origami is encoded in the representation $c : \mathbb{F}_2 \mapsto S_d$; $\sigma_r = c(\alpha_r)$, $\sigma_u = c(\alpha_u)$. Due to the connectedness of S', it turns out that $\mathbb{F}_2$ acts transitively on the set $\{1, \ldots, d\}$.

Conversely, If we have a representation $c : \mathbb{F}_2 \mapsto S_d$ having a transitive action on $\{1, \ldots, d\}$, then we simply consider the surface $\tilde{S}/\Gamma$, $\Gamma \subset \mathbb{F}_2$ where Γ is the stabilizer, say of $1 \in \{1, \ldots, d\}$. To summarize :

Proposition 1. Classes of origamis, up to isomorphism, of degree d are in bijection with representations of $\mathbb{F}_2$ into S_d having a transitive action on $\{1, \ldots, d\}$, up to conjugacy in S_d.

Let γ and λ be the vertical and horizontal loops in the torus $\mathbb{T}^2$. Denote by $\{\gamma_1, \ldots, \gamma_n\}$ and $\{\lambda_1, \ldots, \lambda_m\}$ all the possible lifts of γ and λ via O. This defines a pair of filling multi-curves with the same sign intersection at each possible intersection on the surface S. The converse is also true if we have a pair of multi-curves on S that have unchanged sign between the loops then we can construct an origami having the initial loops as horizontal and vertical loops for O. The fact that we have a decomposition into a cylinder allows us to see that some powers of the matrices below belong to the

Veech group of S.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The converse is also true; if a translation surface has a parabolic element A in its Veech group then the surface is decomposable into flat cylinders in the direction of the fixed vector of A. But for square-tiled surfaces, the following holds:

Theorem 2.2.2. *The Veech group of an origami is a finite index subgroup of $SL(2, \mathbb{Z})$.*

Proof. The fact that the image of the holonomy map of a square-tiled surface lies in $\mathbb{Z}^2$ implies that its Veech group is contained in $\mathsf{SL}(2, \mathbb{Z})$. Now if A is in the Veech group of S then we consider the lift $\tilde{A}$ acting on the universal cover $\tilde{S}$ after assuming that $p = 0 \in \mathbb{T}^2$, up to Inner automorphisms of $\mathbb{F}_2$ we can write:

$$A \in \mathsf{Out}^+_{\pi_1(\mathsf{S}')}(\mathbb{F}_2)$$

Where $\mathsf{Out}_{\pi_1(\mathsf{S}')}(\mathbb{F}_2)$ is the group of automorphisms preserving $\pi_1(S')$, which is exactly the Veech group of O. The fact that $\pi_1(S')$ is a finite index subgroup of $\mathbb{F}_2$ implies that $\mathsf{Out}^+_{\pi_1(\mathsf{S}')}(\mathbb{F}_2)$ is of finite index in $\mathsf{Out}^+(\mathbb{F}_2) = \mathsf{SL}(2, \mathbb{Z})$.

□

2.2.8 Constructing pseudo-Anosov homeomorphisms

As we mentioned before, decomposing the square-tiled surface into cylinders allows us to realize a product of some powers of the Dehn twists along the loops generating the cylinders as a parabolic element in the Veech group of the square-tiled surface. Pseudo-Anosov elements can be constructed out of square-tiled surfaces by considering hyperbolic matrices in the Veech group of such surfaces. For instance, we can take products of two parabolic matrices arising from two different directions. But to realize two products of Dehn-twist along two multi-curves without any powers, we need a more subtle result:

Theorem 2.2.3 (Thurston's construction). *Let $\gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\lambda = \{\lambda_1, \dots, \lambda_m\}$ be two filling multi-curves in a surface. Then there exists $\mu \in \mathbb{R}$ and a half-translation surface on S such that:*

$\tau_{\gamma_1} \circ \cdots \circ \tau_{\gamma_n}$ and $\tau_{\lambda_1} \circ \cdots \circ \tau_{\lambda_m}$ are respectively realized by the affine transformations:

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

Proof. For each intersection $p^k_{i,j}$ between γ_i and λ_j consider a rectangle $R^n_{i,j}$ of height h_j and width l_i (the dimensions depend only on the curves γ_i and λ_j), as shown in Figure 5.3 below

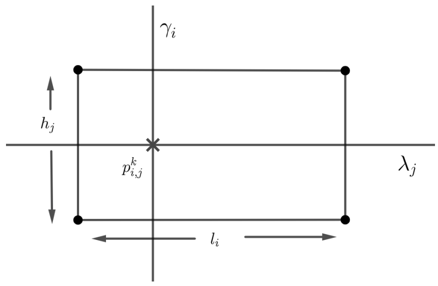


Figure 2.5: The rectangle $R^n_{i,j}$

Let us now consider the matrix $M = (m_{i,j})^{1 \le j \le m}_{1 \le i \le n} \in M_{n,m}(\mathbb{R})$ such that:

$$m_{i,j} = i(\gamma_i \cup \lambda_j)$$

The matrix $\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$ realize the Dehn twist γ_i if and only if

$$\sum_{j=1}^{j=m} m_{i,j}.h_j = \mu.l_i$$

Similarly, the matrix $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ realize the Dehn twist λ_j if and only if

$$\sum_{i=1}^{i=n} m_{i,j}.l_i = \mu.h_j$$

The matrix $M^\intercal M$ is positive, more precisely, if we denote by $(c_{i,j})^{1 \le j \le m}_{1 \le i \le m}$ its coefficients, then $c_{i,j}$ is strictly positive if and only if λ_i and λ_j are connected by some γ_k. By induction, we deduce that the entry (i,j) of $(M^\intercal M)^l$ is strictly positive if and only if there exists a sequence made of $l+1$ curves

$\{\lambda_{r_1}, \ldots, \lambda_{r_{l+1}}\}$ such that $\lambda_{r_1} = \lambda_i$, $\lambda_{r_{l+1}} = \lambda_j$ and λ_{r_k} is connected to $\lambda_{r_{k+1}}$ via some curve in γ. The connectedness of the multi-curve $\gamma \cup \lambda$ implies that some power of $M^\intercal M$ is strictly positive. Frobenius-Perron theorem applies here, therefore $M^\intercal M$ admits a positive eigenvalue t with a positive corresponding eigenvector v. Taking $(h_1, \ldots, h_m)^\intercal = v$, $(l_1, \ldots, l_n)^\intercal = M.v$ and $\mu = \sqrt{t}$ yields the task.

□

2.3 Character varieties

One way of defining the character varieties of a closed surface S is through the notion of flat connections on SU(2)-bundles over the surface; Let $P : X \to S_g$ be an SU(2)-bundle, locally i.e. on a small open set $U_i \subset S$, one can write $p : V_i = U_i \times \mathrm{SU}(2) \to U_i$ where p is simply the projection into the surface component, in such a way that the transition map between V_i and V_j is given by a map $(x, g) \in V_i \mapsto (\phi_{i,j}(x), g_x.g)$, where $\phi_{i,j}$ is the transition homeomorphism on the surface and $g_.$ is a smooth map from U_i to SU(2). It has a meaning to define the map $P^{-1}\{p\} \times P^{-1}\{p\} \to \mathrm{SU}(2)$ which associate to (x, y) the element $y^{-1}x$ that translates y to x. Demanding that the connection preserves this quantity for any x and y in any fiber implies that the connection which is a horizontal distribution H on X is right invariant. Notice that right multiplication by SU(2) is well defined on X. Since H is right invariant, we can describe it using a 1-form $\alpha \in \Omega^1(S, \mathfrak{su}(2))$ that takes values in the Lie algebra $\mathfrak{su}(2)$, for this purpose consider a section s of P, the 1-form α satisfies then $\alpha(T_x S) = H(x, g)(s(x))$ in some local coordinate where $s = 1$. Let X and Y be two vector fields on S such that $[X, Y] = 0$, these two vector fields can be lifted to $\tilde{X} \in H$ and $\tilde{Y} \in H$ which can be written locally as follows: $\tilde{X}(x, g) = X + \alpha(X).g$ and $\tilde{Y}(x, g) = Y + \alpha(Y).g$. The curvature form which is the commutator between $\tilde{X}$ and $\tilde{Y}$ can be computed as follow:

$$[\tilde{X}, \tilde{Y}](x, g) = [X + \alpha(X).g, Y + \alpha(Y).g]$$

Developing the right-hand side we get

$$[X, Y] + [\alpha(X), \alpha(Y)].g + \mathcal{L}_X(\alpha(Y).g) - \mathcal{L}_Y(\alpha(X).g)$$

The fact that the flows of X and Y commute implies that $[\tilde{X}, \tilde{Y}]$ at $(x, 1)$ is equal to:

$$[\alpha(X), \alpha(Y)] + (\mathcal{L}_X \alpha)Y - (\mathcal{L}_Y \alpha)X = [\alpha(X), \alpha(Y)] + d\alpha(X, Y)$$

Hence, the connection is flat if and only if, the curvature form

$$K = d\alpha + \frac{1}{2}[\alpha \wedge \alpha]$$

vanishes. Notice that the 1-form α is defined up to the choice of a section s_1 of P. Let s_2 be another section. Assume on a local coordinate that $s_1 = 1$ and denote $s_2 = s$. Let γ be a curve in S such that $\gamma(0) = x_0$, let $\tilde{\gamma}_1$ and $\tilde{\gamma}_1$ be lifts of γ such that $\tilde{\gamma}_1(0) = 1$, $\tilde{\gamma}_2(0) = s(x_0)$, therefore $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(t).s(x_0)$. We change the coordinates by left multiplying by s^{-1} so that the section s becomes the constant section 1. At these new coordinates we have $\tilde{\gamma}_2(t) = s(\gamma(t))^{-1}.\tilde{\gamma}_1(t).s(x_0)$. This implies that the new form is equal to $Ad_{s^{-1}}\alpha + ds^{-1}.s$ which gives an action of the maps from S of $\mathsf{SU}(2)$ over the space of 1-forms $\Omega^1(S, \mathfrak{su}(2))$ called the Gauge group action.

The holonomy of a 1-form α together with a section gives a representation $\rho : \pi_1(S) \to \mathsf{SU}(2) \in \mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$ by integrating α over the loops of S (taking lifts then evaluating).

If two 1-forms are equivalent by s then the corresponding representations are the same. Changing the base point yields again two conjugated representations. Conversely, if we start with a representation $\rho \in \mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$ then the fundamental group $\pi_1(S)$ acts on the product $\tilde{S} \times \mathsf{SU}(2)$ as follows $\gamma.(x, g) = (\gamma.x, \rho(\gamma).g)$ where $\pi_1(S)$ acts on the universal cover $\tilde{S}$ by Deck transformations on the group $\mathsf{SU}(2)$ by left multiplication. Two representations conjugated by some in $\mathsf{SU}(2)$ produce equivalent flat bundles. Finally, we deduce that the space of flat $\mathsf{SU}(2)$-bundles is identified to $\Omega^1(S, \mathfrak{su}(2))$ up to the Gauge group action, and also identified to the character variety:

$$X(\pi_1(S), \mathsf{SU}(2)) := \mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))/\mathsf{SU}(2)$$

2.3.1 The representation variety as an algebraic variety

The space of all representations from the fundamental group $\pi_1(S)$ to $\mathsf{SU}(2)$ form an irreducible algebraic variety on $\mathsf{SU}(2)^{2g}$, where g is the genus of the surface S. Notice that

$$\pi_1(S_g) = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots \beta_g | \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1\}$$

Or,

$$\pi_1(N_g) = \{\alpha_1, \dots, \alpha_g | \prod_{i=1}^{g} \alpha_i^2 = 1\}$$

In the case of non-orientable surfaces. Hence Hom$(\pi_1(S_g), \mathsf{SU}(2))$ is the algebraic variety in $\mathsf{SU}(2)^{2g}$ defined by the polynomial function

$$\prod_{i=1}^{g} [A_i, B_i] = 1$$

Where $A_1, \dots, A_g, B_1, \dots, B_g$ are the images of $a_1, \dots, a_g, b_1, \dots, b_g$ respectively. Or by the following polynomial, in the non-orientable settings:

$$\prod_{i=1}^{g} A_i^2 = 1$$

Where $A_1, \dots, A_g$ are the images of $\alpha_1, \dots, \alpha_g$. The evaluation maps defined by $f_\gamma(\rho) := tr(\rho((\gamma))$, for $\gamma \in \pi_1(S)$, generate the ring of conjugacy invariant polynomials over the representation variety (See Procesi [Pr]). The group SU(2) acts on Hom$(\pi_1(S), \mathsf{SU}(2))$ and the character variety $X(\pi_1(S), \mathsf{SU}(2))$ is defined to be the algebraic quotient by SU(2), in other words, $\rho_1, \rho_2 \in$ Hom$(\pi_1(S), \mathsf{SU}(2))$ are in the same class if $f_\gamma(\rho_1) = f_\gamma(\rho_2)$, for every $\gamma \in \pi_1(S)$.

2.3.2 Maps between character varieties and relative character varieties

When the surface Σ is compact, then Σ is homeomorphic to $S_{g,n}$ where g is the genus of the surface and n is the number of boundary components. The character variety $X(\pi_1(S_{g,n}), \mathsf{SU}(2))$ is foliated by the levels of the function $(f_{b_1}, \dots, f_{b_n})$, where $b_1, \dots, b_n$ are the boundary components of the surface. Let B be a level of function $(f_{b_1}, \dots, f_{b_n})$, the relative character $X_B(\pi_1(S_{g,n}), \mathsf{SU}(2))$ is defined to be the space all representations in B up to conjugation by SU(2).

Let S_1 be a closed surface and $p_1, \dots, p_n$ be distinct points in S_1. Assume that there exists a ramified cover $P : S_2 \to S_1$ over the points $p_1, \dots, p_n$ of multiplicities $(d_1, d_2, \dots, d_m)$ corresponding to the reciprocal singular points $(q_1, \dots, q_m) = (P^{-1}(p_1), \dots, P^{-1}(p_n))$, using Riemann-Hurwitz formula, the cover surface has Euler characteristic $\chi(S_2) = d.\chi(S_1) - \sum_{i=1}^{i=m}(d_i - 1)$, where d is the cover's degree. Consider now the punctured surfaces $\Sigma_1 = S_1 \setminus \{p_1, \dots, p_n\}$ and $\Sigma_2 = S_2 \setminus \{q_1, \dots, q_m\}$. The restriction of P on Σ_2 is a cover of degree d that induces a map $P_* : \pi_1(\Sigma_2) \mapsto \pi_1(\Sigma_1)$ defining a map between the character varieties:

$$P^* : X(\pi_1(\Sigma_1), \mathsf{SU}(2)) \longrightarrow X(\pi_1(\Sigma_2), \mathsf{SU}(2))$$
$$[\rho] \mapsto [\rho \circ P^*]$$

The fact that $P_*(\pi_1(\Sigma_2))$ is of finite-index in $\pi_1(\Sigma_1)$ implies that P^* is an embedding of $X(\pi_1(\Sigma_1), \mathsf{SU}(2))$ into $X(\pi_1(\Sigma_2), \mathsf{SU}(2))$. More precisely, it embeds the relative character variety $X_B(\pi_1(\Sigma_1), \mathsf{SU}(2))$, where $B = (\kappa_1, \dots, \kappa_n)$ into $X_{B'}(\pi_1(\Sigma_1), \mathsf{SU}(2))$, where $B' = (\mu_1, \dots, \mu_m)$, such that if $P(p_i) = q_j$ then the loop around q_j is sent to the loop around p_i to the power d_j, therefore:

$$\mu_j = 2T_{d_j}(\frac{\kappa_i}{2})$$

Where T_n is the Tchebychev polynomial of degree n.

