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Projections régulières, structure de Lipschitz des ensembles définissables et faisceaux de Sobolev

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Résumé

Dans cette thèse, nous abordons des questions autour de la structure métrique des ensembles définissables dans les structures o-minimales. Dans la première partie, nous étudions les projections régulières au sens de Mostowski, nous prouvons que ces projections n'existent que pour les structures polynomialement bornées, nous utilisons les projections régulières pour refaire la preuve de Parusiński de l'existence des recouvrements réguliers. Dans la deuxième partie de cette thèse, nous étudions les faisceaux de Sobolev (au sens de Lebeau). Pour les fonctions de Sobolev de régularité entière positive, nous construisons ces faisceaux sur le site définissable d'une surface en nous basant sur des observations de base des domaines définissables dans le plan.

Mots clés : Geometrie métrique des singularities, Projections régulières, Structures o-minimales, Theorie des singularities réels, Geometrie semialgebrique et sous analytique, faisceaux de Sobolev.

Regular projections, Lipschitz structure of definable sets and Sobolev sheaves

Abstract

In this thesis we address questions around the metric structure of definable sets in o-minimal structures. In the first part we study regular projections in the sense of Mostowski, we prove that these projections exists only for polynomially bounded structures, we use regular projections to re perform Parusiński's proof of the existence of regular covers. In the second part of this thesis, we study Sobolev sheaves (in the sense of Lebeau). For Sobolev functions of positive integer regularity, we construct these sheaves on the definable site of a surface based on basic observations of definable domains in the plane.

Key words: Metric geometry of singularities, Regular projections, O-minimal structures, Real singularity theory, Semialgebraic and subanalytic geometry, Sobolev sheaves.

To the soul of my mother

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Introduction

The metric study of singular spaces has been initiated by Mostowski in his paper '*Lipschitz equisingularity*' (see[19]) to construct Lipschitz stratification for complex singular spaces. The reel case has been explored by Parusiński (see [21]), where new techniques such as L-regular decompositions and the preparation theorem were introduced. This techniques have been useful for many applications (for example in the proof of Isoperimetric inequality on subanalytic sets [25], approximation of geodesics on singular sets [1], C^m semialgebraic sections over the plane [3], the construction of sheaves on the subanalytic site [12], and series of works of Valette on the extension of PDE's theory on singular spaces (see [27], [28], [29], and [30])). The metric structure of real singular spaces have been also studied by Valette in his paper [31], where he introduced metric triangulation, regular systems of hypersurfaces and the notion of good direction, that has been extremely useful for application in different directions (intersection (co)homology and L^p cohomology of singular spaces, algebraic geometry, and PDE's on singular spaces).

In this thesis we are interested in questions around the metric study of singular spaces definable in o-minimal structures on the real field $(\mathbb{R}, +, \cdot)$. In Chapter 2 we study Regular projections in the sense of Mostowski. The regular projection theorem was first introduced by Mostowski (see [19]) for complex analytic hypersurfaces, and then later a subanalytic version was proved by Parusiński (see [21]). The theorem states that for any compact subanalytic set $X \subset \mathbb{R}^n$ there exists a finite set of regular projections (see Definition 2.1.1 below). In [21], Parusiński showed that a generic choice of $n+1$ projections is sufficient. The theorem has many applications, it leads to the proof of some important metric proprieties of subanalytic sets, it has been used by Parusiński in [23] to prove the existence of Lipschitz stratification of subanalytic sets, more precisely, to show that every compact subanalytic set can be decomposed in a finite union of L-regular sets in such a way that it is easy to glue Lipschitz stratifications of the pieces. The regular projection theorem has been also used recently by Parusiński in [22] to prove the existence of regular cover for subanalytic relatively compact open subsets which is used in [6] and [12] to construct Sobolev sheaves on the subanalytic site. In this Chapter we prove that regular projections exists in any polynomially bounded o-minimal structures and we show also that the result fails in non-polynomially bounded structures. We use this result to adapt the proof of the existence of regular covers given in [22], and as a consequence we have the existence of Sobolev sheaves (in the derived sense and for negative regularity (see [12])) on any definable site.

In Chapter 3 we focus on the sheafification of Sobolev spaces on the subanalytic site. The problem of the sheafification of Sobolev spaces on the subanalytic site is motivated by Kashiwara's proof of the Riemann-Hilbert correspondence [8] based on the construction of the sheaf of tempered distribution on the subanalytic site (Grothendieck topology formed by bounded open subanalytic sets and finite coverings). The sheafification in the derived sense for negative fractional Sobolev spaces was given by Lebeau [12] based on the results of Guillermou and Schapira in [6] and the existence of good subanalytic covers in [22]. In this chapter we give an easy explicit construction of Sobolev sheaves (in the usual sense) for Sobolev spaces with regularities in $\mathbb{N}$ on definable surfaces based on basic properties of definable domains in $\mathbb{R}^2$. This is a first part of a project on the geometric construction of Sobolev sheaves the definable site of $\mathbb{R}^n$.

Notation:

- $\mathcal{P}(X)$: the set of subsets of X.

- $grad(f)$: is the gradient of a C^1 function f.

- If $f : A \times B \to C$ is a map, then for $b \in B$ we denote by $f(\cdot, b)$ (or also by f_b) the map

$$\begin{aligned} f(\cdot, b) : A &\to C \\ x &\mapsto f(x, b). \end{aligned}$$

- $B(v, r)$ is the open ball of radius r and center v, and $\overline{B}(v, r)$ is the closed ball of radius r and center v. We may also use the notations $B_r(v)$ and $\overline{B}_r(v)$.

- $Reg^p(X)$ is the set of points $x \in X$ such that X is a C^p manifold near x, and $Sing^p(X) = X \setminus Reg^p(X)$.

- For $v \in \mathbb{R}^{n-1}$, $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the linear projection parallel to $Vect((v, 1))$.

- For a set $A \subset \mathbb{R}^n \times \mathbb{R}^m$, for $x_0 \in \mathbb{R}^n$, we denote by A_{x_0} the set

$$A_{x_0} = \{y \in \mathbb{R}^m \ : \ (x_0, y) \in A\ \}.$$

- If $F(y, x) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is C^1, we denote by $D_1 F$ or $D_y F$ the differential of F with respect to the first variable, and we denote by $D_x F$ or $D_2 F$ the differential of F with respect to the second variable.

- $\overline{A}$ denotes the topological closure of A.

- For a set $U \subset \mathbb{R}^n$, we denote by $\partial U := \overline{U} \setminus \mathring{U}$, where $\mathring{U}$ is the interior of U.

- $B \subset \mathbb{R}^k$ is called an open (closed) box if it can be written as a product

$$B = I_1 \times .... \times I_k,$$

where the I_i's are open (closed) intervals in $\mathbb{R}$ with $\mathring{I}_i \neq \emptyset$ for each i.

- We denote by $\mathbb{N}$ the set of nonnegative integers.

- If $A \subset \mathbb{R}$, we denote by A_+, A_-, and A^* as the following

$$\begin{aligned} A_+ &= \{x \in A \ : \ x \geqslant 0\}, \\ A_- &= \{x \in A \ : \ x \leqslant 0\}, \\ A^* &= \{x \in A \ : \ x \neq 0\}. \end{aligned}$$

We set $A_+^* := A_+ \cap A^*$ and $A_-^* := A_- \cap A^*$.

- A map $f : \mathbb{R} \to \mathbb{R}$ is said to be ultimately equal to 0 if there is $M \in \mathbb{R}$ such that $f(x) = 0$ for all $x > M$.

- For $n > m$ we denote by $\pi^n_m : \mathbb{R}^n \to \mathbb{R}^m$ the standard projection, and we will use π for the case of $m = n - 1$.

- For a map $f : A \to B$, Γ_f denotes the graph of f.

- For two functions $f \leqslant g : A \to \mathbb{R}$, we denote by $\Gamma(f, g)$ the set:

$$\Gamma(f, g) := \{(x, y) \in A \times \mathbb{R} \ : \ f(x) \leqslant y \leqslant g(x)\}.$$

We will also use the notation $\Gamma(A, f, g)$ if it is necessary to specify the set A.