Remark 2.3.1. In particular, this procedure defines embeddings from the relative character varieties X_B to the character varieties of closed surfaces, provided that all multiplicities over a singular point p_i are all equal and B is of the form

$$(2\cos(2\pi.r_1), \dots, 2\cos(2\pi.r_n))$$

With r_i rational of the form $\frac{k}{d_j}$, for some j such that $P(q_j) = p_i$.

Example 2.3.1. Let us consider the case of origamis in the stratum $\mathcal{H}(2g-2)$. The relative character variety of the punctured torus $X_{2\cos(\frac{\pi}{g})}(\pi_1(S_{1,1}), \mathsf{SU}(2))$, which is diffeomorphic to a two-dimensional sphere (See next chapter for more details), embeds via O into $X(\pi_1(S_g), \mathsf{SU}(2))$.

2.3.3 Tangent spaces and symplectic structure

Fix a smooth curve $d : I \subset \mathbb{R} \to X(\pi_1(S), \mathsf{SU}(2))$, it is possible to quantify the speed of d at a given representation $\rho_0 \in X(\pi_1(S), \mathsf{SU}(2))$; assume that $d(0) = \rho_0$. For any γ we can evaluate d at γ, this yields a curve in $\mathsf{SU}(2)$ that we denote d_γ defined near 0. After left translation by $\rho_0(\gamma)^{-1}$ we get the curve $d_\gamma \rho(\gamma)^{-1}$ that starts at the identity of $\mathsf{SU}(2)$. Now we define

$$c(\gamma) := \frac{\partial}{\partial t} d_\gamma(t).\rho_0(\gamma)^{-1} \mid_{t=0} \in \mathfrak{su}(2)$$

The evaluation at each γ induces a map $c : \pi_1(S) \to \mathfrak{su}(2)$ that takes values in the Lie algebra of SU(2). The map c satisfies the following

$$c(\gamma_1.\gamma_2) = \frac{\partial}{\partial t} d_{\gamma_1.\gamma_2}(t).\rho_0(\gamma_1.\gamma_2)^{-1} \mid_{t=0}$$

Writing $d_{\gamma_1.\gamma_2} = d_{\gamma_1}.d_{\gamma_2}$, we get

$$c(\gamma_1.\gamma_2) = \frac{\partial}{\partial t} d_{\gamma_1}(t) d_{\gamma_2}(t).\rho_0(\gamma_2)^{-1}.\rho_0(\gamma_1)^{-1} \mid_{t=0}$$

This computation shows that

$$c(\gamma_1.\gamma_2) = \mathsf{Ad}_{\rho_0(\gamma_1)} c(\gamma_2) + c_{\gamma_1}$$

The set of 1-cocycles is identified to the tangent space of Hom$(\pi_1(S), \mathsf{SU}(2))$ at ρ_0. If the curve d consists of equivalent representations then there exists a curve $g \in \mathsf{SU}(2)$ such that $d(t) = g(t).\rho_0.g(t)^{-1}$ such that $g(0) = 1$. The associated cocycle c satisfies the following

$$c(\gamma) = \frac{\partial}{\partial t} d_{\gamma}(t)\rho_0(\gamma)^{-1} \mid_{t=0} = \frac{\partial}{\partial t} g(t)\rho_0(\gamma) g(t)^{-1}.\rho_0(\gamma)^{-1} \mid_{t=0}$$

Set v to be the speed of g at $t = 0$, then we get:

$$c(\gamma) = v - \mathsf{Ad}_{\rho_0(\gamma)} v$$

If we denote by $Z^1(S, \mathfrak{su}(2))$ the space of cocycle and by $B^1(S, \mathfrak{su}(2))$ the space of trivial cocycle then the tangent space of the character variety is identified to

$$Z^1(S, \mathfrak{su}(2))/B^1(S, \mathfrak{su}(2))$$

Interpreting this identification in terms of the twisted cohomology group allows the definition of a symplectic form on the character variety; For a surface S consider a triangulation of the surface and denote the vertices of this triangulation by $\{v_1, \ldots, v_m\}$, the edges by $\{e_1, \ldots, e_n\}$ and the faces by $\{f_1, \ldots, f_r\}$. Consider $\tilde{S}$ to be the universal cover of the surface and $\{\tilde{v}_k\}_{k \in K}$, $\{\tilde{e}_i\}_{i \in I}$ and $\{\tilde{f}_j\}_{j \in J}$ to be the lifts of the vertices, edges, and faces, respectively. The fundamental group $\pi_1(S)$ acts on the universal cover by Deck transformations. In particular, it acts on the lifted cells of the triangulation.

Fix a vertex $v_0 \in S$ and a lift $\tilde{v_0} \in \tilde{S}$ and identify the oriented edge $\tilde{e_i}$ relating $\tilde{v_0}$ to $\gamma_i(\tilde{v_0}))$ with $\gamma_i \in \pi_1(S)$ as shown in Figure 2.6 below. Consider $C_i(S, \pi_1(S))$ to be the ring generated by the cells of dimension $i \in \{0, 1, 2\}$ of the triangulation with coefficient in $\pi_1(S)$. Let A be the ring $\pi_1(S)[\mathfrak{su}(2)]$. For a representation $\rho \in \mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$. One has a natural map $r : A \to \mathfrak{su}(2)$ such that $r(\gamma.v) = \mathsf{Ad}_{\rho(\gamma)}v$. Let $C^i(S, A)$ be the space of morphisms from C_i to A and the $C^i(S, \mathsf{Ad}_\rho)$ be the morphisms of $C^i(S, A)$ composed with r. The composition with ∂ defines a sequence on the cohomology :

$$C^0(S, \mathsf{Ad}_\rho) \xrightarrow{d} C^1(S, \mathsf{Ad}_\rho) \xrightarrow{d} C^2(S, \mathsf{Ad}_\rho)$$

Let $c = d(c_0) \in d(C^0(S, \mathsf{Ad}_\rho))$, then one has $c(\gamma_i) = c(\tilde{e_i}) = c_0(\partial\tilde{e_i}) = c_0(\gamma_i(v_0) - v_0) = Ad_{\rho(\gamma_i)}c_0(v_0) - c_0(v_0)$, thus the image spans the space of trivial cocycles $B^1(S, \mathfrak{su}(2))$. If $c \in Ker(d)$, then as shown in Figure 2.6, one has $c(\tilde{e_1}) + c(\gamma_1(\tilde{e_2})) - c(\tilde{e_3}) = c(\gamma_1) + \mathsf{Ad}_{\rho(\gamma_1)}c(\gamma_2) - c(\gamma_1.\gamma_2) = 0$. Therefore the kernel is exactly the space $Z^1(S, \mathfrak{su}(2))$ and the the tangent space at ρ is identified to the cohomology $H^1(S, \mathsf{Ad}_\rho)$.

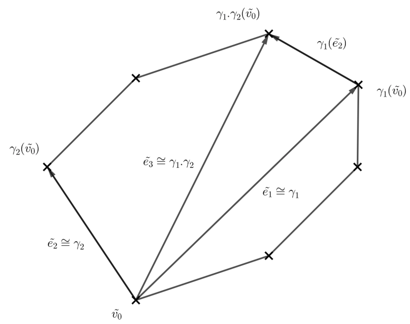


Figure 2.6: The universal cover $\tilde{S}$

When the surface is orientable we can define a 2-form on the character variety, for this purpose consider a bilinear form $\mathbb{B}$ invariant by the adjoint action on Lie algebra $\mathfrak{su}(2)$, the following 2-form defined on the character variety is symplectic (See [G1] for more details).

$$H^1(S, \mathsf{Ad}_\rho) \times H^1(S, \mathsf{Ad}_\rho) \xrightarrow{\cup} H^2(S, \mathsf{Ad}_\rho \otimes \mathsf{Ad}_\rho) \xrightarrow{\mathbb{B}} H^2(S, \mathbb{R}) \cong \mathbb{R}$$

Chapter 3

Dynamics on the moduli space of triangles

3.0.1 Moduli space of Euclidean triangles

On the moduli space of Euclidean triangles $M_{\mathbb{E}^2}$ i.e. the space of all triangles in $\mathbb{E}^2$ with enumerations which we denote by $T_{\mathbb{E}^2}$ up to isometries of the plane, let us consider the following dynamical system; A triangle (A, B, C) is sent to the triangle (A', B', C'), via these simple rules A jumps over B defining A', B jumps over C defining B' and C jumps over A' defining C' (See the Figure 3.1 below), in other words:

$$(A', B', C') = (R_B(A), R_C(B), R_{R_B(A)}(C))$$

Where R_X is the reflection about the point X. This procedure defines a map Φ on the space of Euclidean triangles, that descends to the moduli space $M_{\mathbb{E}^2} := T_{\mathbb{E}^2}/\mathsf{E}(2)$ since the procedure commutes with any transformation of the isometry group $\mathsf{E}(2)$.

The space of Euclidean triangles is identified to $\mathbb{C}^3$ by considering the coordinates of each point of the triangle. The transformation Φ is then identified to a linear map in $\mathsf{GL}(3, \mathbb{C})$:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 2z_2 - z_1 \\ 2z_3 - z_2 \\ 4z_2 - 2z_1 - z_3 \end{pmatrix}$$

We notice that this procedure is decomposable by three steps i.e. Φ is generated by three transfor-

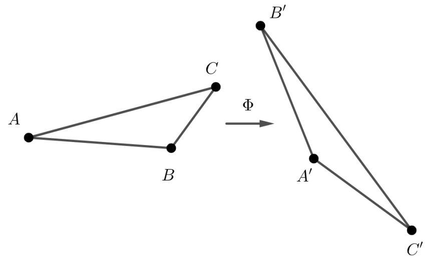


Figure 3.1: Example of a leapfrog

mations; The jump of A over B via the map:

$$\sigma_1(A, B, C) = (R_B(A), B, C)$$

The jump of B over C via the map:

$$\sigma_2(A, B, C) = (A, R_C(B), C)$$

And the jump of C over A via the map:

$$\sigma_3(A, B, C) = (A, B, R_A(C))$$

For example the transformation Φ is the product $\sigma_1 \circ \sigma_2 \circ \sigma_3$.

In the Euclidean plane, the group Γ generated by σ_1, σ_2 and σ_3 acts on the moduli space of triangles preserving the area of the triangles. In other words, the function $(A, B, C) \mapsto (B - A) \wedge (C - A)$ is Γ-invariant. We denote by M_κ the moduli space of Euclidean triangles with volume κ.

Elements of E(2) acts diagonally on $T_{\mathbb{E}^2}$, if we start with the quotient by translations, then the intermediary space $T'_{\mathbb{E}^2}$ which is identified to $\mathbb{C}^2$ is foliated by levels of the area function $\kappa(z_1, z_2) := z_1 \wedge z_2 = Im(\bar{z}_1 z_2)$. Where each level T'_κ is preserved by Γ. Therefore one has:

$$\mathbb{C}^2 \cong T'_{\mathbb{E}^2} = \bigcup_{\kappa \in \mathbb{R}} T'_\kappa$$

The spaces T'_κ, for $\kappa > 0$, are all identical via some real homothety. We notice here that the action of Γ commutes with homotheties. While the space of degenerate triangles $T'_0 \subset T'_{\mathbb{E}^2}$ corresponds in $\mathbb{C}^2$

to the "real linear space" $\{(r_1.z, r_2.z) \mid (z, r_1, r_2) \in \mathbb{C} \times \mathbb{R}^2\}$. Taking the quotient by the $\mathbb{S}^1$ action together with normalizing the volume we get:

$$M_1 \cup M_{-1} = (T'_{\mathbb{E}^2} \setminus T'_0)/\mathbb{C}^* \cong \mathbb{C}P^1 \setminus \mathbb{R}P^1$$

Hence the moduli space M_1 (Or any M_κ,for $\kappa > 0$) is identified to the hyperbolic plane $\mathbb{H}^2$. The actions of σ_1, σ_2 and σ_3 on $T'_{\mathbb{E}^2}$ are given respectively by the linear parabolic matrices on $\mathbb{C}^3 = \mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3$:

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

On the intermediary quotient space $T'_{\mathbb{E}^2} \cong \mathbb{C}^3/\mathbb{C}.(e_1 + e_2 + e_3) \cong \mathbb{C}^2$. The transformations σ_1,σ_2 and σ_3 acts by the following parabolic matrices for the basis $\{\tilde{e}_1, \tilde{e}_2\}$:

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

The group generated by the three elements is exactly SL$(2, \mathbb{Z})$ acting linearly on $\mathbb{C}^2$ which descends to an isometric action on $\mathbb{H}^2$. To summarize:

Proposition 2. For $\kappa > 0$, the moduli space M_κ is identified to the hyperbolic plane, and the action of Γ is nothing but the isometric action of PSL$(2, \mathbb{Z})$ on $\mathbb{H}^2$.

The previous procedure is well-defined for any Lie group G or more generally for any symmetric space X. Since one only needs to define a reflection about any points in G or X.

Can we describe the action of Γ for any symmetric space X, in particular, can one find an analogous for the volume κ in these settings?

3.0.2 The character variety of the punctured torus

Let us now consider the surface $S_{1,1}$ which is a punctured torus, recall that its fundamental group is a free group generated by two elements α and β (See Figure 6.1 below). The mapping class group which is the group of orientation-preserving homeomorphisms that preserve the boundary loop b of $S_{1,1}$ up to isotopies is isomorphic to Out$^+(\mathbb{F}_2)$ = SL$(2, \mathbb{Z})$. The SU(2)-character variety of $S_{1,1}$ denoted by

$X(\mathbb{F}_2, \mathsf{SU}(2))$ is the space of all representations from $\pi_1(S_{1,1}) = \mathbb{F}_2$ to $\mathsf{SU}(2)$ which is identified to $\mathbb{S}^3$ with the standard Lie group structure on it.

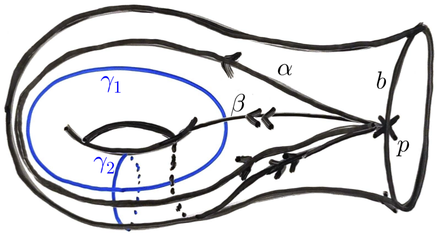


Figure 3.2: The once-punctured $S_{1,1}$

The modular group $\mathsf{SL}(2, \mathbb{Z})$ viewed as the mapping class group has a well-defined action on $X(\mathbb{F}_2, \mathsf{SU}(2))$ by pre-composition i.e. if $[\rho] \in X(\mathbb{F}_2, \mathsf{SU}(2))$ and $[\phi] \in \mathsf{SL}(2, \mathbb{Z})$, then $[\phi].[\rho] = [\rho \circ \phi^{-1}]$. Since the class of the boundary loop b is preserved then the evaluation of the map $\kappa(\rho) = f_b(\rho) := tr(\rho(b))$ is invariant by the action. In fact, the character variety is homeomorphic to a ball B_3 foliated by the levels of κ denoted S_κ which are spheres that reduce to a point when $\kappa = -2$ and corresponds to the abelian representations when $\kappa = 2$. If we consider the evaluation maps f_α, f_β and $f_{\alpha\beta}$, where $f_\gamma(\rho) = tr(\rho(\gamma))$. Then κ can be expressed by means of the previous functions: $\kappa = f_\alpha^2 + f_\beta^2 + f_{\alpha.\beta}^2 - f_\alpha f_\beta f_{\alpha\beta} - 2$. Let us use the parallelogram identity $tr(XY) + tr(XY^{-1}) = tr(X).tr(Y)$ which holds for any $X, Y \in \mathsf{SL}(2, \mathbb{C})$. One has:

$$tr(\rho(\alpha.\beta.\alpha^{-1}\beta^{-1})) + tr(\rho(\alpha.\beta.\alpha^{-1}\beta)) = tr(\rho(\beta)).tr(\rho(\beta)) = f_\beta^2(\rho)$$

$$tr(\rho(\alpha.\beta.\beta^{-1}\alpha)) + tr(\rho(\alpha.\beta.\alpha^{-1}\beta)) = tr(\rho(\alpha.\beta)).tr(\rho(\alpha^{-1}.\beta))$$

$$tr(\rho(\alpha.\beta)) + tr(\rho(\alpha.\beta^{-1})) = tr(\rho(\alpha)) + tr(\rho(\beta)) = (f_\alpha + f_\beta)(\rho)$$

Combining the last three identities, we get:

$$\kappa = f_\beta^2 - f_{\alpha.\beta}.(f_\alpha + f_\beta - f_{\alpha.\beta}) + f_{\alpha^2} = f_\alpha^2 + f_\beta^2 + f_{\alpha.\beta}^2 - f_\alpha f_\beta f_{\alpha\beta} - 2$$

The spheres S_κ can be visualized in $\mathbb{R}^3$ (Figure 3.3) using the following fact (we postpone the proof to the end of the note)

Proposition 3 (Vogt-Fricke). $(f_\alpha, f_\beta, f_{\alpha\beta})$ is an embedding of the character variety $X(\mathbb{F}_2, \mathsf{SU}(2))$ in $[-2,2]^3$ and the image is the semi-algebraic set defined by the inequality $-2 \leq \kappa \leq 2$.

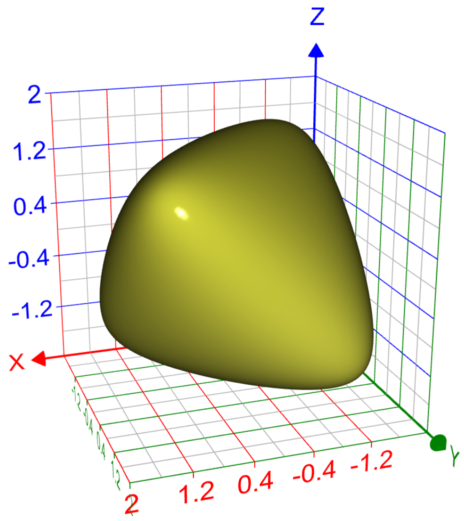


Figure 3.3: The sphere S_1

We can better understand a representation ρ by looking at its κ, for instance, if we consider the set $\Delta \subset \pi_1(S_{1,1})$ that consists of simple closed loops of the punctured torus, then:

Proposition 4. For a representation $[\rho]$, if $\kappa(\rho)$ is close to -2, then the evaluation of elements $\gamma \in \Delta$ i.e. $\gamma \mapsto f_\gamma(\rho)$ on the set Δ is contained in a set of small measure in $[-2,2]$. Equivalently, the set $\{\rho(\gamma) \mid \gamma \in \Delta\} \subset \mathsf{SU}(2)$ is contained in a small cylinder based on the imaginary elements (traceless elements).

Proof. Assume that $-2 < \kappa(\rho) < -2 + \epsilon$ for a small ϵ. Let $\gamma \in \Delta$ then there exists $\phi \in \mathsf{SL}(2,\mathbb{Z})$ such that $\phi(\gamma) = \alpha$. The fact that the sphere S_κ is close to $\{(0,0,0)\} = S_{-2}$ implies that the image set $f_\alpha(S_\kappa)$ is contained in a small neighborhood of zero. Since $\rho \in S_k$ then $\phi.\rho \in S_\kappa$, we conclude the proof by noticing that

$$f_\alpha([\phi.\rho]) = tr(\phi.[\rho](\alpha)) = tr(\rho(\phi^{-1}(\alpha))) = tr(\rho(\gamma)) = f_\gamma(\rho)$$

□

The modular group $\mathsf{SL}(2,\mathbb{Z})$ acts on these spheres by polynomial automorphisms. For instance, the

Dehn-twist τ_{γ_1} along the simple closed curve γ_1 induces the following automorphisms on $\mathbb{F}_2$: $\alpha \mapsto \alpha$ and $\beta \mapsto \beta.\alpha$. Therefore the action τ_{γ_1} seen on the local coordinates $(x, y, z) = (f_\alpha, f_\beta, f_{\alpha.\beta})$ induces the polynomial automorphisms:

$$(x, y, z) \mapsto (x, z, xz - y)$$

The action of hyperbolic elements is by far the most interesting one. In this context, we shall state the result of Brown [Br]:

Theorem (R. Brown)**.** *Let $f \in$ SL$(2, \mathbb{Z})$ be a hyperbolic element. Then there exists $\epsilon > 0$ such that $\forall \kappa \in] - 2, -2 + \epsilon[$, there exists either an elliptic fixed point or an elliptic period 2 point on S_κ.*

An elliptic fixed point p of f is a fixed point by f where the derivative $d_p f$ is conjugated to a rotation. Brown showed that varying κ implies a variation of the rotation angle of the elliptic fixed point. Therefore a theorem by Rüssmann [R] ensures the stability of such fixed points.

This implies the non-ergodicity of hyperbolic elements for the spheres S_κ where κ is close to -2. For a larger group, for instance, if we consider the action of the modular group, Goldman proved that S_κ inherits a symplectic form and he showed [GX1]:

Theorem (W. Goldman)**.** *The group SL$(2, \mathbb{Z})$ acts ergodically on the spheres S_κ with respect to its symplectic measure.*

3.0.3 Moduli space of spherical triangles

Once we discussed the SU(2)-character variety of the punctured torus, we are ready to discuss the moduli space of spherical triangles $M_{\mathbb{S}^2}$. Since $\mathbb{S}^2 =$ SO(3)/SO(2) is a symmetric space then one can define, as in the Euclidean case, the action of the group Γ generated by σ_1,σ_2 and σ_3 where the action of σ_i on the triangles (A_1, A_2, A_3) fixes the components that are different from i and reflects the A_i along the point A_{i+1}, in a cyclic manner. As an example, we consider the transformation $\Phi = \sigma_1 \circ \sigma_2 \circ \sigma_3$ (See Figure 3.4 below)

It is possible to embed the moduli space $M_{\mathbb{S}^2}$ into $\mathbb{R}^3$. If (A, B, C) is a triangle in $\mathbb{S}^2$, then we consider the spherical lengths "or the angles" $\theta_1 = dis_{\mathbb{S}^2}(B, C)$, $\theta_2 = dis_{\mathbb{S}^2}(A, C)$ and $\theta_3 = dis_{\mathbb{S}^2}(A, B)$. A choice of $(\theta_1, \theta_2, \theta_3)$ determines the triangle up to isometries of $\mathbb{S}^2$. Hence, the map from of the moduli space $M_{\mathbb{S}^2}$ to $[0, \pi]^3$ given by the length functions $(\theta_1, \theta_2, \theta_3)$ is an embedding. The

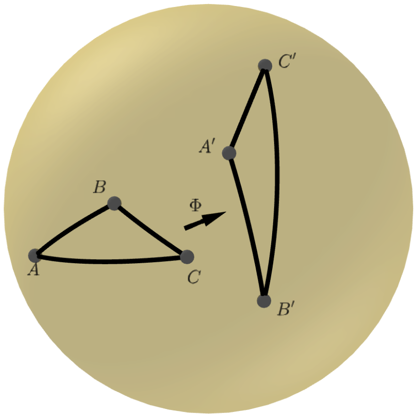


Figure 3.4: The transformation Φ

image is exactly the polyhedra $\mathcal{H}$ (as illustrated in Figure 3.5) defined by the triangle inequalities in the sphere i.e.

$$\begin{cases} \theta_2 + \theta_3 \leq \theta_1 \\ \theta_1 + \theta_3 \leq \theta_2 \\ \theta_1 + \theta_2 \leq \theta_3 \\ \theta_1 + \theta_2 + \theta_3 \leq 2\pi \end{cases}$$

Note that the last inequality is deduced by reflecting the first point of a triangle about the origin and then applying the usual triangle inequality on the new triangle.

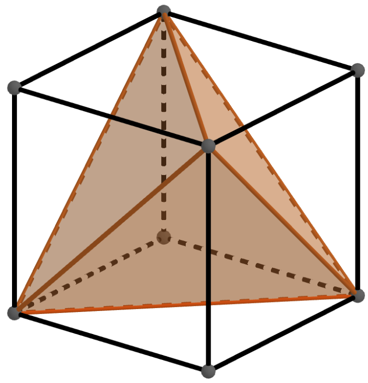


Figure 3.5: Polyhedra $\mathcal{H}$ in $[0,\pi]^3$

For the spherical case, one has the following identification:

Proposition 5. The map from the SU(2)-character variety of the punctured-torus to the moduli space of spherical triangles $M_{\mathbb{S}^2}$ given by $[\rho] \mapsto [1, \rho(\alpha), \rho(\beta)]$ is an identification between the two spaces.

Using this identification we can express the maps σ_1, σ_2 and σ_3 as elements of the modular group of the punctured torus; The map σ_1 sends (A, B, C) to $(R_B(A), B, C)$, so in terms of representations we have:

$$\sigma_1([1, \rho(\alpha), \rho(\beta)]) = [1, \rho(\alpha^{-1}), \rho(\beta.\alpha^{-2})]$$

The map σ_2 sends (A, B, C) to $(A, R_C(B), C)$, in terms of representations, we have:

$$\sigma_2([1, \rho(\alpha), \rho(\beta)]) = [1, \rho(\beta.\alpha^{-1}.\beta), \rho(\beta)]$$

And σ_3 sends (A, B, C) to $(A, B, R_A(C))$, so:

$$\sigma_3([1, \rho(\alpha), \rho(\beta)]) = [1, \rho(\alpha), \rho(\beta^{-1})]$$

Seen on the group $GL(2,\mathbb{Z})$, the action of σ_1,σ_2 and σ_3 corresponds respectively to the action of:

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For instance, Φ acts via the hyperbolic matrix $\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$ on the character variety.

Proposition 6. The action of the group Γ on $M_{\mathbb{S}^2}$ is nothing but the action of a finite index subgroup of $\mathrm{GL}(2,\mathbb{Z})$ on the $\mathrm{SU}(2)$-character variety of the punctured torus $S_{1,1}$.

One can notice that the volume $Area(A,B,C) = det(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC})$ of the simplex $OABC$ is invariant by such a procedure (See Figure 3.6 Below). Therefore the function $Area$ is invariant by the group Γ. Using the result of Goldman, we deduce that the function $Area$ is a reparametrization of κ defined in the previous section.

Now we can see that the sphere of the abelian representations S_2 corresponds to the set of degenerate triangles, while the point S_{-2} corresponds to the right-angled triangle (all the angles are $\frac{\pi}{2}$) which have the maximum volume among the simplices $OABC$ for A, B and C points in $\mathbb{S}^2$.

Proof of Proposition 3: The relation between an element $A \in \mathrm{SU}(2)$ and the spherical distance $dis_{\mathbb{S}^3}(Id, A)$ is given by the trace function; if θ is the distance between 1 and A then $tr(A) = 2cos(\theta)$. Using the identification, we deduce that the image of the map $(f_\alpha, f_\beta, f_{\alpha\beta})$ defined in Proposition 3 is exactly the image of $\mathcal{H}$ by the homeomorphism from $[0,\pi]^3$ to $[-2,2]^3$ given by:

$$(\theta_1, \theta_2, \theta_3) \mapsto (2cos(\theta_1), 2cos(\theta_2), 2cos(\theta_3))$$

It is sufficient now to show that the image of the boundary of the Polyhedra $\mathcal{H}$ is exactly the sphere S_2, we recall that S_2 is defined by the polynomial $x^2+y^2+z^2-xyz-4$. Without loss of generality let us consider the image of the face $\theta_1+\theta_2=\theta_3$ by the homeomorphism. Applying the cosine function we get:

$$cos(\theta_1+\theta_2) = cos(\theta_3)$$

Hence:

$$sin(\theta_1)sin(\theta_2) = cos(\theta_1)cos(\theta_2) - cos(\theta_3)$$

Taking the square now we get:

$$(1-cos^2(\theta_1))(1-cos^2(\theta_2)) = (cos(\theta_1)cos(\theta_2) - cos(\theta_3))^2$$

We replace $2cos(\theta_1)$, $2cos(\theta_2)$ and $2cos(\theta_3)$ by x, y and z, respectively, we get:

$$(1-\frac{x^2}{4})(1-\frac{y^2}{4}) = (\frac{xy}{4}-\frac{z}{2})^2$$

Which is exactly the equation of the sphere S_2.

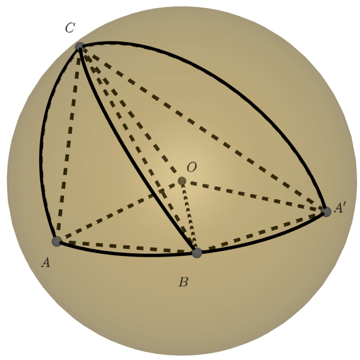


Figure 3.6: Reflection of the simplex $OABC$

Chapter 4

Elliptic fixed points

4.1 Two one-parameter subgroups of parabolic matrices

In this section, we address the question of the existence of elliptic matrices in subgroups generated by two parabolic matrices of the special linear group SL$(2,\mathbb{R})$.

In SL$(2,\mathbb{R})$,conjugacy classes of matrices of SL$(2,\mathbb{R})$ are characterized via their traces i.e two matrices are conjugate if and only if they share the same trace, let $A \in$ SL$(2,\mathbb{R})$

- If $|tr(A)| > 2$ then A has two distinct real eigenvalues λ, λ^{-1}, in this case A is called hyperbolic.
- If $|tr(A)| < 2$ then A is conjugate to a rotation in the Euclidean plane with angle $\arccos(\frac{tr(A)}{2})$, A is then called elliptic.
- If $|tr(A)| = 2$ then A has 1 as eigenvalue and A is called parabolic.

Let $C_1, C_2 : I \mapsto$ SL$(2,\mathbb{R})$ be two different parabolic one-parameter subgroups in $SL(2,\mathbb{R})$, so the two subgroups fix different vectors in $\mathbb{R}^2$, and after changing the basis of $\mathbb{R}^2$, we can write :
$C_1(t) = \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix}$ and $C_2(t) = \begin{pmatrix} 1 & 0 \\ bt & 1 \end{pmatrix}$, for some real numbers a and b.