- $W^{s,2}(\mathbb{R}^n)$: the space of Sobolev functions on $\mathbb{R}^n$.

- $\|x\|_E$ the norm of x in a normed space $(E, \|\cdot\|)$.

- If we have functions $\phi_1 < \phi_2 : A \to \mathbb{R}$, then $\Gamma(\phi_1, \phi_2)$ denotes the set

$$\Gamma(\phi_1, \phi_2) := \{(x, y) \in A \times \mathbb{R} \ : \ \phi_1(x) < y < \phi_2(y)\}.$$

- $\mathbb{S}^{n-1}$ denotes the standard sphere of $\mathbb{R}^n$, that is the set of points in $\mathbb{R}^n$ with distance equal to 1 from the origin.

- $\mathcal{S}(\mathbb{R}^n)$: the space of Schwartz functions on $\mathbb{R}^n$.

- $\mathcal{D}(\mathbb{R}^n)$: the space of compactly supported functions on $\mathbb{R}^n$.

- $\mathcal{D}'(\mathbb{R}^n)$: the topological dual of $\mathcal{D}(\mathbb{R}^n)$, that is the space of distributions.

Contents

Chapter 1

O-minimality

O-minimal geometry is a branch of mathematics that has been introduced in model theory. It gives a beautiful context for tame geometry, a geometry where pathological phenomena are not allowed but we still keeping the analytic and algebraic objects with more flexibility. In this chapter we will give a brief introduction to the basic concept of o-minimality. we choose here to follow the spirit of axiomatizing the suitable properties of semialgebraic geometry rather than going throw the model theoretical language.

An o-minimal structure on the field $(\mathbb{R}, +, \cdot)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, we have:

(1) $\mathcal{D}_n \subset \mathcal{P}(\mathbb{R}^n)$ is stable by complement and finite union.

(2) For any $P \in \mathbb{R}[X_1, ..., X_n]$, we have $Z(P) \in \mathcal{D}_n$, where

$$Z(P) = \{x \in \mathbb{R}^n \ : \ P(x) = 0\}.$$

(3) $\pi(\mathcal{D}_n) \subset \mathcal{D}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection.

(4) The sets in $\mathcal{D}_1$ are precisely the finite unions of intervals and points.

For a fixed o-minimal structure $\mathcal{D}$, we have the definitions:

(•) Elements of $\mathcal{D}_n$ are called definable sets.

(•) If $A \in \mathcal{D}_n$ and $B \in \mathcal{D}_m$, then a map $f : A \to B$ is called a definable map if its graph is a definable set.

(•) The o-minimal structure $\mathcal{D}$ is said to be polynomially bounded if for each definable function $f : \mathbb{R} \to \mathbb{R}$, there is some $n \in \mathbb{N}$ such that $|f(x)| \leq x^n$ for x big enough.

(•) For $p \in \mathbb{N} \cup \{\infty\}$ a definable C^p-manifold is a manifold with finite atlas of definable transition maps and definable domains, see e.g. [4].

Note that if $\mathcal{A} = \{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is a collection of sets $\mathcal{A}_n \subset \mathcal{P}(\mathbb{R}^n)$, then the smallest structure on $(\mathbb{R}, +, \cdot)$ that contain $\mathcal{A}$ make sense since there is at least one structure that contain $\mathcal{A}$ (the collection of all subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$). We denote this structure by $(\mathbb{R}, +, \cdot, \mathcal{A})$.

Example 1.0.1. we present here some classical examples of o-minimal structures:

(1) $\mathbb{R}_{\text{alg}} := \{\mathcal{S}_n\}$ the o-minimal structure of semialgabraic sets, that means $A \in \mathcal{S}_n$ if there exist a finite number of polynomials $P_{i,j}, Q_i \in \mathbb{R}[x_1, ..., x_n]$ such that

$$A = \cup_j \cap_i \{x \in \mathbb{R}^n \ : \ Q_i(x) = 0 \text{ and } P_{i,j}(x) > 0\}$$

By the Taski-Seidenberg theorem $\mathbb{R}_{\text{alg}}$ is an o-minimal structure, and it is the smallest one by definition.

(2) $\mathbb{R}_{\text{an}} := (\mathbb{R}, +, \cdot, \mathcal{A})$, where $\mathcal{A}_{n+1}$ is the collection of graphs of restricted analytic functions on $[-1,1]^n$. $(\mathbb{R}_{\text{an}})_n$ is the set of globally subanalytic subsets of $\mathbb{R}^n$. It follows from Grabrielov's complement theorem that $\mathbb{R}_{\text{an}}$ is an o-minimal structure.

(3) $\mathbb{R}_{\exp} := (\mathbb{R}, +, \cdot, \exp)$ the smallest structure that contains the graph of the exponential function $x \mapsto e^x$. It was proven by L. van den Dries [34] (by Model theory) that $\mathbb{R}_{\exp}$ is an o-minimal structure (see also Wilkie [37] and Khovanskii [9]). It follows from Wilkie [37] that $A \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{\exp}$ if and only if there are $N \geqslant n$ and $P \in \mathbb{R}[x_1, ..., x_k, \exp(x_{k+1}), ..., \exp(x_N)]$ such that

$$A = \pi_n^N(Z(P)).$$

(4) $\mathbb{R}_{\text{an,exp}} := (\mathbb{R}, +, \cdot, \mathbb{R}_{\text{an}}, \exp)$ the smallest structure that contains globally subanalytic sets and the graph of the exponential function. It was proven by van den Dries and Miller [35] that $\mathbb{R}_{\text{an,exp}}$ is o-minimal.

(5) Take $f_1, ..., f_k : \mathbb{R}^n \to \mathbb{R}$ a *Pfaffian chain*, and let $\mathcal{P}(f_1, ..., f_k)$ be the class of *Pfaffian functions*, that is $h \in \mathcal{P}(f_1, ..., f_k)$ if there is $Q \in \mathbb{R}[x_1, ..., x_n, y_1, ..., y_k]$ such that

$$h(x) = Q(x, f_1(x), ..., f_k(x)).$$

By Wilkie [36] and Khovanskii [9] the structure $\mathbb{R}_{f_1,...,f_k} := (\mathbb{R}, +, \cdot, \mathcal{P}(f_1, ..., f_k))$ is o-minimal.

Theorem 1.0.2. *(Miller's Dichotomy [18]). An o-minimal structure $\mathcal{D}$ is either polynomially bounded or contains the graph of the exponential function.*

From Miller's Dechotomy it's clear that $\mathbb{R}_{\text{alg}}$ and $\mathbb{R}_{\text{an}}$ are polynomially bounded.

Definition 1.0.3. Let $\mathcal{A}$ an o-minimal structure on $(\mathbb{R}, +, \cdot)$. We define first order formula (in the structure $\mathcal{A}$) as follows:

(i) If $A \in \mathcal{A}_n$ (for some $n \in \mathbb{N}$), then $\varphi(x) :=$ "$x \in A$" is a first order formula.

(ii) If $\varphi(x)$ and $\phi(x)$ are first order formulas, then "*not* $\varphi(x)$", "$\varphi(x)$ *and* $\phi(x)$", and "$\varphi(x)$ *or* $\phi(x)$" are first order formulas.

(iii) If $\varphi(x, y)$ is a first order formula and A definable sets, then "$\forall y \in A : \ \varphi(x, y)$" and "$\exists y \in A : \ \varphi(x, y)$" are first order formulas.

The following theorem is indispensable in practice.