Let $\mathcal{R}$ be the SL$(2,\mathbb{R})$-character variety of the free group generated by two words A and B. Consider the curve C_t defined in $\mathcal{R}$ as follow, $C_t(A) = C_1(t)$ and $C_t(B) = C_2(t)$.

Question 1. For which word $W \in \langle A, B\rangle$ and $t \in \mathbb{R}$, the matrix $C_t(W) \in$ SL$(2,\mathbb{R})$ is elliptic?

Let us first answer this question near the point $t = 0$, we need to define the following quantity : Let W be a word in the free group $\langle A, B\rangle$ and assume that the word W is written in a minimal way using A,B,A^{-1} and B^{-1}, and denote a_+ the number of appearance of A in W, a_- the number of appearance of A^{-1} in W, b_+ the number of appearance of B in W and b_- the number of appearance of B^{-1} in W. Finally we denote $K_W := \frac{1}{2}(a_+b_+ + a_-b_- - a_+b_- - a_-b_+) \in \mathbb{Q}$.

Proposition 7. If abK_W is negative, then there exists a small neighborhood of 0 such that $C_t(W)$ is elliptic, for t in the neighborhood.

Proof. It is sufficient to study the behavior of the polynomial function

$$t \mapsto \text{tr}(C_t(W))$$

near 0; so using the Leibniz rule we get :

$$\frac{\partial}{\partial t}\text{tr}(C_t(W)|_{t=0} = tr(\sum_{L\in W}\frac{\partial}{\partial t}L(t))|_{t=0} = 0$$

Where L spans the letters of W. Using the fact that for any letter L in W, the second derivative at zero vanishes. If L and M are two letters is W then the quantity $tr((\frac{\partial}{\partial t}L(t)|_{t=0}).(\frac{\partial}{\partial t}N(t)|_{t=0}))$ is equal to $ab, -ab, ab, -ab$, whenever $\{L, M\} = \{A, B\}$, $\{L, M\} = \{A^{-1}, B\}$, $\{L, M\} = \{A^{-1}, B^{-1}\}$, $\{L, M\} = \{A, B^{-1}\}$, respectively, thus;

$$\frac{\partial^2}{\partial t^2}\text{tr}(C_t(W)|_{t=0} = tr(\sum_{L\in W, N\in W}(\frac{\partial}{\partial t}C_t(L)|_{t=0}).(\frac{\partial}{\partial t}C_t(N)|_{t=0})) = abK_W$$

Negativity of abK_W implies that $t \mapsto \text{tr}(C_t(W))$ is concave, hence $C_t(W)$ is elliptic in a neighborhood of 0.

□

For instance, we can check that

$$C_t(AB^{-1}) = \begin{pmatrix} 1 & 0 \\ bt & 1 \end{pmatrix}\begin{pmatrix} 1 & -at \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -at \\ bt & 1 - abt^2 \end{pmatrix}$$

or

$$C_t(AB) = \begin{pmatrix} 1 & 0 \\ bt & 1 \end{pmatrix}\begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & at \\ bt & 1 + abt^2 \end{pmatrix}$$

is elliptic, whenever $0 < |t| < \frac{2}{\sqrt{|ab|}}$. Hence we deduce that we have an elliptic matrix whenever

$0 < |t| < \frac{2}{\sqrt{ab}}$.

In fact, except at the parameter $|t| = \frac{2}{\sqrt{|ab|}}$, this condition is necessary for the existence of elliptic matrices in the image of C_t. To see this we focus on the action of the group $\langle C_1(t), C_2(t)\rangle$ on the hyperbolic plane $\mathbb{H}^2$. Let us assume that $|t| > \frac{2}{\sqrt{ab}}$. The hyperbolic polygon P, as shown in Figure 4.1, is a Dirichlet region for $\langle C_1(t), C_2(t)\rangle$ with an infinite area and the quotient is a pair of pants, therefore :

Proposition 8. The group $\langle C_1(t), C_2(t)\rangle$ is a torsion free Fuchsian group if $|t| > \frac{2}{\sqrt{|ab|}}$.

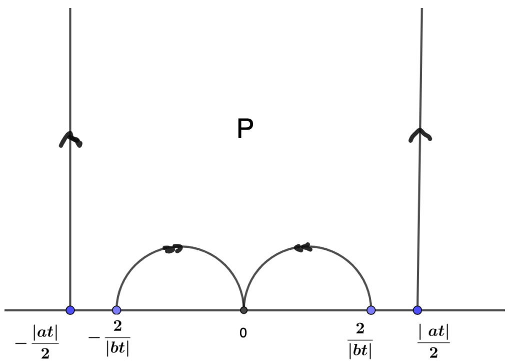


Figure 4.1: Dirichlet Region P

Now let us consider Another situation that is more or less equivalent to the discussion before. Let P and R be two polynomial functions defined on an open interval I around 0 such that P and R and $P.R$ are not constant and P vanishes at 0. For $t \in I$, set $C_1(t) = \begin{pmatrix} 1 & 0 \\ P(t) & 1 \end{pmatrix}$ and $C_2(t) = \begin{pmatrix} 1 & R(t) \\ 0 & 1 \end{pmatrix}$. If $PR > 0$, for $t > 0$, up to change of basis, we can assume that:

$$C_1(t) = \begin{pmatrix} 1 & \mu(t) \\ 0 & 1 \end{pmatrix}$$

and,

$$C_2(t) = \begin{pmatrix} 1 & 0 \\ \mu(t) & 1 \end{pmatrix}$$

Otherwise i.e $PR < 0$, we can assume that:

$$C_1(t) = \begin{pmatrix} 1 & -\mu(t) \\ 0 & 1 \end{pmatrix}$$

and,

$$C_2(t) = \begin{pmatrix} 1 & 0 \\ \mu(t) & 1 \end{pmatrix}$$

Using the previous discussion and seeing μ as the variable instead of t, we deduce that:

Proposition 9. When $t > 0$, if the sign K_W is opposite to the sign of PR, then $C_t(W)$ is elliptic in a small neighborhood of 0,

4.2 SU(2)-Character variety of the one punctured torus $S_{1,1}$

Let $S_{1,1}$ be the punctured torus, its fundamental group is a free group generated by two elements γ_1 and γ_2. The character variety $X(\mathbb{F}_2, \mathsf{SU}(2))$ is defined to be the space of all representations of $\mathbb{F}_2$ into SU(2) up to conjugacy of SU(2).

As defined before, if $\gamma \in \mathbb{F}_2$, we define the evaluation function $f_\gamma([\rho]) := \mathrm{tr}(\rho(\gamma))$ of $[\rho] \in X(\mathbb{F}_2, \mathsf{SU}(2))$ at γ. Let us consider the functions $x = f_{\gamma_1}$, $y = f_{\gamma_2}$, $z = f_{\gamma_1.\gamma_2}$. The mapping class group of $S_{1,1}$ acts on the character variety preserving the conjugacy class of the boundary loop b, therefore the trace of the boundary loop which is denoted by $\kappa := f_b$ is invariant under the action of the mapping class group. Since $b = [\gamma_1, \gamma_2]$, then κ can be expressed by means of the previous functions $x = f_{\gamma_1}$, $y = f_{\gamma_2}$, $z = f_{\gamma_1.\gamma_2}$ using the following identity that holds in SL(2, $\mathbb{C}$), if $M, N \in$ SL(2, $\mathbb{C}$) then

$$tr(M.N^{-1}) + tr(M.N) = tr(M).tr(N)$$

So we get that

$$\kappa = x^2 + y^2 + z^2 - xyz - 2$$

The levels of κ denoted S_κ are two-dimensional spheres except for the level $\kappa = -2$ where it reduces to a point. (Figure 02).

In particular, the level $\kappa = 2$ is the abelian locus of the character variety $X(\mathbb{F}_2, \mathsf{SU}(2))$ i.e those representations for which $\rho(\gamma_1)$ and $\rho(\gamma_2)$ commute and since we are working on SU(2), then $\rho(\gamma_1)$

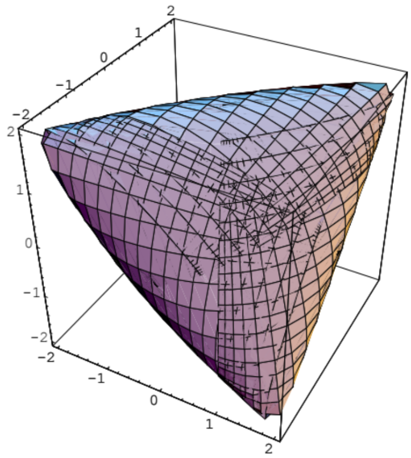


Figure 4.2: Tetrahedral Pillow

and $\rho(\gamma_2)$ lie inside the same one parameter subgroup. Up to conjugacy by some element in SU(2), one can always assume that the image of the representation lies in a fixed one-parameter subgroup H, therefore we can identify the abelian locus with Hom$(\mathbb{Z}^2, \mathbb{S}^1)/\langle g \rangle$, where g is any matrix that belongs to the normalizer of H. This gives an identification between the abelian locus and $\mathbb{T}^2/\langle -Id \rangle$ which is homeomorphic to a sphere, as illustrated in Figure 3.

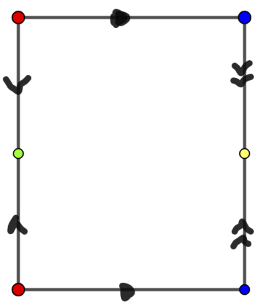


Figure 4.3: Abelian Locus S_2

The irreducible locus of $X(\mathbb{F}_2, \text{SU}(2))$ is exactly $\kappa^{-1}([-2, 2[)$.

Proposition 10 (Vogt-Fricke). $\Phi = (f_{\gamma_1}, f_{\gamma_2}, f_{\gamma_1.\gamma_2})$ is an embedding of the character variety $X(\mathbb{F}_2, \text{SU}(2))$ into $[-2, 2]^3$.

.

The mapping class group acts on the κ levels since it preserves κ and it turns out that the action of $Mod(S_{1,1})$ is ergodic on S_κ. [GX1].

Theorem 4.2.1 (W. Goldman). Mod($S_{1,1}$) *acts ergodically on the spheres* S_κ *with respect to its symplectic measure.*

We say that a diffeomorphism admits an elliptic fixed point p, if p is fixed by f and the derivative at p of f is elliptic.

R. Brown Studied the dynamics of a single pseudo-Anosov homeomorphism f (See [Br]) and he proved the existence of some elliptic fixed points for f near the singular point $\kappa = -2$. The idea of Brown relies on the fact that if the pseudo-Anosov f fixes a point in the Abelian locus then the derivative of f at this point should be hyperbolic and from the other side the point $(0,0,0)$ ($\kappa = -2$) is fixed by f and the derivative of f lies in SO(3), once we use the trace coordinates. So f should admit an elliptic fixed point of any angle near $\kappa = -2$. Together with the study of the set of fixed points of f, he showed :

Theorem 4.2.2 (R. Brown). *Let* $f \in$ Mod($S_{1,1}$) *be a pseudo-Anosov. Then there exists* $\epsilon > 0$ *such that* $\forall \kappa \in]-2, -2+\epsilon[$, *there exists either an elliptic fixed point or an elliptic period* 2 *point on* S_κ.

Mateus noticed that since S_κ is a two-dimensional sphere and f is preserving the area induced by ω, then one can use Rüssmann theorem to deduce non-ergodicity of a pseudo-Anosov homeomorphism action near $\kappa = -2$. (See [R] for more details on the subject)

Theorem 4.2.3 (H. Rüssmann). *Any Brjuno elliptic periodic point of a real-analytic area-preserving map is stable.*

Here, we point out that Brjuno numbers are of full measure in U(1).

4.3 Powers of Dehn twists

The action of some power of a single Dehn twist on $X(\pi_1(S_{1,1}), \mathrm{SU}(2))$ or on $X(\pi_1(S_{0,4}), \mathrm{SU}(2))$ fixes a "large" set of representations and even if we consider two Dehn twists, the set of fixed points of the group generated by these powers is still considerable and it would be reasonable to ask whether there exist some elliptic point among those fixed points, for some pseudo-Anosov homeomorphism in such groups.

4.3.1 Once-punctured torus

In the case of the once punctured-torus, let τ_1 to be the Dehn twist along γ_1, τ_2 to be the Dehn twist along γ_2 and consider the group $\Gamma_n = \langle \tau_1^n, \tau_2^4 \rangle$ with $n > 2$.

In fact, τ_1 and τ_2 have a simple action on the algebraic level; $\tau_1^*(\gamma_1, \gamma_2) = (\gamma_1, \gamma_1.\gamma_2)$ and $\tau_2^*(\gamma_1, \gamma_2) = (\gamma_2.\gamma_1, \gamma_2)$ and we can express their dynamics on the trace coordinates $x = f_{\gamma_1}$, $y = f_{\gamma_2}$, $z = f_{\gamma_1.\gamma_2}$. Hence one would have that the two Dehn twist acts by polynomial automorphisms in $\mathbb{R}^3$, once seen in these coordinates, and we have

$$\tau_1(x, y, z) = (x, z, xz - y)$$

and,

$$\tau_2(x, y, z) = (z, y, yz - x)$$

We can write τ_1 in a convenient way noticing that if x is fixed, then τ_1 becomes linear i.e

$$\tau_1(x, y, z) = (x, M_x \begin{pmatrix} y \\ z \end{pmatrix})$$

where $M_x = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$. Now τ_1^n can be expressed as follows

$$\tau_1^n(x, y, z) = (x, M_x^n \begin{pmatrix} y \\ z \end{pmatrix})$$

We notice here that M_x is elliptic, and if $x_0 = f_{\gamma_1}(\rho) = 2\cos\left(\frac{2\pi p}{n}\right)$ for some $p \in \mathbb{Z}$, then $[\rho]$ is fixed by τ_1^n. As a consequence

Proposition 11. Let L_p be the segment line defined to be the intersection between the line $x = 2\cos\left(\frac{2\pi p}{n}\right)$, $y = 0$ with the character variety $X(\mathbb{F}_2, \mathsf{SU}(2))$. Then L_p consists of fixed points of Γ_n.

Along the points of L_p, we have that

$$\frac{\partial}{\partial x} \begin{pmatrix} x \\ M_x^n \begin{bmatrix} y \\ z \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ \left(\frac{\partial}{\partial x} M_x^n\right) \begin{bmatrix} y \\ z \end{bmatrix} \end{pmatrix}$$

we also have

$$\frac{\partial}{\partial y}\begin{pmatrix} x \\ M_x^n \begin{bmatrix} y \\ z \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and,

$$\frac{\partial}{\partial z}\begin{pmatrix} x \\ M_x^n \begin{bmatrix} y \\ z \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Define the polynomial vector $\begin{pmatrix} P(y,z) \\ Q(y,z) \end{pmatrix} := (\frac{\partial}{\partial x} M_x^n) \begin{bmatrix} y \\ z \end{bmatrix}$, and deduce that the derivative of τ_1^n at any point in the segment L_p is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ P & 1 & 0 \\ Q & 0 & 1 \end{pmatrix}$$

Similarly, if we set $\begin{pmatrix} R(x,z) \\ T(x,z) \end{pmatrix} := (\frac{\partial}{\partial y} M_y^4) \begin{bmatrix} x \\ z \end{bmatrix}$, we get that the derivative of τ_2^4 at a point in the segment L_p is of the form

$$\begin{pmatrix} 1 & R & 0 \\ 0 & 1 & 0 \\ 0 & T & 1 \end{pmatrix}$$

If placed on the segment L_p, in particular, we have that $y = 0$. We notice here that the polynomial

$$P(0,z) = (\frac{\partial}{\partial x} M_x^n) \begin{bmatrix} 0 \\ z \end{bmatrix}$$

vanishes at $z = 0$. Therefore the setting of the first section applies here, by considering the first 2×2 blocks of $d\tau_1$ and $d\tau_2$ i.e $\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$, respectively.

Now let f be a pseudo-Anosov element in the group Γ_n, we express f as a minimal word using the letters τ_1^n and τ_2^4 and define K_f as in section 2. Along the segment L_p in a small neighborhood of $z = 0$, when $z > 0$, assume that $P.R.K_f$ is of negative sign, and denote by $\epsilon_n \in \{-1, 1\}$ the sign of PR for $z > 0$ close to 0. Since $P(0, z)$ converges to 0 when z converges to 0, hence applying Proposition 1 and Remark 3.1, we deduce that df along the segment L_p in a small neighborhood of $z = 0$, when $z > 0$, is an elliptic matrix with a non-constant angle depending on z. Applying Rüssmann theorem on those levels κ near $4\cos^2(\frac{2\pi p}{n}) - 2$, we deduce that f does not act ergodically on S_κ. To summarize:

Theorem 4.3.1. *Let f be a pseudo-Anosov element in Γ_n, such that K_f is of opposite sign to ϵ_n (the sign of PR in a small neighborhood of L_p near $z = 0$, for $z > 0$). Then f does not act ergodically on the corresponding levels S_κ which are close to the level $\kappa = 4\cos^2(\frac{2\pi p}{n}) - 2$, moreover if we vary n, p such κ's cover a dense subset of positive measure in $[-2, 2]$.*

4.3.2 Four-punctured sphere

A similar result to the previous one holds in the case of the four-punctured sphere. The four-punctured sphere $S_{0,4}$ is a topological sphere with 4 open disks removed, its fundamental group is generated by the boundary loops $\alpha, \beta, \gamma, \delta$ of S and it is isomorphic to the free group $\mathbb{F}^3$ and it can be expressed as follows:

$$\pi_1(S_{0,4}) = \{\alpha, \beta, \gamma, \delta | \alpha\beta\gamma\delta = 1\}$$

The mapping class group of $S_{0,4}$, $\mathrm{Mod(S_{0,4})}$ is generated by the Dehn twists along the simple closed loops in $S_{0,4}$ which are the loops $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$.

Using the trace coordinates, the $SU(2)$-character varieties of the four punctured sphere $X(\pi_1(S_{0,4}), SU(2))$ can be embedded in $\mathbb{R}^7$ as an algebraic variety. If we denote $a = f_\alpha$, $b = f_\beta$, $c = f_\gamma$, $d = f_\delta$, $x = f_{\alpha\beta}$, $y = f_{\beta\gamma}$ and $z = f_{\alpha\gamma}$, then $X(\pi_1(S_{0,4}), SU(2))$ is the zero locus in $[-2, 2]^7$ of:

$$x^2 + y^2 + z^2 + xyz = px + qy + rz + s$$

Where, $p = ab + cd$, $q = bc + ad$, $r = ac + bd$ and $s = 4 - a^2 - b^2 - c^2 - d^2 - abcd$. Since the mapping class group preserves the boundary loops of $S_{0,4}$ then the parameter $\tau = (a, b, c, d)$ is an invariant function, hence the mapping class group acts on the levels of τ which is an algebraic surface, in this case, denoted $S_{(a,b,c,d)}$.

If we consider the Dehn twist along $\alpha\beta$ then its action on the relative character variety $S_{(a,b,c,d)}$

using the coordinates $a = f_\alpha$, $b = f_\beta$, $c = f_\gamma$, $d = f_\delta$, $x = f_{\alpha\beta}$, $y = f_{\beta\gamma}$ and $z = f_{\alpha\gamma}$, is as follow:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ (x^2-1)y + xz + q - xr \\ xy - z + r \end{pmatrix}$$

If x is fixed, then it becomes an affine map i.e:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ A_x \begin{pmatrix} y \\ z \end{pmatrix} \end{pmatrix}$$

Where $A_x.V = \begin{pmatrix} (x^2-1) & x \\ x & -1 \end{pmatrix}.V + \begin{pmatrix} q - xr \\ r \end{pmatrix}$. We notice that the linear part of A_x is elliptic.

Dehn twists action

In general, the action of a Dehn twist along a separating curve c can be expressed in a simple way by writing:

$$\pi_1(S) = \{\pi_1(\Sigma_1), \pi_1(\Sigma_1) \mid c_1 = c_2\}$$

here Σ_1 and Σ_2 are the resulting surfaces after cutting along c, where c_1 and c_2 are their boundary components respectively. Hence the Dehn twist along α acts on the character variety as follows:

$$\tau_c^*(\rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \rho(c_2).\rho(\gamma).\rho(c_2)^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2) \end{cases}$$

This ensures that if $f_c(\rho) = 2\cos(\frac{2\pi p}{n})$ then ρ is a fixed point of τ_c^n.

Consider the group $\Gamma_n = \langle \tau_{\alpha\beta}^n, \tau_{\beta\gamma}^4 \rangle$, denote by L_p a line contained in the intersection of a relative character variety where $c = d = 0$ with the subspace $x = 2cos(\frac{2\pi p}{n})$, $y = 0$. The last paragraph ensures that :

Proposition 12. L_p consists of fixed points of Γ_n.

Before jumping to the computations of the derivative of $\tau_{\alpha\beta}^n$ and $\tau_{\beta\gamma}^4$, we notice that over the locus where $c = d = 0$, both r and q vanishes, therefore the affine map A_x becomes a linear elliptic map.

Similarly to the case of the punctured torus, along the line segment L_p and using the fact that $A_x^n = Id$ we get that:

$$d\tau_{\alpha\beta}^n = \begin{pmatrix} 1 & 0 & 0 \\ P & 1 & 0 \\ Q & 0 & 1 \end{pmatrix}$$

Where P and Q are polynomial functions such that

$$\begin{pmatrix} P(0,z) \\ Q(0,z) \end{pmatrix} = (\frac{\partial}{\partial x} A_x^n) \begin{pmatrix} 0 \\ z \end{pmatrix}$$

On the other hand, we have that

$$d\tau_{\beta\gamma}^4 = \begin{pmatrix} 1 & R & 0 \\ 0 & 1 & 0 \\ 0 & S & 1 \end{pmatrix}$$

Where R and S are polynomial functions such that

$$\begin{pmatrix} R(x,z) \\ S(x,z) \end{pmatrix} = (\frac{\partial}{\partial y} A_y^4) \begin{pmatrix} x \\ z \end{pmatrix}$$

We notice here that $P(0,0) = 0$ since A_x is linear, and we deduce in a similar way to the discussion of the punctured torus case, the following:

Theorem 4.3.2. *Let f be a pseudo-Anosov element in Γ_n, such that K_f is of opposite sign to PR (the sign here depends only on n) in a small neighborhood of L_p near $z = 0$, for $z > 0$, then f does not act ergodically on the corresponding levels $S(a,b,0,0)$ for a subset of parameters of positive measure on L_p.*

4.4 Constructing elliptic fixed points for the twice-punctured torus

Let α, β be two curves in a surface S such that the geometric intersection between them is 1, we denote the levels of f_α defined on $X(\pi_1(S), \mathsf{SU}(2))$ by L_α. The following fact will be useful for what is next:

Proposition 13. If $f_{\alpha\beta}([\rho]) = 0, f_\alpha = 0$ and $f_\beta = 2cos(\frac{2\pi p}{n})$, then ρ is a fixed point for τ_β^n and $D_{[\rho]}\tau_\beta^n(T_{[\rho]}L_\alpha) = T_{[\rho]}L_\alpha$, for all $n > 2$.

Proof. The fact that ρ is a fixed point follows from the HNN extension by writing: $\pi_1(S) = \{\pi_1(S \setminus \beta) \bigcup d \mid d\beta^+ d^{-1} = \beta^-\}$ here β^+ and β^- are the resulting boundary loops after cutting along β. So we can write

$$\tau_\beta^*(\rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(S \setminus \beta) \\ \rho(\gamma).\rho(\alpha^+) & \text{if } \gamma = d \end{cases}$$

Now it is obvious that ρ is fixed whenever $f_\beta = 2cos(\frac{2\pi p}{n})$.

It is possible to add a system of coordinates $(f_{\gamma_1}, \dots, f_{\gamma_N})$, for some $N \in \mathbb{N}$, to f_α, f_β and $f_{\alpha\beta}$ such that the γ_i's do not intersect β which implies that $(f_{\gamma_1}, \dots, f_{\gamma_N})$ is invariant by τ_β, now it is sufficient to reduce the proof to the case of the one-punctured torus and using the computation of the derivative in the last section we see that the condition $f_{\alpha\beta}([\rho]) = 0$ ensures that $D_{[\rho]}\tau_\beta^n(T_{[\rho]}L_\alpha) = T_{[\rho]}L_\alpha$. □

Let us consider the following configuration, as shown in Figure 4, in a twice-punctured torus $S_{1,2}$.

We define the two subgroups of the mapping class group on $S_{1,2}$:

$$\Gamma_n = \langle \tau_\delta^4, \tau_\gamma^n \rangle$$

and,

$$\Delta = \langle \tau_\alpha^4, \tau_\beta^4 \rangle$$

Where α, β, γ and δ are curves in $S_{1,2}$, as shown in Figure 4. We notice here that the four curves fill the surfaces $S_{1,1}$. So one can generate pseudo-Anosov homeomorphisms from the group $\langle \Gamma_n, \Delta \rangle$. One way to do so would be to use Fathi's theorem:

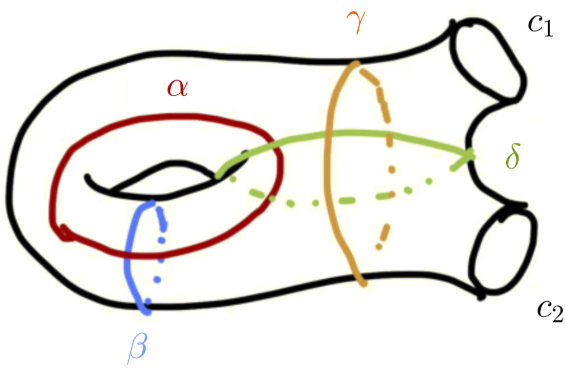


Figure 4.4: System of curves of $S_{1,2}$

Theorem 4.4.1. *Let us consider a family of distinct curves* $\{\gamma_1, ..., \gamma_n\}$ *filling* S. *Then* $\exists N \in \mathbb{N}$ *such that* $\forall (n_1, ..., n_k) \in \mathbb{Z}^k$, *if* $|n_i| > N$, $\forall i$. *Then* $\tau_{\gamma_1}^{n_1} \circ \cdots \circ \tau_{\gamma_k}^{n_k}$ *is a pseudo-Anosov homeomorphism.*

Fix $\Phi \in \langle \Gamma_n, \Delta \rangle$ a pseudo-Anosov homeomorphism of the form $\Phi = f_1 \circ f_2$ for some $f_1 \in \Gamma_n$ that satisfies the conditions of theorem 5.2 once seen on a small neighborhood of γ with δ and $f_2 \in \Delta$ that satisfies the conditions of theorem 5.1 once seen on a small neighborhood of α and β.

4.4.1 The character variety

The fundamental group of the twice-punctured torus $S_{1,2}$ is isomorphic to a free group with three generators, it can be expressed as follows:

$$\pi_1(S_{1,2}) = \{\alpha, \beta, c_1, c_2 \mid [\alpha, \beta] = c_1.c_2\}$$

Since the fundamental group is a free group of three generators, the character variety of the twice punctured torus $X(\pi_1(S_{1,1}), \mathsf{SU}(2))$ is the same as the character variety of the four-punctured sphere.

Let us denote $x = f_\alpha$, $y = f_\beta$, $z_1 = f_{\alpha.\beta}$, $\kappa_1 = f_{c_1}$, $\kappa_2 = f_{c_2}$, $z_2 = f_{\alpha.c_2}$, $z_3 = f_{\beta.c_2}$ and $w = f_{\beta.\alpha.c_2}$. Then $X(\pi_1(S_{1,1}), \mathsf{SU}(2))$ is embedded in $\mathbb{R}^8$, in addition, it is defined by the zero locus of:

$$y.z_3 - x.z_3.z_1 + w.z_1 + x^2.\kappa_2 - x.z_2 - \kappa_2 = \kappa_1$$

and,

$$z_1^2 + z_2^2 + z_3^2 + z_1 z_2 z_3 = p.z_1 + q.z_2 + r.z_3 + s$$

where, $p = xy + \kappa_2.w$, $q = x.\kappa_2 + y.w$, $r = y.\kappa_2 + xw$ and $s = 4 - y^2 - x^2 - \kappa_2^2 - w^2 - xy.\kappa_2 w$.

Now the relative character varieties $X_{(\kappa_1,\kappa_2)}(\pi_1(S_{1,2}), \mathsf{SU}(2))$ are exactly the levels of (κ_1, κ_2) i.e. the algebraic variety in $\mathbb{R}^6$ of dimension 4 defined by the previous polynomials.

Theorem 4.4.2. *For a homeomorphism $f_2 \in \Delta$ satisfying the conditions of Theorem 4.3.1, there exists $N \in \mathbb{N}$ such that for any $n > N$, for any $f_1 \in \Gamma_n$ satisfying the conditions of Theorem 4.3.2, the homeomorphism $\Phi = f_1 \circ f_2$ is pseudo-Anosov. Moreover, Φ admits a line of elliptic fixed points of irrational frequency vector over the corresponding relative character varieties $X_B(\pi_1(S_{1,2}), \mathsf{SU}(2))$.*

Remark 4.4.1. Here, we mean by an elliptic fixed point of a symplectomorphism on a $2n$-dimensional symplectic manifold, a fixed point p for f, where the derivative at p has a spectrum of the form $\{e^{2\pi i \omega_j} : 1 \leq j \leq n\}$ for some frequency vector $\omega = (\omega_1, \dots, \omega_n)$.

Consider the representation $[\rho_0] \in X(\pi_1(S_{1,2}), \mathsf{SU}(2))$ such that $\rho_0(\alpha) = i$, $\rho_0(\beta) = j$ and $\rho_0(c_1) = \rho_0(c_2) = i$. At ρ_0 we have $f_\alpha = f_\beta = f_\delta = f_{\delta.\beta} = f_{\alpha\beta\alpha^{-1}c_2} = f_{\alpha.\beta} = 0$ and $f_\gamma = -2$. We notice that $[\rho_0]$ is an irreducible representation.