Theorem 1.0.4. *If $\varphi(x_1, ..., x_n)$ is a first order formula, then the set of $(x_1, ..., x_n)$ in $\mathbb{R}^n$ which satisfy $\varphi(x_1, ..., x_n)$ is definable.*

1.0.1 Cell decomposition.

Take $p \in \mathbb{N}$. A definable set $C \subset \mathbb{R}^n$ is said to be a C^p-cell (with respect to a fixed system of coordinates in $\mathbb{R}^n$) if:

- *Case* $n = 1$*:* C is either a point or an open interval.
- *Case* $n \geq 2$*:* C is one of the following:
  - $C = \Gamma_\phi$ (the graph of ϕ), where $\phi : B \to \mathbb{R}$ is a C^p definable function, where B is a C^p-cell in $\mathbb{R}^{n-1}$.
  - $C = \Gamma(\phi, \varphi) = \{(x_1, ..., x_n) \in B \times \mathbb{R} \ : \ \phi(x_1, ..., x_{n-1}) < x_n < \varphi(x_1, ..., x_{n-1})\}$, where ϕ and φ are two C^p definable functions on a C^p-cell B, such that $\phi < \varphi$ with the possibility of $\phi = -\infty$ or $\varphi = +\infty$.

A C^p-cell decomposition of $\mathbb{R}^n$ (with respect to a fixed system of coordinates) is defined by induction as follows:

- A C^p-cell decomposition of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ by points and open intervals.
- A C^p-cell decomposition of $\mathbb{R}^n$ is a finite partition $\mathcal{A}$ of $\mathbb{R}^n$ by C^p-cells, such that $\pi(\mathcal{A})$ is a C^p-cell decomposition of $\mathbb{R}^{n-1}$, where

$$\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$$

is the standard projection and $\pi(\mathcal{A})$ is the family

$$\pi(\mathcal{A}) = \{\pi(A) \ : \ A \in \mathcal{A} \ \}.$$

Theorem 1.0.5. *Let $p \in \mathbb{N}$ and $\{X_1, ..., X_k\}$ be a finite family of definable sets of $\mathbb{R}^n$. Then there is a C^p-cell decomposition of $\mathbb{R}^n$ compatible with this family, i.e. each X_i is a union of some cells.*

Proof. See [2] or [33]. □

Now we can define the dimension of a definable set. Take X a definable subset of $\mathbb{R}^n$ and $\mathcal{C}$ a cell decomposition of $\mathbb{R}^n$ compatible with X. Then we define the dimension of X by

$$\dim_{\mathcal{C}}(X) = \max\{\dim(C) \ : \ C \subset X \ and \ C \in \mathcal{C}\}.$$

This number does not depend on $\mathcal{C}$, we denote it by $\dim(X)$.
From the cell decomposition theorem we get the following fundamental tools in o-minimal geometry:

Lemma 1.0.6. ***(Monotonicity theorem).*** *Let $f :]a, b[\to \mathbb{R}$ be a definable map. Then there are points $a = a_0 < a_1 < .. < a_k = b$ such that $f \restriction_{]a_i, a_{i+1}[}$ is either constant or strictly monotone on $]a_i, a_{i+1}[$ for each $i = 0, ..., k-1$.*

Proof. See [2] or [33]. □

Lemma 1.0.7. ***(Definable choice).*** *Take $A \subset \mathbb{R}^n \times \mathbb{R}^m$ definable and $\pi_n^{n+m} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the standard projection. Then there is a definable map $f : \pi_n^{n+m}(A) \to \mathbb{R}^m$ such that for any $x \in \pi(A)$ we have $(x, f(x)) \in A$.*

Lemma 1.0.8. ***(The curve selection Lemma).*** *Let A be a definable subset of $\mathbb{R}^n$ and $a \in \overline{A} \setminus A$. Let $p \in \mathbb{N}$. Then there exists a C^p definable curve $\gamma :]0, 1[\longrightarrow A \setminus \{a\}$, such that $\lim_{t \to 0^+} \gamma(t) = a$.*

Proof. See [2] or [33]. □

1.0.2 Lipschitz cell decomposition.

L-regular cell (Lipschitz cells) were introduced by A.Parusiński to prove the existence of Lipschitz stratification of subanalytic sets ([23], see also [10]). We give here the definition and some basic proprieties of L-regular cells. We fix $\mathcal{A}$ an o-minimal structure on $(\mathbb{R}, +, \cdot)$

Definition 1.0.9. Let $X \subset \mathbb{R}^n$ be a definable subset. We say that X is L-regular if :

- X is a point in the case of $\dim(X) = 0$.
- X is an open interval if $\dim(X) = 1$ and $n = 1$.
- If $\dim(X) = n$ (with $n > 1$), then there is $X' \subset \mathbb{R}^{n-1}$ L-regular and two C^1 definable functions with bounded derivatives $\phi_1, \phi_2 : X' \longrightarrow \mathbb{R}$ with $\phi_1 < \phi_2$, such that

$$X = \{(x', x_n) \in X' \times \mathbb{R} \ : \ \phi_1(x') < x_n < \phi_2(x')\}.$$

- If $\dim(X) = k < n$, then X is the graph of a C^1 definable map $\phi : X' \longrightarrow \mathbb{R}^{n-k}$ with bounded derivatives on $Int(X')$, where $X' \subset \mathbb{R}^k$ is L-regular and of dimension k.

We will also say that A is L-regular if it is so after a linear change of coordinates.

Theorem 1.0.10. *Let $X_1, ..., X_l$ definable subsets of $\mathbb{R}^n$, then there exists a finite definable partition $(L_k)_k$ of $\bigcup_i X_i$ compatible with each X_i such that each element L_k is L-regular.*

Proof. See [23] or [10]. □

Here we recall some basic properties of L-regular cells, the proof is obvious by induction on the dimension of L-cells.

Proposition 1.0.11. *Let $X \subset \mathbb{R}^n$ be L-regular.*

(1) *If $\dim(X) = k < n$ then the standard projection $\pi_k^n : \mathbb{R}^n \to \mathbb{R}^k$ induces a bi-Lipschitz mapping from X to $X' = \pi_k^n(X)$.*

(2) *$X \setminus \partial X$ is definably homeomorphic to the open ball $B(0, 1) \subset \mathbb{R}^{\dim(X)}$.*

(3) *The inner and the outer metrics on X are equivalent, it means that for any $x, y \in X$ there is a definable continuous curve $\gamma : [0, 1] \to X$ and a constant $C > 0$ such that $\gamma(0) = x$, $\gamma(1) = y$, and*

$$Length(\gamma) \leqslant C\,|x-y|.$$

(4) *If* $\dim(X) = n$ *and* $\phi : X \to \mathbb{R}$ *is a* C^1 *function with bounded derivative, then* ϕ *is Lipschitz.*

Chapter 2

Regular projections and applications

In [22], Parusiński asked if the regular projection theorem and the existence of regular covers can be shown in any o-minimal structures. A first answer was given by Nguyen [20], he gives a Lipschitz version of this theorem, that is we can always find a finite set of regular projections after applying a bilipschitz homeomorphism of the ambient space. Nguyen's proof is based on the cell decomposition Theorem and Valette's result on the existence of a good direction (see [31]). A direct consequence of this is the existence of regular covers in any o-minimal structure.
In this chapter we give an answer to Parusiński's question, we will prove that the regular projection theorem works in any polynomially bounded o-minimal structure, and we will give an example showing that this is no longer true in non-polynomially bounded o-minimal structures. We will also give a weak version of the regular projection theorem in any o-minimal structure, which is a straightforward consequence of the techniques used in [20]. We will use this weak version to adopt the proof in [22] to show the existence of regular covers in any o-minimal structure. Our fundamental tools for the proof will be the cell decomposition and Miller's result ([17]) to find a replacement for the Puiseux with parameter argument used in [21].

2.1 Regular projection in the sense of Mostowski.

Let $X \subset \mathbb{R}^n$ be a definable subset of $\mathbb{R}^n$. For any $\lambda \in \mathbb{R}^{n-1}$, we denote by $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$, the projection parallel to the vector $(\lambda, 1) \in \mathbb{R}^n$. Fix ε and C positive real numbers and $p \in \mathbb{N}^*$. For $v \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}^n$ we define the cone $C_\varepsilon(x, v)$ by

$$C_\varepsilon(x, v) = \{x + t(v', 1) \ : \ t \in \mathbb{R}^* \text{ and } v' \in B(v, \varepsilon)\}.$$

Definition 2.1.1. We say that the projection π_λ is (ε, p)-weak regular at a point $x \in \mathbb{R}^n$ (with respect to X) if:

(1) $(\pi_\lambda)_{|X}$ is finite.

(2) the intersection of X with the cone $C_\varepsilon(x, \lambda)$ is either empty or a disjoint finite union of sets of the form

$$A_{f_i} = \{x + f_i(\lambda')(\lambda', 1) \ : \ \lambda' \in B(\lambda, \varepsilon)\},$$

where the f_i's are non-vanishing C^p functions defined on $B(\lambda, \varepsilon)$.

We say that the projection π_λ is (ε, C, p)-regular at a point $x \in \mathbb{R}^n$ (with respect to X) if moreover we have

(3) $\|grad(f_i)\| \leq C\,|f_i|$ on $B(\lambda, \varepsilon)$ for all i.

Remark 2.1.2. The notion of regular projection introduced by Mostowski in [19] is not exactly the same as the notion of good direction (regular projection in the sense of Valette in [31], see also [20] for comparing both notions). One can show that a projection is regular in the sense of Mostowski if and only if it is weak regular and good.

2.1.1 Weak regular projection theorem.

Let $\mathcal{A}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$ (no condition on $\mathcal{A}$).

Theorem 2.1.3. *Let X be a definable subset of $\mathbb{R}^n$ such that $\dim(X) \leqslant n-1$, and $p \in \mathbb{N}^*$. Then there are $\varepsilon > 0$ and $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is i such that π_{v_i} is (ε, p)-weak regular at x with respect to X.*

For the proof we need a few results. The following lemma was proved in [20] to show a bi-Lipschitz version of the regular projection in arbitrary o-minimal structures. It states that we can avoid definable sets with empty interior up to a decomposition and product in a chosen box. This lemma is central in our proof even for the case of regular projections in polynomially bounded structures.

Lemma 2.1.4. *Take C a definable subset of $\mathbb{R}^n$ and let B be a box in $\mathbb{R}^k$. Let Δ be a definable subset of $\mathbb{R}^n \times \mathbb{R}^k$ such that $\dim(\Delta_x) \leqslant (k-1)$, for all $x \in C$. Then there exists a finite definable partition $\mathcal{C}$ of C, such that for each $D \in \mathcal{C}$ there is a box $B_D \subset B$ such that we have*

$$(D \times B_D) \cap \Delta = \emptyset.$$

Proof. Let's first prove the following special case:

Lemma 2.1.5. *Take $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$. Let $a = \xi_0 < \xi_1 < ... < \xi_k = b$ be continuous definable functions on a definable set $C \subset \mathbb{R}^n$. Then there is a definable partition $\mathcal{C}$ of C such that for any $D \in \mathcal{C}$ there exist $[a_D, b_D] \subset [a, b]$ such that*

$$D \times [a_D, b_D] \subset \Gamma(C, \xi_i, \xi_{i+1}) \text{ for some } i \in \{1, ..., k\}.$$

Proof. Take $a_1 < a_2 < ... < a_k$ real numbers in $[a, b]$. Define

$$\xi_1^+ = \{(x, y) \in \Gamma_{\xi_1} \ : \ y \geqslant a_1\} \text{ and } \xi_1^- = \{(x, y) \in \Gamma_{\xi_1} \ : \ y < a_1\},$$
$$H_1^+ = \pi(\xi_1^+) \times]a, a_1[\ , \ H_1^- = \pi(\xi_1^-) \times]a_1, +\infty[,$$
$$C_1^+ = \pi(\xi_i^+), \ C_1^- = \pi(\xi_1^-).$$

For $i \in \{2, ..., k\}$ we define

$$\xi_i^+ = \{(x,y) \in \Gamma_{\xi_i} \ : \ y \geqslant a_i\} \cap H_{i-1}^- \text{ and } \xi_i^- = \{(x,y) \in \Gamma_{\xi_i} \ : \ y < a_i\} \cap H_{i-1}^-,$$

$$H_i^+ = \pi(\xi_i^+) \times]a_{i-1}, a_i[\text{ and } H_i^- = \pi(\xi_i^-) \times]a_i, +\infty[,$$

$$C_i^+ = \pi(\xi_i^+) \text{ and } C_i^- = \pi(\xi_i^-).$$

Clearly we have

$$C = C_1^+ \sqcup C_1^- \text{ and } C_i^- = C_{i+1}^+ \sqcup C_{i+1}^-.$$

So $\mathcal{C} = \{C_i^+\}_i$ is a partition of C and we have that $H_i^+ = C_i^+ \times]a_{i-1}, a_i[\subset \Gamma(C, \xi_i, \xi_{i+1})$, so $[a_{C_i^+}, b_{C_i^-}]$ can be chosen in $]a_{i-1}, a_i[$, and this finish the proof. □

Take $\mathcal{U}$ a cell decomposition of $\mathbb{R}^{n+k}$ compatible with $C \times B$ and $\mathcal{U}_0 = \pi_n^{n+k}(\mathcal{U})$ the induced cell decomposition on $\mathbb{R}^n$. For each $S \in \mathcal{U}_0$ we put

$$\mathcal{U}(S) = \{A \in \mathcal{U} \ : \ S = \pi_n^{n+k}(A) \text{ and } \dim(A) = \dim(S) + k\}.$$

$\mathcal{U}(S)$ is the set of cells above S with fibers of maximal dimension, and it's clear that $\mathcal{U}(S)$ is not empty for any $S \in \mathcal{U}_0$. By refining cell decomposition, it's easy to see that we need just to prove the claim

Claim: There is a refinement $\mathcal{C}$ of $\mathcal{U}_0$ such that for any $D \in \mathcal{C}$ there is a box $B_D \subset B$ such that $D \times B_D$ is in some element of $\mathcal{U}(S)$ for some $S \in \mathcal{U}_0$ such that $D \subset S$.
We prove by induction on k. The case of $k = 0$ is obvious. Assume $k > 0$ and the claim is true for any $l < k$. Take the cell decomposition $\mathcal{U}' = \pi_{n+k-1}^{n+k}(\mathcal{U})$ of $C \times B'$ where $B' = \pi_{k-1}^k(B)$. For $S \in \mathcal{U}_0$ we define

$$\mathcal{U}'(S) := \{A \in \mathcal{U}' \ : \ S = \pi_n^{n+k-1}(A) \text{ and } \dim(A) = \dim(S) + k - 1\} = \pi_{n+k-1}^{n+k}(\mathcal{U}(S)).$$

Take $S \in \mathcal{U}_0$ and $L \in \mathcal{U}'(S)$. By Lemma1.1.4 we get a partition $\sum_L$ of L such that for any $D \in \sum_L$ there is $[a_D, b_D]$ such that $D \times [a_D, b_D]$ is contained in an element of $\mathcal{U}(S)$.
Take $\mathcal{B}$ a cell decomposition of $C \times B'$ compatible with

$$\mathcal{U}' \bigcup_{L \in \mathcal{U}'(S),\, S \in \mathcal{U}_0} \sum_L .$$

Put

$$\mathcal{B}_0 := \pi_n^{n+k-1}(\mathcal{B}).$$

By the inductive applied to $\mathcal{B}$ we get a partition $\mathcal{C}$ compatible with $\mathcal{B}_0$ such that for any $D' \in \mathcal{C}$ there is a box $B_{D'} \subset B'$ such that $D' \times B_{D'}$ is contained in an element $\mathcal{B}(K)$ for some $K \in \mathcal{B}_0$ and $D' \subset K$. Finally $\mathcal{C}$ is the wanted partition in our claim. □

Remark 2.1.6. In Lemma 2.1.4 we can replace $\mathbb{R}^n$ by $\mathbb{S}^n$ (we need this in the proof of Lemma 2.1.16). Indeed take a definable subset $C \subset \mathbb{S}^n$ and $\Delta \subset \mathbb{S}^n \times \mathbb{R}^k$ such that for any $x \in C$

$$\dim(\Delta_x) < k.$$

Fix $B \subset \mathbb{R}^k$ a box. If $C = \{x\}$ consists of one point x then clearly Δ_x has empty interior (because $\dim(\Delta_x) < k$). So we can find a box $B_x \subset B$ such that $(\{x\} \times B_x) \cap \Delta = \emptyset$. Then we can assume that $C \subset \mathbb{S}^n \setminus \{N_0 = (0, 0, ..., 1)\}$ and we go back to the case of Lemma 2.1.4 by identifying definably $\mathbb{S}^n \setminus \{N_0\}$ and $\mathbb{R}^n$.