Lemma 1. There exist a small neighborhood of $[\rho_0]$ for which :

$$(f_\alpha, f_\beta, f_\delta, f_{\delta.\alpha}, f_{\alpha\beta\alpha^{-1}c_2}, f_{\alpha.\beta})$$

is a homeomorphism into a small neighborhood of $(0, 0, 0, 0, 0, 0)$.

Proof. The map

$$f : X(\pi_1(S_{1,2}), \mathsf{SU}(2)) \longrightarrow [-2, 2]^6$$
$$[\rho] \mapsto (f_\alpha(\rho), f_\beta(\rho), f_\delta(\rho), f_{\delta.\alpha}(\rho), f_{\alpha\beta\alpha^{-1}c_2}(\rho), f_{\alpha.\beta}(\rho))$$

is continuous, it would be sufficient to show that f admits an inverse in a neighborhood of $[\rho_0]$. Let us consider $(x, y, d_1, d_2, d_3, z) \in [-2, 2]^6$ near $(0, 0, 0, 0, 0, 0)$, for the case of $\mathbb{F}_2$ we know, using

Proposition 2 (Vogt-Fricke), that the coordinates (x, y, z) give local coordinates for $X(\mathbb{F}_2, \mathrm{SU}(2))$. i.e for (x, y, z) near $(0, 0, 0)$ there exists exactly one representation $[\rho'] \in X(\mathbb{F}_2, \mathrm{SU}(2))$ for which $f_\alpha = x$, $f_\beta = y$, $f_{\alpha.\beta} = z$. We want to prove that for $[\rho'] \in X(\mathbb{F}_2, \mathrm{SU}(2))$ there exists unique extension $[\rho] \in X(\pi_1(S_{1,2}), \mathrm{SU}(2))$ near $[\rho_0]$, knowing that $(f_\delta(\rho), f_{\delta.\alpha}(\rho), f_{\alpha\beta\alpha^{-1}c_2}(\rho)) = (d_1, d_2, d_3)$. To do so, we write the curves $(\delta, \delta.\alpha, \alpha.\beta.\alpha^{-1}.c_2)$ using the generators $\langle \alpha, \beta, c_2 \rangle = \pi_1(S_{1,2})$ i.e

$$(\delta, \delta.\alpha, \alpha.\beta.\alpha^{-1}.c_2) = (c_2.\beta, c_2.\beta.\alpha, \alpha.\beta.\alpha^{-1}.c_2)$$

Hence (d_1, d_2, d_3) determines the spherical distances in $\mathrm{SU}(2)$:

$$(d(\rho(c_2^{-1}), \rho(\beta)), d(\rho(c_2^{-1}), \rho(\beta.\alpha)), d(\rho(c_2^{-1}), \rho(\alpha.\beta.\alpha^{-1})))$$

Generically the intersection of three spheres is two points. For $(0, 0, 0, 0, 0, 0)$ we have exactly two representations $[\rho_0]$ and $[\rho'_0]$ such that $\rho'_0(\alpha) = i$, $\rho'_0(\beta) = j$, $\rho'_0(c_2) = -i$. Finally, we deduce that f is a bijection from a small neighborhood of $[\rho_0]$ into a small neighborhood of $(0, 0, 0, 0, 0, 0)$.

□

Now we are ready to prove the second part of Theorem 6.2:

Proof. Using the coordinate system $(f_\alpha, f_\beta, f_\delta, f_{\delta.\alpha}, f_{\alpha\beta\alpha^{-1}c_2}, f_{\alpha.\beta})$ around the point $[\rho_0]$. We consider for sufficiently big n the line :

$$L = \{(0, 0, 0, 0, t, 2\cos(\frac{(n-1)\pi}{2n})) \mid 0 < t < \epsilon\}$$

Along L we have that $f_\gamma = f_{\alpha.\beta}^2 - 2$ hence $f_\gamma = 2\cos(\frac{(n-1)\pi}{n})$.

We shall prove that L consists of elliptic fixed points. We first notice that since $f_\alpha = f_\beta = 0$, Δ fixes any point in L. Therefore, Δ acts trivially on the level $(f_\alpha, f_\beta) = (0, 0)$. Since $f_{\delta.\beta} = 0$ on L and applying Proposition 2, we have that τ_α^4 preserves the infinitesimal levels of f_δ. With the fact α and β do not intersect γ and β do not intersect δ, we deduce that Δ preserves the infinitesimal levels of (f_δ, f_γ). To summarize Δ at any point in L:

- acts trivially on any level of (f_α, f_β).
- preserves any level of (f_γ, f_δ).

Now we fix $n \in \mathbb{N}$ such that $f_{\alpha.\beta} = 2\cos(\frac{(n-1)\pi}{2n})$ is small enough to ensure that Df_2 is elliptic.

In a similarly way, Γ_n fixes any point in L and we have that Γ_n along L:

- preserves any level of (f_α, f_β).
- acts trivially on any level of (f_γ, f_δ).

Now we choose ϵ close enough to zero that Df_2 is elliptic. This implies that along L, on the local coordinates $(f_\alpha, f_\beta, f_\gamma, f_\delta, f_{c_1}, f_{c_2})$ we have:

$$Df_2 = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & A & 0 \\ 0 & \# & I_2 \end{pmatrix}$$

and,

$$Df_1 = \begin{pmatrix} B & 0 & 0 \\ 0 & I_2 & 0 \\ * & 0 & I_2 \end{pmatrix}$$

Hence,

$$D\Phi = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ *' & \#' & I_2 \end{pmatrix}$$

Cutting along γ, the resulting surface is homeomorphic to a punctured torus and applying theorem 5.1 we deduce that B is conjugated to a rotation with an angle of the form $\arccos(P(2\cos(\frac{(n-1)\pi}{n}))$ for some polynomial P. Cutting along β, we get a four-punctured sphere, and applying theorem 5.2 for ϵ small enough, we deduce that A is conjugate to a rotation with angle $\Theta(t)$ that depends on t.

We conclude that $D\Phi$ has two invariant two-dimensional sub-vector spaces such that the restriction of $D\Phi$ over the first and second subspace is conjugated to a rotation of angle $\arccos(P(2\cos(\frac{(n-1)\pi}{n}))$ and $\Theta(t)$, respectively. For some polynomial P depending only on f_1.

The first part, i.e. Φ is a pseudo-Anosov element, is a consequence of the second part; it is clear that if Φ is isotopic to a periodic homeomorphism then $D\Phi$ along L does not have an irrational frequency vector. Now it remains to check that Φ is not reducible. It is sufficient to prove:

Lemma 2. If a homeomorphism Φ of $S_{g,n}$ is reducible then it does not admit an elliptic fixed point with irrational frequency on any relative character variety $X_B(\pi_1(S_{g,n}), \mathsf{SU}(2))$.

Proof. If Φ is reducible then some of its powers preserve a simple closed curve γ, therefore the levels of the function f_γ are Φ-invariant which yields that $D\Phi$ at a fixed point can not be elliptic with irrational frequency. □

□

Chapter 5

Invariant functions

5.1 Multi-twists and square-tiled surfaces

As discussed in the Background, one way of generating pseudo-Anosov homeomorphisms on a surface S is by considering the group generated by Dehn-twists along two filling multi-curves.

Definition 5.1.1. A square-tiled surface is a finite collection of squares on $\mathbb{C}$, where edges are glued together two by two via a translation or a half-translation (i.e. a similarity with linear part -1).

The square-tiled surfaces are naturally endowed with a half-translation structure i.e. a structure where the transition maps are of the form $z \mapsto \pm z + c$, for some $c \in \mathbb{C}$. An affine transformation of S is then a transformation such that it is affine for the charts of the half-translation structure. The group generated by all the linear parts of such transformations is called the Veech group of S. We shall point out here that SL$(2, \mathbb{R})$ (in particular SL$(2, \mathbb{Z})$) acts on the set of square-tiled surfaces by post-composition i.e. If S is a half-translation surface and $A \in$ SL$(2, \mathbb{R})$ then $A.S$ is the half-translation surface obtained by composing the charts of S with A. In particular, we have that the stabilizer of a half-translation surface is its Veech group. The following then holds for square-tiled surfaces (See [GJ]):

Proposition 14. The Veech group of a square-tiled surface is a finite index subgroup of SL$(2, \mathbb{Z})$.

From a square-tiled surface, one can obtain two multi-curves that consist of horizontal curves $\gamma_1, \ldots, \gamma_n$ and vertical ones $\lambda_1, \ldots, \lambda_m$. In fact the surface S is decomposed in two different ways via vertical and horizontal cylinders and the curves $\gamma_1, \ldots, \gamma_n$ (resp. $\lambda_1, \ldots, \lambda_m$) are exactly the generators of the fundamental groups of the horizontal cylinders (resp. vertical cylinders).

The converse is also true, let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ and $\lambda = \{\lambda_1, \ldots, \lambda_m\}$ be two filling multi-curves on an orientable surface S that is their complementary set is a union of disks. Then we can construct a square-tiled surface by simply considering a square on each intersection between the curves centered at the intersection point, and the gluing is deduced according to the combinatorial data of γ and λ. We conclude:

Proposition 15. On orientable surfaces, the set of two filling multi-curves is in bijection with the set of square-tiled surfaces, up to homeomorphisms.

In the case where the intersections between the curves of γ and λ have the same sign with respect to the orientation of S, then the corresponding square-tiled surface has only translations as identifications between the edges of the squares, hence we get an origami i.e. a ramified cover of the torus with only one singular point at the basis (in this case the torus). To encode the origami we need two permutations: let $S_1, \ldots, S_d$ be the squares forming the origami S (Notice here that d is the degree of the ramified cover). For $i \in \{1, \ldots d\}$ denote by a_i the edge to the left of the square S_i and by b_i the edge to the bottom of S_i. To determine the origami it is sufficient to decide which edge a_{σ_i} is to the right to the square S_i and which edge $b_{\sigma'_i}$ is to the top of the square S_i. Therefore one has:

Proposition 16. The set of pair of permutations that acts transitively on $\{1, \ldots, d\}$, up to conjugation of the permutation group of d elements, is in bijection with the set of connected origamis of degree d, up to homeomorphisms.

For instance, consider $d = 4$, $\sigma = (1, 2, 3, 4)$ and $\sigma' = (1, 3, 2, 4)$. In this example, we get a genus 2 surface where the corresponding two multi-curves γ and λ are exactly two curves, the vertical curve γ and the horizontal curve λ, as illustrated in Figure 5.1.

The vertical curve γ (resp. λ) generates the fundamental group of the vertical (resp. horizontal) cylinder made by the four squares. Viewing the surface in this way allows one to see that the matrix $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$) realizes an affine transformation on the surface that is in the isotopy class of the Dehn-twist along the curve γ (resp. λ).

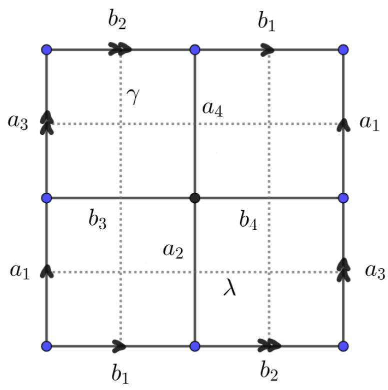


Figure 5.1: Two filling curves in S_2

5.2 SU(2)-character varieties and Goldman's flow

The simplest actions on Hom$(\pi_1(S), \mathrm{SU}(2))$ are those defined via a Dehn-twist along a simple closed curve. If a curve α is non-separating, then its action on Hom$(\pi_1(S), \mathrm{SU}(2))$ can be expressed in a simple way using the HNN extension by writing:

$$\pi_1(S) = \{\pi_1(S \setminus \alpha) \cup \beta \mid \beta\alpha^+\beta^{-1} = \alpha^-\}$$

With α^+ and α^- being the resulting boundaries loops after cutting along α. So, on the Hom level, we can write:

$$\tau_\alpha^*(\rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(S \setminus \alpha) \\ \rho(\gamma).\rho(\alpha^+) & \text{if } \gamma = \beta \end{cases}$$

If we denote the one-parameter subgroup of SU(2) of velocity 1 that passes through $\rho(\alpha^+)$ (choosing the shortest path between Id and $\rho(\alpha^+)$) by $t \mapsto \xi_{\rho(\alpha^+)}(t)$, then we have a natural flow Ξ_α of $\mathbb{S}^1$ on Hom$(\pi_1(S), \mathrm{SU}(2))$:

$$\Xi_\alpha^t(\rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(S \setminus \alpha) \\ \rho(\gamma).\xi_{\rho(\alpha^+)}(t) & \text{if } \gamma = \beta \end{cases}$$

It is a well-defined flow on the character variety except when $tr(\rho(\alpha^+)) = \pm 2$. We restate here a version of Theorem 4.3 in [G2] in the case where α is a non-separating curve:

Proposition 17. The Hamiltonian flow of the function $\rho \mapsto tr(\rho(\alpha))$ with respect to the symplectic form is a reparametrization of the previous flow Ξ_α.

Using the HNN extension, the restriction to $\pi_1(S \backslash \alpha)$ defines a projection $P_\alpha : \mathsf{Hom}(\pi_1(S), \mathsf{SU}(2)) \mapsto \mathsf{Hom}(\pi_1(S \setminus \alpha), \mathsf{SU}(2))$. Due to the above description, one has:

Proposition 18. The projection P_α defines an $\mathbb{S}^1$-bundle almost everywhere (except when $\rho(\alpha^+) = \pm Id$) over its image. Moreover, its fibers coincide with the orbits of Ξ_α.

Proof. We notice first that the projection P_α is invariant under the flow Ξ_α. Now we need to prove that the orbits of Ξ_α are exactly the fibers of P_α.

Let ρ_0 be a representation in $\mathsf{Hom}(\pi_1(S \setminus \alpha), \mathsf{SU}(2))$, We observe that the image of P_α is the algebraic subset defined by the polynomial function $\rho \mapsto tr(\rho(\alpha^+)) - \mathsf{tr}(\rho(\alpha^-))$. Therefore we assume that $tr(\rho_0(\alpha^+)) = tr(\rho_0(\alpha^-))$. In order to find an extension of ρ_0 in $\mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$, it is sufficient to determine $\rho(\beta)$. The condition $\beta.\alpha^+.\beta^{-1} = \alpha^-$ implies that $\rho(\beta)$ lies in a big circle $S_\alpha \subset \mathsf{SU}(2)$ that depends only on $\rho_0(\alpha^+)$ and $\rho_0(\alpha^-)$.

□

The situation is not very different for a non-separating multi-curve $\alpha = \{\alpha_1, \dots, \alpha_n\}$. In fact the flows $\Xi_{\alpha_1}, \dots, \Xi_{\alpha_n}$ commutes, therefore we have an action of an n-dimensional torus $\mathbb{T}^n$ defined almost everywhere on the representation variety.

Similarly, we denote by P_α the restriction to $\pi_1(S \setminus \alpha)$. Let us now denote M_α the group generated by $\tau_{\alpha_1}, \dots, \tau_{\alpha_n}$ and consider a homeomorphism $f = \tau_{\alpha_{k_1}}^{i_1} \circ \dots \circ \tau_{\alpha_{k_n}}^{i_n}$ in M_γ, for some non-zero integers $i_1, \dots, i_n$, then one has:

Proposition 19. The ergodic components of f, the ergodic components of M_α, and the fibers of P_α are almost everywhere equal.