We need Transversality theorem with parameters. It was used in [15] and the proof is a consequence of Sard's theorem in the setting of o-minimal structures.

Sard's theorem in the o-minimal category: Take N and M two C^1 definable manifolds, $f : N \to M$ a definable C^1 map, and $s \in \mathbb{N}$. Take

$$C_s = \{x \in N \ : \ \mathrm{rank}(df(x)) < s\}.$$

Then $f(C_s)$ is definable and $\dim(f(C_s)) < s$.

Proof. It's clear that $f(C_s)$ is definable since C_s is defined by a first order formula and f is definable. Assume now that $\dim(f(C_s)) \geqslant s$. By the definable choice (Lemma 1.0.7) there are a C^1 cell $X \subset f(C_s)$ and a definable C^1 map $g : X \to C_s$ such that:

$$f \circ g(y) = y \text{ for all } y \in X.$$

This implies that $\mathrm{rank}(df(g(y)) \circ dg(y)) \geqslant s$ for any $y \in X$, and so $\mathrm{rank}(df(t)) \geqslant s$ for any $t \in g(X) \subset C_s$ and this is a contradiction. □

Lemma 2.1.7. *Let M, J, and N be definable manifolds, and let*

$$f : M \times J \to N$$

be a C^1 definable submersion. Take $\mathcal{S}$ a finite collection of C^1 definable submanifolds of N. Then

$$\rho(f, \mathcal{S}) = \{s \in J \ : \ f(\cdot, s) \textit{ is transverse to all elements of } \mathcal{S}\}$$

is a definable set and we have $\dim(J \setminus \rho(f, \mathcal{S})) < \dim(J)$.

Proof. It's enough to assume that $\mathcal{S}$ consists of one single manifolds $X \subset N$. We have that $V = f^{-1}(X)$ is a definable submanifold of $M \times J$ (because f is a submersion). Let $\pi : M \times J \longrightarrow J$ be the projection on J. By Sard's theorem we have that:

$$\dim(\{s \in J \ : \ s \text{ is regular value of } \pi_{|V}\}) < \dim(J).$$

But a simple computation shows that if s is regular value of $\pi_{|V}$ then f_s is transverse to X, and this finish the proof. □

Proof of Theorem 2.1.3 : Take $m \in \mathbb{N}$. Let $P(x, v, \varepsilon)$ be a property on $(x, v, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^*_+$, i.e a map $P : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^*_+ \to \{0, 1\}$. Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ the standard projection.

Lemma 2.1.8. *If we have:*

(1) for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$, ε *and* ε' *such that* $\varepsilon < \varepsilon'$, *we have*

$$P(x, v, \varepsilon') \text{ is true} \Rightarrow P(x, v, \varepsilon) \text{ is true.}$$

(2) $\mathcal{X} = \{(x, v, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^*_+ \ : \ P(x, v, \varepsilon) \text{ is true}\}$ *is definable and for all* $x \in \mathbb{R}^n$ $\dim(\mathcal{B}_x) < m$, *where* $\mathcal{B}$ *is defined by*

$$\mathcal{B} = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ \pi^{-1}(x, v) \cap \mathcal{X} = \emptyset\}.$$

Then there are $\varepsilon_0 > 0$ *and* $\mathbb{A} = \{v_1, ..., v_k\} \subset \mathbb{R}^m$ *such that for all* $x \in \mathbb{R}^n$ *there is some* i *such that* $P(x, v_i, \varepsilon_0)$ *is true.*

Proof. By the assumptions (1) and (2), the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ given by

$$F(x, v) = \begin{cases} \sup\{\varepsilon : \ P(x, v, \varepsilon) \ is \text{ true}\} & \text{if } \{\varepsilon : \ P(x, v, \varepsilon) \ is \text{ true}\} \neq \emptyset, \\ 0 & \text{if } \{\varepsilon : \ P(x, v, \varepsilon) \ is \text{ true}\} = \emptyset. \end{cases}$$

is well defined and definable.
Now take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ compatible with $\mathbb{R}^n \times \mathbb{R}^m \times \{0\}$, $\mathcal{B} \times \{0\}$ and Γ_F (the graph of F).
If $\mathcal{D}_{n+m}$ is the cell decomposition of $\mathbb{R}^n \times \mathbb{R}^m$ determined by $\mathcal{D}$ (it means that $\pi(\mathcal{D}) = \mathcal{D}_{n+m}$), then for any $C \in \mathcal{D}_{n+m}$ of maximal dimension, some of the bands over C (at least one) are in $\mathcal{X}$ and the others in $\mathcal{X}^c$. Hence, by Lemma 2.1.4, we can find a cell decomposition $\mathcal{D}'_{n+m}$, finer than $\mathcal{D}_{n+m}$, such that for all $C \in \mathcal{D}'_{n+m}$ with $\dim(C) = n + m$ there is a box $[a_C, b_C] \subset \mathbb{R}^*_+$ such that $C \times [a_C, b_C] \subset \mathcal{X}$.
Now, since $\dim(\mathcal{B}_x) < m$ for all $x \in \mathbb{R}^n$, we can apply Lemma 2.1.4 again and find a partition $\mathcal{P}$ of $\mathbb{R}^n$ such that for all $P \in \mathcal{P}$ there is a box $B_P = [v_P(1), v'_P(1)] \times ... \times [v_P(m), v'_P(m)] \subset \mathbb{R}^m$ with $P \times B_P$ included in some band C_P of $\mathcal{D}'_{n+m}$, and hence

$$P \times B_P \times [a_{C_P}, b_{C_P}] \subset \mathcal{X}.$$

Finally we can take $\mathbb{A} = \{v_P = (v_P(1), ..., v_P(m))\}_{P \in \mathcal{P}}$ and $\varepsilon_0 = \min_{P \in \mathcal{P}}\{a_{C_P}\}$.

□

Now we define $P_X(x, v, \varepsilon)$ as follows:

$$P_X(x, v, \varepsilon) \text{ is true if } \pi_v \text{ is } (\varepsilon, p)\text{-weak regular at } x \text{ with respect to } X.$$

It is obvious that P_X satisfies (1) of the precedent lemma. Let's prove that also (2) holds true for P_X. We have

$$\begin{aligned} \mathcal{X} &= \{(x, v, \varepsilon) \ : \ \pi_v \ is \ \varepsilon\text{-weak regular at } x \text{ with respect to } X\} \\ &= \mathcal{X}_1 \cup \mathcal{X}_2, \end{aligned}$$

where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined below.

$$\begin{aligned} \mathcal{X}_1 &= \{(x, v, \varepsilon) \ : \ C_\varepsilon(x, v) \cap X = \emptyset\} \\ &= \{(x, v, \varepsilon) \ : \ x + t(v', 1) \notin X \text{ for any } (t, v) \in \mathbb{R}^* \times B(v, \varepsilon)\}. \end{aligned}$$

Hence $\mathcal{X}_1$ is defined by a first order formula, and therefore $\mathcal{X}_1$ is a definable set.

$$\begin{aligned}\mathcal{X}_2 = \{(x, v, \varepsilon) \ : \ & \text{such that } C_\varepsilon(x, v) \cap X \neq \emptyset,\ C_\varepsilon(x, v) \cap X \subset Reg^p(X), \\ & \text{and } \forall y \in C_\varepsilon(x, v) \cap X \text{ there are } t \in \mathbb{R}^* \text{ and } v' \in B(v, \varepsilon) \text{ with} \\ & y = x + t(v', 1) \text{ and the line through } x \text{ directed by } (v', 1) \text{ is} \\ & \text{transverse to } X \text{ at } y\}.\end{aligned}$$