Proof. The proof relies on the relation between the action of a single Dehn-twist τ_γ and the flow Ξ_γ, one has the following (See section 2 in [GX1] for more details):

$$\tau_\gamma^*(\rho) = \Xi_\gamma^{\theta(\rho(\gamma^+))}(\rho)$$

Where $\theta(X)$ is the angle of the matrix X, more precisely, $\theta(X) = \arccos(\frac{tr(X)}{2})$. Since the curves do not intersect, f preserves the functions $\rho \mapsto \theta(\rho(\alpha_i^+))$, for all $i \in \{1, \ldots, n\}$. Hence we deduce that for a generic representation ρ, the automorphism f acts by the same translation inside the Ξ_α-orbit of ρ which is generically diffeomorphic to a torus $\mathbb{T}^n$. The fact that $\{\theta(\rho(\alpha_i^+)\}_{i=1}^{i=n}$ can be chosen freely yields the first part. The proof of the second part i.e. the relation between P_α and M_α is the same as the proof of Proposition 18.

□

The last proposition says that one can not distinguish measurably between the action of such an $f \in M_\gamma$ and the action of the whole group M_γ.

Remark 5.2.1. If $\alpha = \{\alpha_i\}_{i=1}^{i=n}$ is a non-separating multi-curve then the image of P_α is the algebraic variety defined to be the zero locus in Hom$(\pi_1(S \setminus \alpha), \text{SU}(2))$ of the polynomials $\rho \mapsto tr(\rho(\alpha_i^+)) - tr(\rho(\alpha_i^-))$, for $i \in \{1, \ldots n\}$.

5.3 Foliations on the intersection of quadrics

To analyze the action of two multi-twists, let us reflect on a larger family of algebraic foliations. On the affine space $\mathbb{R}^n \times \mathbb{R}^n$, let $\{f_i\}_{i=1}^{i=m}$ be a collection of bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$ and denote by f the bilinear map $(f_1, \ldots, f_m)$. Set V to be the zero locus of the maps f. Over the algebraic variety V we derive two natural foliations:

For $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^n$, let $\mathcal{A}$ and $\mathcal{B}$ be the intersection of V with the affine subspaces $\{A\} \times \mathbb{R}^n$ and $\mathbb{R}^n \times \{B\}$ i.e. the levels of the maps $X \mapsto f(A, X)$ and $X \mapsto f(X, B)$, respectively. Let $\mathcal{C}$ to be the equivalence relation generated by the two foliations. Let $P_A : V \mapsto \mathbb{R}^n$ and $P_B : V \mapsto \mathbb{R}^n$ be the projection to the first and the second components, respectively.

At this point, one can ask whether the saturation $\mathcal{C}$ fills the variety V or not. The answer depend on the ability to separate the two variables A and B from each other, in other words:

Proposition 20. If V admits a non-constant function that factors through P_A and P_B simultaneously, then the function is constant on the $\mathcal{C}$-classes.

Example 5.3.1. Let $V \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the quadric defined by the bilinear map $f(a_1, a_2, b_1, b_2) = a_1 b_1 + a_2 b_2$. Then the function $\frac{a_1}{a_2} = -\frac{b_2}{b_1}$ is constant on the $\mathcal{C}$-classes.

Example 5.3.2. If we consider $\mathbb{R}^3$ instead of $\mathbb{R}^2$ that is the quadric $V \subset \mathbb{R}^3 \times \mathbb{R}^3$ defined by $f(a_1, a_2, a_3, b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$, then no such function exists. We simplify the task by considering the zero locus of $a_1b_1 + a_2b_2 = -1$ on the product of the projective spaces of the two components. Therefore the new foliations $\mathcal{A}$ and $\mathcal{B}$ are the integral curves of the vector fields $(0, 0, b_2, -b_1)$ and $(a_2, -a_1, 0, 0)$, respectively.

On the $\mathsf{SU}(2)$-representation variety

Let γ and λ be two filling multi-twists with one disk as complementary and n positive intersections in total (intersections having the same sign with respect to the orientation of S). The dynamics of the group generated by M_γ and M_λ on the representation variety $\mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$ ties into the previous discussion. Let S be the origami associated to γ and λ. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be the generators of $\pi_1(S \setminus \gamma)$ and $\pi_1(S \setminus \lambda)$, respectively.

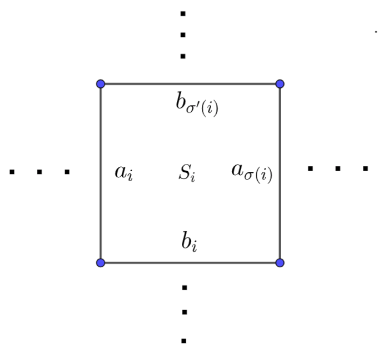


Figure 5.2: Square S_i in the Square-tiled surface

The fundamental groups $\pi_1(S \setminus \gamma)$, $\pi_1(S \setminus \lambda)$ and $\pi_1(S)$ are over-generated by $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ and $\{a_1, b_1, \ldots, a_n, b_n\}$, respectively. The relations between the generators $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are given by the square relations:

$$a_i.b_{\sigma'(i)} = b_i.a_{\sigma(i)}$$

for $i \in \{1, \ldots n\}$. For a representation ρ set $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$, for $i \in \{1, \ldots, n\}$. Since the complementary of λ and γ is one disk, the corresponding surface S belongs to the stratum $\mathcal{H}(2g-2)$, using Proposition 19, we get:

Proposition 21. Let S be a square-tiled surface that belongs to the minimal stratum $\mathcal{H}(2g-2)$ with all vertices identified to one point then the map $\rho \mapsto (A_1, \ldots, A_n, B_1, \ldots, B_n)$ is an embedding of $\mathrm{Hom}(\pi_1(S), \mathrm{SU}(2))$ into $\mathrm{SU}(2)^{2n}$ and the image V is an algebraic variety defined by the square relations:

$$A_i B_{\sigma'(i)} - B_i A_{\sigma(i)} = 0$$

In addition, the foliations $\mathcal{A}$ and $\mathcal{B}$ defined on $V \subset \mathrm{SU}(2)^n \times \mathrm{SU}(2)^n$ are exactly the ergodic component of the subgroups M_γ and M_λ.

5.4 Invariant functions

In what follows, we view $\mathrm{SU}(2)$ as the unit sphere of the quaternion numbers $\mathbb{H}$. Denote by $\mathbb{H}^0$ the subspace of imaginary vectors. We endow $\mathbb{H}$ with the canonical scalar product $(X, Y) \mapsto tr(X.\overline{Y})$ which is bi-invariant, i.e. If $A, B \in \mathrm{SU}(2)$, then the linear map:

$$X \mapsto A.X.B$$

is an isometry of $\mathbb{H}$. Conversely, every isometry of $\mathbb{H}$ can be expressed in this way.

Let us consider the following specific rectangle made with one vertex, two horizontal edges b_I, b_J, and two vertical edges a_I and a_J as illustrated in Figure 5.3 below.

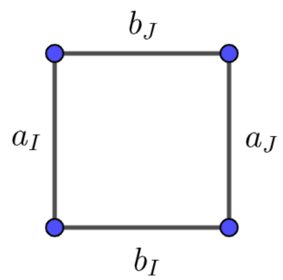


Figure 5.3: A rectangle in the square-tiled surface

The rectangle relation writes $a_I.b_J = b_I.a_J$, so if ρ is a representation, then we set $A_I = \rho(a_I)$,

$A_J = \rho(a_J)$, $B_I = \rho(b_I)$ and $B_I = \rho(b_I)$. Now the rectangle relation writes:

$$A_I.B_J = B_I.A_J$$

In general, it is not possible to separate the variables (A_I, A_J) from (B_I, B_J) i.e. to find a function that factors through P_A and P_B simultaneously. However, under some conditions the task becomes possible.

Lemma 3. If a_I is conjugated to a_J and b_I is conjugated to b_J then the two directions $[A_I - A_J]$ and $[B_I - B_J]$ defined on $P(\mathbb{H}^0)$ are equal.

Proof. Consider the following linear map, which is a function that depends only on A_I and A_J :

$$\psi \colon \mathbb{H} \longrightarrow \mathbb{H}$$
$$X \mapsto A_I.X.A_J^{-1} - X$$

The fact that A_I is conjugated to A_J implies that the kernel of ϕ is of rank 2 since non-zero elements in the kernel are those who conjugate A_I to A_J. So the image of ϕ, denoted $Im(\phi)$, is of rank 2. Observe that $Im(\phi)$ can not only contain traceless matrices; if $tr \circ \phi$ vanishes then, in particular, $\phi(1) = A_I.A_J^{-1} - 1$ would be traceless which would imply that $A_I = A_J$ which does not hold in general. From the rectangle relation, we deduce that

$$B_I - B_J = A_I.B_J.A_J^{-1} - B_J$$

Therefore $B_I - B_J \in Im(\phi)$. Since B_I and B_J are conjugate then

$$B_I - B_J = Im(\phi) \cap \mathbb{H}^0$$

The conjugacy assumption of A_I and A_J together with the fact that $\phi(A_J) = A_I - A_J$ allow us to deduce that:

$$[A_I - A_J] = [B_I - B_J]$$

□

5.4.1 Examples on the representation variety of S_2

(Proof of Theorem 1.2.1)

In what follows, Γ will denote the group generated by M_γ and M_λ. Using the previous lemma, one can see that the following examples admit a function on Hom$(\pi_1(S), \text{SU}(2))$ that factors through P_A and P_B, simultaneously.

For instance, let us consider the square-tiled surface S' below (Figure 5.4) made of three squares with only one vertex and hence S' has genus 2. We enumerate the squares of S' from bottom to top.

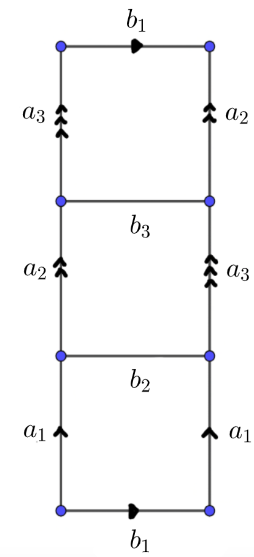


Figure 5.4: The surface S'

Using the relation of square 1, i.e. $a_1.b_2 = b_1.a_1$, we deduce that b_1 is conjugated to b_2. The curve $a_2.a_3$ is always conjugated to $a_3.a_2$. By looking at the rectangle made by squares 2 and 3, using the relation $a_2.a_3.b_1 = b_2.a_2.a_3$, we conclude that for a representation ρ, Lemma 3 applies and hence one has:

$$[A_2.A_3 - A_3.A_2] = [B_1 - B_2]$$

In other words, the direction orthogonal to 1, A_2, and A_3 is exactly the direction of $B_1 - B_2$.

Remark 5.4.1. In this example, one can see that the affine transformations of S' given by:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

are elements of Γ.

Lemma 3 applies to different surfaces of genus 2, For example, we can consider the famous square-tiled surface S (Figure 5.5 below). The group Γ is generated by four Dehn-twists along two horizontal loops and two vertical ones as defined in section 5.1. A finite index subgroup of the Veech group of S is contained in Γ, this is a consequence of the fact that the two affine transformations of S:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a finite index subgroup of $\mathsf{SL}(2, \mathbb{Z})$.

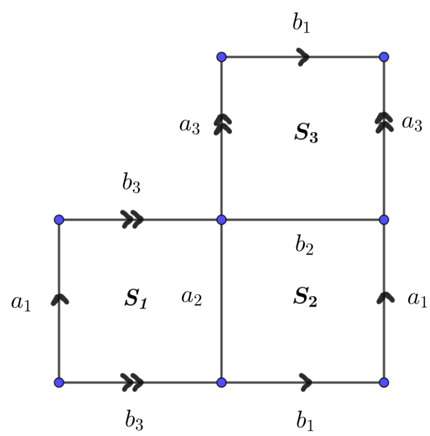


Figure 5.5: The genus two surface S

After applying Lemma 3 on the second square knowing from the first and the third squares that a_1 is conjugated to a_2 and b_1 is conjugated to b_2, we deduce that $[A_1 - A_2] = [B_1 - B_2]$, hence $\rho \mapsto [A_1 - A_2]$ is Γ-invariant on $\mathsf{Hom}(\pi_1(S), \mathsf{SU}(2))$.

5.4.2 Examples on the character variety of N_4

(Proof of Theorem 1.2.2)

In this section, we will slightly modify the previous discussion by considering an arbitrary family of filling curves $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ on a surface Σ (not necessarily consisting of two multi-curves).

Without loss of generality, one can assume that there are no three curves that intersect at a single point. Now one can reconstruct the closed surface using the combinatorial data of the curves $\gamma_1, \ldots, \gamma_n$. Let γ be an abstract family of curves, for each point of intersection p consider a square centered at p with edges transversal to the curves that perform the intersection at p. The gluing among the squares is then deduced by following the paths of the curves. As a result, if the surface is orientable, we get a $\frac{1}{4}$-translation surface i.e a Euclidean surface with identifications of the form $z \mapsto R_{\frac{k.\pi}{2}}(z) + c$, where $R_{\frac{k.\pi}{2}}$ is a rotation of angle $\frac{k\pi}{2}$, for some $k \in \mathbb{Z}$ and $c \in \mathbb{C}$. To summarize:

Proposition 22. The set of families of filling curves (possibly with self-intersections) is in bijection with the square-tiled $\frac{1}{4}$-translation surfaces, up to homeomorphisms.

The fact that any $\frac{1}{4}$-translation surface has a half-translation surface as a double ramified cover implies:

Corollary 5.4.1. Any family of filling curves is the image of two filling multi-curves via a ramified cover of degree two.

Remark 5.4.2. For a general closed surface Σ (orientable or not), we shall add a reflection along the x-axis to the group of rotations and translations to get a structure corresponding to any family of curves on Σ.

On the SU(2)-representation variety Hom$(\pi_1(\Sigma), \mathrm{SU}(2))$. The group Γ generated by the Dehn-twists along $\gamma_1, \ldots, \gamma_n$ acts on it. Let us now denote the projection defined on Section 5.2 associated with γ_i by P_i. The previous discussion can be adapted to this situation:

Lemma 4. A function that factors through P_i, for all $i \in \{1, \ldots, n\}$, is a Γ-invariant function on Hom$(\pi_1(\Sigma), \mathrm{SU}(2))$.