Hence $\mathcal{X}_2$ is also defined by a first order formula, thus it is a definable set. Finally $\mathcal{X}$ is definable set. Let's prove that $\dim(\mathcal{B}_x) < n - 1$ for all $x \in \mathbb{R}^n$. Assume that this is not true. For $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n-1}$ we denote by $L(x, v)$ the line that passes through x and is directed by $(v, 1)$. For all $x \in \mathbb{R}^n$ we have

$$\mathcal{B}_x = \mathcal{B}_x^1 \cup \mathcal{B}_x^2,$$

where

$$\mathcal{B}_x^1 = \{v \in \mathbb{R}^{n-1} \ : \ \text{such that } L(x, v) \text{ is not transverse to } Reg^p(X)\}$$

and

$$\mathcal{B}_x^2 = \{v \in \mathbb{R}^{n-1} \ : \ L(x, v) \cap Sing^p(X) \neq \emptyset\ \}.$$

We define the definable map

$$\begin{aligned}\phi : \mathbb{R}^{n-1} \times \mathbb{R}^* &\to \mathbb{R}^n \\ (v, t) &\mapsto \phi(v, t) = x + t(v, 1).\end{aligned}$$

A simple calculation shows that ϕ is a submersion, hence it is a local diffeomorphism, but since $\dim(Sing^p(X)) < (n - 1)$, we can deduce that

$$\dim(\phi^{-1}(Sing^p(X))) < n - 1,$$

therefore this means that

$$\dim(\pi(\phi^{-1}(Sing^p(X)))) < n - 1,$$

hence

$$\dim(\mathcal{B}_x^2) < n - 1.$$

But by our assumption there must be an $x_0 \in \mathbb{R}^n$ such that $\dim(\mathcal{B}_{x_0}^1) = n - 1$, and

$$\mathcal{B}_{x_0}^1 = \{v \in \mathbb{R}^{n-1} \ : \ \text{such that } \phi(v, \cdot) \text{ is not transverse to } Reg^p(X)\}.$$

Finally, by Lemma 2.1.7 we deduce that $\dim(\mathcal{B}_{x_0}^1) < n - 1$, and this is a contradiction with $\dim(\mathcal{B}_{x_0}^1) = n - 1$. This completes the proof of Theorem 2.1.3.

□

Remark 2.1.9.(i) By the proof of Theorem 2.1.3 we can require $\{v_1, ..., v_k\}$ to be chosen in a given open subset of $(\mathbb{R}^{n-1})^k$.

(ii) If $\dim(X) < (n-1)$, then we can see that $\mathcal{X}_2 = \emptyset$, hence this means that we can find a finite set $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ and $\varepsilon > 0$ such that for any $x \in \mathbb{R}^n$ there is an i such that

$$C_\varepsilon(x, v_i) \cap X = \emptyset.$$

Question 1. From the proof of Theorem 2.1.3 we can minimize the number of the projections by

$$\begin{aligned} k_m := \min\{k \in \mathbb{N} :\ & k = |\mathcal{P}| \text{ where } \mathcal{P} \text{ is a partition of } \mathbb{R}^n \\ & \text{such that for all } P \in \mathcal{P} \text{ there is a box} \\ & B_P = [v_P(1), v'_P(1)] \times ... \times [v_P(m), v'_P(m)] \subset \mathbb{R}^m \\ & \text{such that } P \times B_P \text{ is included in some band } C_P \text{ of } \mathcal{D}'_{n+m}\}. \end{aligned}$$

Clearly this number depends on n and X, therefore we ask the question:
Can the number of projections be chosen independently of X?

Question 2. Can the set of projections be chosen in an open definable dense subset of $(\mathbb{R}^{n-1})^k$?

2.1.2 Counterexample in non-polynomially bounded o-minimal structures.

Fix $\mathcal{A}$ a non-polynomially bounded o-minimal structure on $(\mathbb{R}, +, \cdot)$. Assume that the Regular projection theorem is true for this structure, and consider the set X defined by

$$X = X_1 \cup X_2,$$

where

$$X_1 = \{(x, ax^{a+1}, x^{a+1}) \ : \ x > 0 \text{ and } a \in \mathbb{R}\}$$
$$X_2 = \{(x, -ax^{a+1}, x^{a+1}) \ : \ x > 0 \text{ and } a \in \mathbb{R}\}.$$

For all $p \in \mathbb{N}^*$ we have : $Reg^p(X) = X \setminus \{(x, 0, x) \ : \ x > 0\}$, and the connected components of $Reg^p(X)$ are C^p-manifolds.

By Miller's Dichotomy (Theorem 1.0.2), the graph of the exponential function $x \mapsto \exp(x)$ is a definable set, and therefore X is a definable set.
Take the germ of the curve $x(s) = (s, 0, 0)$. By the cell decomposition and the assumption there are $\delta > 0$, a vector $v = (v_1, v_2) \in \mathbb{R}^2$, $\varepsilon > 0$ and $C > 0$ such that for all $s \in [0, \delta[$ we have one of the following two cases:

(1) $C_\varepsilon(x(s), v) \cap X = \emptyset$.

(2) $C_\varepsilon(x(s), v) \cap X = \sqcup A_{f_s^i}$, where $f_s^i : B(v, \varepsilon) \to \mathbb{R}^*$ are C^p-regular definable ordered functions such that for all s, i, and $\lambda \in B(v, \varepsilon)$ we have

$$\frac{\|grad\ f_s^i(\lambda)\|}{|f_s^i(\lambda)|} \leq C.$$

We are interested in the second case, because for s small enough we have

$$C_\varepsilon(x(s), v) \cap X \neq \emptyset.$$

To obtain a contradiction, let's prove the next two facts:

Fact 1 : For $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$, the functions $s \mapsto f_s^i(\lambda)$ are characterized by the functional equation

$$f_s^i(\lambda) = (s + \lambda_1 f_s^i(\lambda))^{\pm\lambda_2+1}.$$

Indeed, take $s \in [0, \delta[$.
Since $x(s) + f_s^i(\lambda)(\lambda, 1) \in C_\varepsilon(x(s), v) \cap X$, there are some $\widehat{x} \in \mathbb{R}_+$ and $a \in \mathbb{R}$ such that

$$x(s) + f_s^i(\lambda)(\lambda, 1) = (\widehat{x}, \pm a\widehat{x}^{a+1}, \widehat{x}^{a+1}).$$

Hence

$$\begin{aligned} \widehat{x} &= s + f_s^i(\lambda)\lambda_1, \\ f_s^i(\lambda) &= \widehat{x}^{a+1}, \\ \pm a\widehat{x}^{a+1} &= \lambda_2 \widehat{x}^{a+1}. \end{aligned}$$

And this implies

$$\lambda_2 = \pm a \quad \text{and} \quad f_s^i(\lambda) = (s + \lambda_1 f_s^i(\lambda))^{\pm\lambda_2+1}.$$

But since the sets $A_{f_s^i}$ are connected definable C^p-manifolds and the connected components of $Reg^p(X)$ are C^p-definable manifolds we deduce that the sign $\pm$ on λ_2 depends only on i.
Let's take $\lambda \in B(v, \varepsilon)$ and a definable continuous function $s \mapsto t_s \in \mathbb{R}_+^*$ such that

$$t_s = (s + \lambda_1 t_s)^{\pm\lambda_2+1}.$$

Fix $s \in [0, \delta[$. Then we have

$$x(s) + t_s(\lambda, 1) \in C_\varepsilon(x(s), v).$$

Let's take

$$\widehat{x} = s + \lambda_1 t_s \text{ and } a = \pm\lambda_2.$$

Then

$$x(s) + t_s(\lambda, 1) = (\widehat{x}, \pm a\widehat{x}^{a+1}, \widehat{x}^{a+1}).$$

Hence

$$x(s) + t_s(\lambda, 1) \in C_\varepsilon(x(s), v) \cap X.$$

This implies that there are i_s and $\lambda' \in B(v, \varepsilon)$ such that

$$x(s) + t_s(\lambda, 1) = x(s) + f_s^{i_s}(\lambda')(\lambda', 1).$$

And this implies that $\lambda = \lambda'$ and $t_s = f_s^{i_s}(\lambda)$.
By the fact that $f_s^1 < .... < f_s^k$ (because these functions are ordered), and the functions $s \mapsto t_s$ and $s \mapsto f_s^i(\lambda)$ are continuous and definable, we deduce that i_s doesn't depend on s, hence $s \mapsto t_s$ is one of the functions $s \mapsto f_s^i(\lambda)$.
This finishes the proof of **Fact 1**.

Fact 2:

(•) If $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ is such that $\lambda_2 > 0$, then there is a solution $s \mapsto t_s$ of the equation

$$t_s = (s + \lambda_1 t_s)^{\lambda_2 + 1}$$

such that $\lim_{s \to 0} t_s = 0$.
For that, we define the function $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(s,t) = \begin{cases} t - (s + \lambda_1 t)^{\lambda_2 + 1} & \text{if } s + \lambda_1 t > 0, \\ t & \text{if } s + \lambda_1 t \leqslant 0. \end{cases}$$

F is a C^1 function and we have

$$F(0,0) = 0 \text{ and } \tfrac{\partial F}{\partial t}(0,0) = 1.$$

Hence by the Implicit Function Theorem, there is a continuous function $s \mapsto t(s)$ such that $t(0) = 0$ and $F(s, t(s)) = 0$.

(•) If $\lambda \in B(v, \varepsilon)$ is such that $\lambda_2 < 0$, then there is a solution $s \mapsto t_s$ of the equation

$$t_s = (s + \lambda_1 t_s)^{-\lambda_2 + 1}$$

such that

$$\lim_{s \to 0} t_s = 0.$$

We use the same argument as in the first case by applying the Implicit Function Theorem to the function:

$$F(s,t) = \begin{cases} t - (s + \lambda_1 t)^{-\lambda_2 + 1} & \text{if } s + \lambda_1 t > 0, \\ t & \text{if } s + \lambda_1 t \leqslant 0. \end{cases}$$

Now we will discuss the projection in two cases:

• *Case 1:* Assume that $v_2 \geqslant 0$, and take $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ such that $\lambda_2 > 0$. From **Facts 1 and 2** there is an i_0 such that $\lim_{s \to 0} f_s^{i_0}(\lambda) = 0$ and

$$f_s^{i_0}(\lambda) = (s + \lambda_1 f_s^{i_0}(\lambda))^{\lambda_2+1}.$$

Therefore we have:

$$\frac{\partial f_s^{i_0}}{\partial \lambda_2}(\lambda) = \left(\ln\left(s + \lambda_1 f_s^{i_0}(\lambda)\right) + \frac{(\lambda_2+1)\lambda_1 \frac{\partial f_s^{i_0}}{\partial \lambda_2}(\lambda)}{s + \lambda_1 f_s^{i_0}(\lambda)} \right) f_s^{i_0}(\lambda).$$

Then we have:

$$\frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \left(1 - (\lambda_2+1)\lambda_1 \left(s + \lambda_1 f_s^{i_0}(\lambda)\right)^{\lambda_2}\right) = \ln\left(s + \lambda_1 f_s^{i_0}(\lambda)\right).$$

Hence, since $\lambda_2 > 0$ and $\lim_{s\to 0} f_s^{i_0}(\lambda) = 0$, we deduce that

$$\lim_{s\to 0} \left| \frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \right| = +\infty.$$

Thus, this is a contradiction with the fact that

$$\left| \frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \right| \leq C \ \ \forall s.$$

- *Case 2:* Assume that $v_2 < 0$, and take $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ such that $\lambda_2 < 0$. From **Fact 1 and 2** there is an i_0 such that

$$\lim_{s\to 0} f_s^{i_0}(\lambda) = 0$$

and

$$f_s^{i_0}(\lambda) = \left(s + \lambda_1 f_s^{i_0}(\lambda)\right)^{-\lambda_2+1}.$$

We have

$$\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda) = \left(-\ln\left(s + \lambda_1 f_s^{i_0}(\lambda)\right) + \frac{(-\lambda_2+1)\lambda_1 \frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{s + \lambda_1 f_s^{i_0}(\lambda)} \right) f_s^{i_0}(\lambda).$$

Then

$$\frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \left(1 - (-\lambda_2+1)\lambda_1 \left(s + \lambda_1 f_s^{i_0}(\lambda)\right)^{-\lambda_2}\right) = -\ln\left(s + \lambda_1 f_s^{i_0}(\lambda)\right).$$

Hence, since $-\lambda_2 > 0$ and $\lim_{s\to 0} f_s^{i_0}(\lambda) = 0$, we deduce that

$$\lim_{s\to 0} \left| \frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \right| = +\infty.$$

Hence this is a contradiction with the fact that

$$\left| \frac{\frac{\partial f_s^{i_0}}{\partial \lambda 2}(\lambda)}{f_s^{i_0}(\lambda)} \right| \leq C \ \ \forall s.$$

□

2.1.3 Regular projection theorem in polynomially bounded structures.

Let $\mathcal{A}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$ and let $\mathcal{K}$ be the field of exponents of $\mathcal{A}$, it means that

$$\mathcal{K} = \{r \in \mathbb{R} \ : \ x \mapsto x^r \text{ is definable in } \mathcal{A} \ \}.$$

Before beginning the proof of the theorem we recall a few results:

Lemma 2.1.10. ***(Piecewise Uniform Asymptotics).*** *Assume that $\mathcal{A}$ is polynomially bounded. Let $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a definable function. Then there is a finite $S \subset \mathcal{K}$ such that for each $y \in \mathbb{R}^m$ either the function $f(y, \cdot)$ is ultimately equal to 0 or there exists $r \in S$ such that $\lim_{t \to +\infty} \frac{f(y,t)}{t^r} \in \mathbb{R}^*$.*

Proof. This is proven by induction on m.

• *Case $m = 0$:* This is a direct consequence of Miller's Dichotomy, for any definable function $f : \mathbb{R} \to \mathbb{R}$ there is $r \in \mathcal{K}$ such that if f not ultimately 0 then

$$\lim_{t \to +\infty} \frac{f(t)}{t^r} \in \mathbb{R}^\star.$$

A simple computation shows that

$$r = \lim_{t \to +\infty} \frac{tf'(t)}{f(t)}.$$

• *Case $m > 0$:* Assume that the statement is true for definable functions on $\mathbb{R}^k \times \mathbb{R}$ for each $k < m$. Let $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be definable. Take $\mathcal{C} = \{C_1, ..., C_l\}$ a cell decomposition of $\mathbb{R}^m$. So for each $i \in \{1, ..., l\}$ there is some $k_i \leqslant m$ and a definable homeomorphism

$$\phi_i : \mathbb{R}^{k_i} \to C_i.$$

By the induction hypothesis for each i such that $k_i < m$, there is a finite set $\mathcal{S}_i \subset \mathcal{K}$ that satisfies the statement of the lemma for the function $(x, t) \mapsto f(\phi_i(x), t)$. Now take i such that $k_i = m$. This implies that C_i is open connected subset of $\mathbb{R}^m$. By refining this cell decomposition we can assume that for any $y \in C_i$

$$\lim_{t \to +\infty} f(y, t) \notin \mathbb{R}^\star,$$

and the unique definable function

$$r : C_i \to \mathcal{K} \subset \mathbb{R},$$

such that for any $y \in C_i$ we have $\lim_{t \to +\infty} \frac{f(y,t)}{t^{r(y)}} \in \mathbb{R}^\star$, is C^1.
Now we have the definable function

$$C_i \times \mathbb{R}_+ \longrightarrow \mathbb{R}, \qquad (y, t) \mapsto t^{r(y)} = \lim_{x \mapsto +\infty} \frac{f(y, xt)}{f(y, x)}.$$