The following example is a genus 4 non-orientable surface denoted N_4. The curves $\{\gamma_1, \gamma_2, \gamma_3\}$ are in minimal position since the geometric intersection between any pair of curves is one, therefore the curves $\{\gamma_1, \gamma_2, \gamma_3\}$ are filling the surface. As shown in Figure 5.6 below:

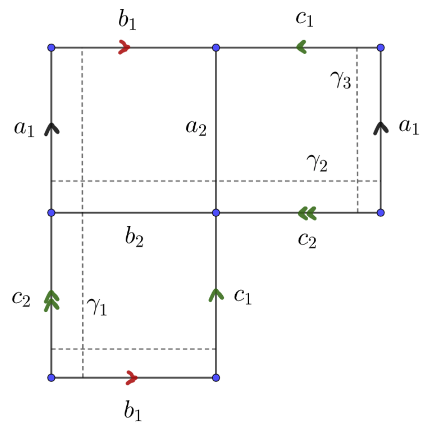


Figure 5.6: The non-orientable surface N_4

The relation coming from the three squares above read:

$$\begin{cases} a_1.b_1 = b_2.a_2 \\ c_2^{-1}.a_1.c_1 = a_2 \\ c_2^{-1}.b_1.c_1 = b_2 \end{cases}$$

For a representation $\rho \in \mathsf{Hom}(\pi_1(N_4), \mathsf{SU}(2))$, we write:

$$\begin{cases} A_1.B_1 = B_2.A_2 \\ C_2^{-1}.A_1.C_1 = A_2 \\ C_2^{-1}.B_1.C_1 = B_2 \end{cases}$$

The group $\Gamma = \langle \tau_{\gamma_1}, \tau_{\gamma_2}, \tau_{\gamma_3} \rangle$ acts on $\mathsf{Hom}(\pi_1(N_4), \mathsf{SU}(2))$, and a function is Γ-invariant once it can be written in terms of each of the following 4-uplets (A_1, A_2, B_1, B_2), (A_1, A_2, C_1, C_2) and (B_1, B_2, C_1, C_2), simultaneously. Consider the isometry of $\mathbb{H}$, $\Phi_C : X \mapsto C_2^{-1}.X.C_1$. The system of equations can be rewritten as:

$$\begin{cases} A_1.B_1 = B_2.A_2 \\ \Phi_C(A_1) = A_2 \\ \Phi_C(B_1) = B_2 \end{cases}$$

Taking the trace of the first equation we get:

$$tr(A_1.B_1) = tr(A_2.B_2)$$

From the last two equations and since Φ_C is an isometry we deduce that the angle between A_1 and B_1 is equal to the angle between A_2 and B_2, in other words:

$$tr(A_1.B_1^{-1}) = tr(A_2.B_2^{-1})$$

Adding the two previous equations and using the parallelogram identity on $\mathsf{SL}(2,\mathbb{C})$ (i.e. $tr(XY) + tr(XY^{-1}) = tr(X).tr(Y)$, for any $X, Y \in \mathsf{SL}(2,\mathbb{C})$), we get:

$$tr(A_1)tr(B_1) = tr(A_2)tr(B_2)$$

Finally, we deduce that the function:

$$\frac{tr(A_1)}{tr(A_2)} = \frac{tr(B_2)}{tr(B_1)}$$

is a Γ-invariant function on the $\mathsf{SU}(2)$-character variety of N_4. What is left to do is to check that the function is not constant, for this purpose, it is sufficient to prove that both A_1 and A_2 can take arbitrary values in $\mathsf{SU}(2)$.

Lemma 5. The projection $\rho \mapsto (A_1, A_2)$ from $\mathsf{Hom}(\pi_1(N_4), \mathsf{SU}(2))$ to $\mathsf{SU}(2)^2$ is surjective.

Proof. For $(A_1, A_2) \in \mathsf{SU}(2)^2$, take B_1 in the sphere

$$\{X \in \mathsf{SU}(2) \mid tr(X.A_1^{-1}) = tr(A_1.X.A_2^{-2})\}$$

Therefore $B_2 = A_1.B_1.A_2^{-1}$. The last condition is equivalent to saying that the angle between A_1 and B_1 is equal to the angle between A_2 and B_2. This ensures the existence of an isometry Φ_C in $\mathsf{SO}(4)$ such that $\Phi_C(A_1) = A_2$ and $\Phi_C(B_1) = B_2$. □

Chapter 6

Ergodicity of the homeomorphism group

Let S_g be the closed surface of genus g, Let us consider the SU(2)-representation variety of S_g which is the space of all homomorphisms from the fundamental group of S_g to the unitary special group SU(2). The fundamental group of S_g is generated by $2g$ elements $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots \beta_g\}$ with only one relation on the commutators $[\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ i.e.

$$\pi_1(S_g) = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots \beta_g | \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1\}$$

For a representations $\rho \in \mathrm{Hom}(\pi_1(S_g), \mathrm{SU}(2))$, let us set $A_i = \rho(\alpha_1)$ and $B_i = \rho(\beta_i)$, for all $i \in \{1, \ldots, g\}$. This gives an embedding of $\mathrm{Hom}(\pi_1(S_g), \mathrm{SU}(2))$ into $\mathrm{SU}(2)^{2g}$. Therefore the representation variety is the algebraic variety defined to be the zero locus in $\mathrm{SU}(2)^{2g}$ of the polynomial $\prod_{i=1}^{g}[A_i, B_i] - 1$. The automorphism group $\mathrm{Aut}^+(\pi_1(\mathrm{S_g}))$ of the fundamental group acts by pre-composition on $\mathrm{Hom}(\pi_1(S_g), \mathrm{SU}(2))$ i.e. if $\phi \in Aut(\pi_1(S_g))$ then $\phi.\rho = \rho \circ \phi^{-1}$. This yields an action of Aut^+ via polynomial automorphisms on $\mathrm{Hom}(\pi_1(S_g), \mathrm{SU}(2))$ that extends to polynomial action on $\mathrm{SU}(2)^{2g}$.

Fix a point $p \in S_g$ in the closed surface S_g. In this chapter, we study the action of the group of orientation-preserving homeomorphisms $\mathrm{Homeo}^+(\mathrm{S_g}, \mathrm{p})$ fixing a base point p, viewed as elements of $\mathrm{Aut}^+(\pi_1(\mathrm{S_g}, \mathrm{p}))$ on the representation variety $\mathrm{Hom}(\pi_1(S_g), \mathrm{SU}(2))$. This gives a version of Goldman and Xia's Theorem [GX1] about the ergodicity of the mapping class group on the SU(2)-character variety but rather on the representation variety:

Theorem. *The group* Homeo$^+$(S$_g$, p) *acts ergodically on the representation variety* Hom$(\pi_1(S_g), \mathrm{SU}(2))$ *with respect to the class of Lebegue measure, More precisely, ergodicity is ensured by the action of a group generated by no more than* $3g-1$ *Dehn twists on* S_g.

We begin the proof by considering the case of the once-punctured torus $S_{1,1}$, we remind here that its fundamental group is a free group generated by two elements. Here as shown in Figure 6.1 the fundamental group at the base point p is generated by the two elements α and β.

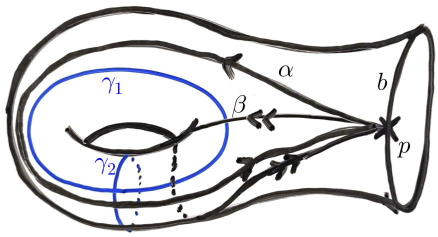


Figure 6.1: The punctured torus

The Dehn-twists along the two closed curves γ_1 and γ_2, up to suitable homeomorphisms isotopic to the identity, give rise to two automorphisms τ_1 and τ_2 on $\langle \gamma_1, \gamma_2 \rangle$ acting as follows: $\tau_1(\alpha) = \alpha$, $\tau_1(\beta) = \beta.\alpha$ and $\tau_2(\alpha) = \alpha.\beta$, $\tau_2(\beta) = \beta$. This defines two polynomial automorphisms on SU$(2)^2$, T_1 and T_2 such that:

$$T_1(A, B) = (A, B.A)$$

And,

$$T_2(A, B) = (A.B, B)$$

.

Let us denote ξ_X^t the one-parameter subgroup in SU(2) of velocity 1 that passes throughs X. When studying the ergodicity on SU$(2)^2$, the dynamics of $\langle T_1, T_2 \rangle$ and the dynamics of the following two flows are not different;

$$\Phi_1^t(A, B) = (A, B.\xi_A^t)$$

And,

$$\Phi_2^s(A, B) = (A.\xi_B^s, B)$$

.

Now consider the projection P_1 (resp. P_2) from SU$(2)^2$ to the first (resp. the second) component and set P to be the projection to the word $b = [\alpha.\beta.\alpha^{-1}.\beta^{-1}]$ i.e. $P(A, B) = ABA^{-1}.B^{-1}$. One notice that the word P is invariant by the flows Φ_1 and Φ_2 and we have:

Lemma 6. The fibers of the projection P are exactly the saturations of Φ_1 and Φ_2.

Proof. A t-step of the flow Φ_1 followed by an s-step of the flow Φ_2 then projected to the first component gives a parameterization of a sphere in SU$(2) \subset \mathbb{H}$, where $\mathbb{H}$ is the field of quaternion numbers, in other words:

$$P_1(\Phi_2^s(\Phi_1^t(A, B))) = P_1(\Phi_2^s(A, B\xi_A^t)) = A.\xi^s_{B.\xi^t_A}$$

On the other hand, $P_1(P^{-1}\{K\})$, for $K \in$ SU(2), is again a sphere in SU$(2) \subset \mathbb{H}$ that is define by the polynomial $tr(AK) - tr(A)$, hence the two spheres must coincide. Now noticing that fixing A and moving B in $P^{-1}\{K\}$ define exactly the integral curves of Φ_1, conclude the lemma. □

Let us now consider the case of the twice-punctured torus together with the curves α, β, γ, and δ as shown in Figure 6.2.

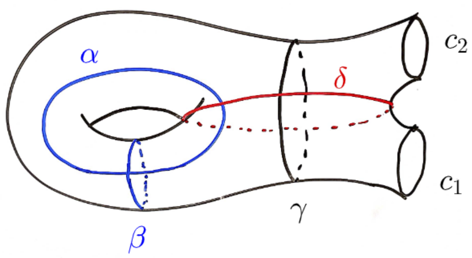


Figure 6.2: The twice-punctured torus

The fundamental group $\pi_1(S_{1,2})$ is free group of rank three generated by α, β and c_1 and we have:

$$\pi_1(S_{1,2}) = \{\alpha, \beta, c_1, c_2 | [\alpha, \beta] = c_1.c_2\}$$

We have that $\gamma = \alpha.\beta.\alpha^{-1}.\beta^{-1} = c_1.c_2$. Let us focus on the action of the Dehn twist along δ on the curve γ, writing $\gamma = c_1.c_2$, we get $\tau_\delta(\gamma) = c_1.\delta.c_2.\delta^{-1}$. Therefore on the representation variety the polynomial T_δ acts on $\rho(\gamma)$ as follows:

$$T_\delta(\rho)(\gamma) = C_1.\rho(\delta)^{-1}.C_2.\rho(\delta) = C_1.Ad_{\rho(\delta)}C_2$$

Where $\rho(c_1) = C_1$ and $\rho(c_2) = C_2$. It is equivalent to replace the action of T_δ by the flow Φ_δ:

$$\Phi^t_\delta(\rho)(\gamma) = C_1.\xi^{-t}_{\rho(\delta)}.C_2.\xi^t_{\rho(\delta)} = C_1.Ad_{\xi^t_{\rho(\delta}}C_2$$

If $\rho(\delta)$ can be chosen freely and C_1 is chosen freely inside a fixed sphere then $\rho(\gamma)$ covers SU(2), in other words:

Lemma 7. Let $K \in$ SU(2) be fixed and $S \subset$ SU(2) is a sphere. The map $E_K : (X, Y) \longrightarrow Y.X.Y^{-1}.K.X^{-1}$, from SU$(2) \times S$ to SU(2), is surjective.

Proof. For a generic $X \in$ SU(2), the map $Y \longrightarrow Y.X.Y^{-1}$ is a surjection to the set of elements conjugated to X. Hence without loss of generality, one can assume that $Y.X.Y^{-1} = X_1$ and $X^{-1} = X_2$, with only condition that $tr(X_1 - X_2) = 0$. In particular, one can assume that $tr(X_1) = tr(X_2) = 0$. Therefore the map $r : K \longrightarrow X_1.K.X_2$ is an isometric involution in $\mathbb{H}$. The fact that for any $L \in$ SU(2) there exists an involution that sends K to L, yields the lemma. □

Now we are ready to prove the Theorem announced before. The surface S_g can be viewed as g handle glued to the sphere (Figure 6.3 is an illustration when $g = 4$). Let us denote these handles by $H_1, \dots, H_g$ enumerated from left to right. For each handle, let α_i, β_i be the generators of the fundamental groups of the once-punctured tori obtained after cutting along a curve isotopic b_i (To contain the base point p) (See Figure 6.3). Now let $\delta_{i,j}$ be a curve that tights the handles i to j (For example, in Figure 6.3 we draw $\delta_{1,2}$). Finally, the curves γ_i and λ_i are the support of the handle H_i.

Proof. If we consider T_i and T'_i to be the polynomial automorphisms induced from the Dehn-twist action along γ_i and λ_i, respectively. And Φ_i, Ψ_i be the induced flow, respectively. Then Lemma 6 implies that the saturation of the flows $\Phi_1, \dots, \Phi_g, \Psi_1, \dots, \Psi_g$ are the fibers of the map $\rho \mapsto (B_1, \dots, B_g)$, where $B_i := \rho(b_i)$, for all $i \in \{1, \dots, g\}$. Because $\prod_{i=1}^g B_i = 1$, one can consider for instance, the map $\rho \mapsto (B_1, \dots, B_{g-1})$. What is left to do is to show that one can change each matrix B_i in SU(2) for $g - 1$ components via some induced flows. For this reason and without loss of generality let us assume that we aim to change $B_1 \in \{B_1, B_3, \dots, B_g\}$. A small neighborhood of $\delta_{1,2}$ and b_1 is a four-punctured sphere, by gluing again the components homotopic to γ_1, we get a twice-punctured torus with one of

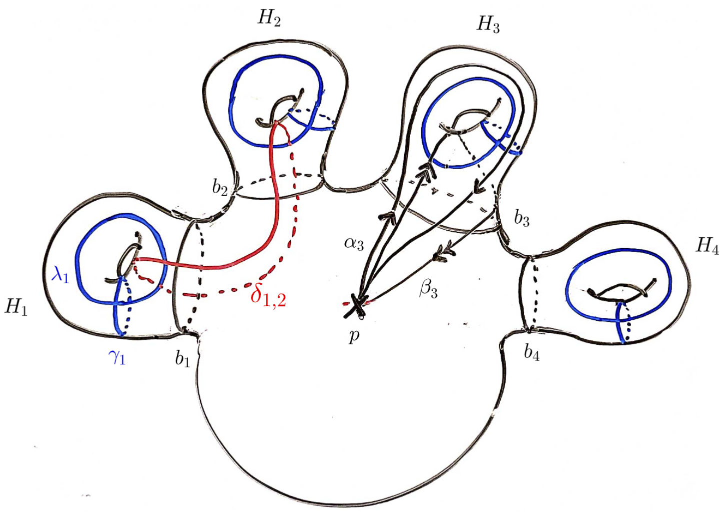


Figure 6.3: Example for $g = 4$

the boundaries on the handle H_2, let us call this boundary c_1. We saw before in the proof of Lemma 6 that $C_1 := \rho(c_i)$ (seen as A) belongs to some sphere. Together with the fact the $\rho(\delta_{1,2})$ can be chosen freely, we conclude using Lemma 7 the proof.

□

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