Hence the function

$$(y, t) \mapsto \frac{\partial r}{\partial y_j}(y) \ln(t) = \frac{\frac{\partial}{\partial y_j}(t^{r(y)})}{t^{r(y)}}$$

is definable. But since the o-minimal structure doesn't define the logarithmic function, then we have

$$grad(r)(y) = 0 \text{ for any } y \in C_i.$$

Since C_i is connected and open, the function r is constant and equal to some $r_i \in \mathcal{K}$. Finally we take

$$\mathcal{S} = \cup_{i\in\{1,\dots,l\}} S_i.$$

□

Lemma 2.1.11. *Let $U \subset \mathbb{R}^k$ be a nonempty definable open set and*

$$M : U\times]0,\alpha[\to \mathbb{R}$$

be a C^1 definable map. Suppose there exists $K > 0$ such that $|M(y,t)| \leq K$, for all $(y,t) \in U\times]0,\alpha[$. Then there are a closed definable subset F of U with $\dim(F) < \dim(U)$ and continuous definable functions $C, \tau : U \setminus F \to \mathbb{R}_+^*$, such that for all $y \in U \setminus F$ we have

$$\|D_y M(y,t)\| \leq C(y) \text{ for all } t \in]0,\tau(y)[.$$

Proof. We assume that the assertion of the lemma is false, so this means that we can find an open definable set $B \subset U$ such that for any $y \in B$ we have that

$$\lim_{t\to 0^+} \|D_y M(y,t)\| = +\infty.$$

By taking a cell decomposition of B compatible with the sets

$$Y_i = \{y \in U \ : \ \lim_{t\to 0^+} \left|\frac{\partial M}{\partial y_i}(y,t)\right| = +\infty\}$$

for $i = 1,...,k$, we can find an open definable $B' \subset B$ and $i \in \{1,....,k\}$ such that for any $y \in B'$ we have

$$\lim_{t\to 0^+} \left|\frac{\partial M}{\partial y_i}(y,t)\right| = +\infty.$$

Now Let $\tau : B' \longrightarrow]0,\alpha[$ be the definable function given by:

$$\tau(y) = \sup\{s \ : \ \left|\frac{\partial M}{\partial y_i}(y,.)\right| \text{ is strictly decreasing on }]0,s[\}.$$

By shrinking B' we can assume that τ is continuous and $\tau > \alpha'$ for some $\alpha' > 0$. We introduce the definable function $\phi :]0,\alpha'[\to \mathbb{R}_+$ given by

$$\phi(t) = \inf\{\left|\frac{\partial M}{\partial y_i}(y,t)\right| \ : \ y \in B'\}.$$

From one side we can shrink B' again, such that $\lim_{t\to 0^+} \phi(t) = +\infty$. From another side we have that for all $y \in B'$ and $t \in]0,\alpha'[$

$$\phi(t) \leqslant \left|\frac{\partial M}{\partial y_i}(y,t)\right|.$$

Therefore this implies that for any $y, y' \in B'$ and any $t \in]0,\alpha'[$ we have

$$\left|M(y,t) - M(y',t)\right| \geqslant \phi(t)\|y - y'\|.$$

Hence this implies $\phi(t) \leqslant \dfrac{2K}{diam(B')}$, and this is a contradiction. □

In all the rest of this section we assume that the o-minimal structure $\mathcal{A}$ is polynomially bounded.

Lemma 2.1.12. *Take Ω a definable open neighborhood of 0 in $\mathbb{R} \times \mathbb{R}^m$, and a definable function*

$$\begin{aligned} f : \Omega &\to \mathbb{R} \\ (x, y) &\mapsto f(x, y), \end{aligned}$$

such that $f^{-1}(0) \subset (\Omega \cap (\{0\} \times \mathbb{R}^m))$ and f is C^1 with respect to y on $\Omega \setminus (\{0\} \times \mathbb{R}^m)$. Then there exists a definable set $W \subset (\Omega \cap (\{0\} \times \mathbb{R}^m))$ with $\dim(W) < m$, such that for all $(0, y) \in \Omega \setminus W$ there are some $\varepsilon > 0$ and $C > 0$ such that we have on $B((0, y), \varepsilon) \cap (\Omega \setminus (\{0\} \times \mathbb{R}^m))$

$$\|D_y f\| \leqslant C\,|f|.$$

Proof. We can assume that $f \geqslant 0$. Let's assume that the statement is not true. Then we can find $O \subset (\Omega \cap (\{0\} \times \mathbb{R}^m))$ with $\dim(O) = m$ and for all $(0, y) \in O$ we have

$$(*) \qquad \lim_{x \to 0^+} \frac{\|D_y f(x, y)\|}{\|f(x, y)\|} = +\infty.$$

Now we can find an open definable set $U \subset O$ and $\alpha > 0$ such that $]0, \alpha[\times U \subset \Omega$ (here we identify U with an open subset of $\mathbb{R}^m$). Since $f^{-1}(0) \subset (\Omega \cap (\{0\} \times \mathbb{R}^m))$, by Lemma 2.1.10 there exists an open set $B \subset U$ and $r \in \mathcal{K}$ such that

$$f(x, y) = c(y)x^r + \phi(x, y)x^r, \text{ for } y \in B \text{ and } \alpha' > x > 0 \text{ for some } \alpha' < \alpha,$$

where c is a definable function on B with $c(y) \neq 0$ for all $y \in B$, ϕ definable such that $\lim_{x \to 0} \phi(x, y) = 0$ for all y in B. Therefore for any $y \in B$ and $x \in]0, \alpha'[$ we have

$$\frac{\|D_y f(x, y)\|}{\|f(x, y)\|} = \frac{\|Dc(y) + D_y\phi(x, y)\|}{\|c(y) + \phi(x, y)\|} \leqslant \frac{\|Dc(y)\| + \|D_y\phi(x, y)\|}{\|c(y) + \phi(x, y)\|}.$$

But by Lemma 2.1.11 and the fact that $\lim_{x \to 0} \phi(x, y) = 0$ for all y in B, we can shrink B and $]0, \alpha'[$ and assume that there is a continuous definable function $y \mapsto M(y) > 0$ such that for any $(x, y) \in]0, \alpha'[\times B$ we have

$$\|D_y\phi(x, y)\| \leqslant M(y).$$

Hence for any $y \in B$

$$\lim_{x \to 0^+} \frac{\|D_y f(x, y)\|}{\|f(x, y)\|} \leqslant \frac{\|Dc(y)\|}{\|c(y)\|} + \frac{M(y)}{\|c(y)\|},$$

and we can shrink B such that c is bounded away from 0, M is bounded, and $\|Dc\|$ is bounded. Finally this contradicts $(*)$.

□

Lemma 2.1.13. *Let $f : [0, \varepsilon[\times \Omega \to \mathbb{R}$ be a definable function, C^1 with respect to y on $]0, \varepsilon[\times \Omega$, where Ω is an open definable subset of $\mathbb{R}^m$. Then we can find a definable subset $W \subset \Omega$ with $\dim(W) < m$ and such that for every y_0 in $\Omega \setminus W$ there are $r > 0$, $\varepsilon' > 0$, and $C > 0$ such that we have*

$$\|D_{y_0} f(x,y)\| \leqslant C\,|f(x,y)| \text{ for all } (x,y) \in]0,\varepsilon'[\times B(y_0,r).$$

Proof. Take $Z = f^{-1}(0)$, and take $\mathcal{C}$ a cell decomposition of $[0,\varepsilon[\times\Omega$ compatible with $\{0\}\times\Omega$ and $Z\cap(]0,\varepsilon[\times\Omega)$. Hence Ω is a finite disjoint union of cells of $\mathcal{C}$. Take $O\in\mathcal{C}$ such that $O\subset\Omega$ and $\dim(O)=m$. The cells of dimension smaller than m will be chosen to be in the set W. We discuss two cases:

- *Case 1:* There is a cell $B\subset Z$ of maximal dimension (i.e $\dim(B)=m+1$) such that $O\subset\overline{B}$. In this case at each point $(0,y)\in O$ we can find a neighborhood of $(0,y)$ in $]0,\varepsilon[\times\Omega$, where $f\equiv 0$ on this neighborhood, hence in this neighborhood $\|D_y f\|\equiv 0$, then the result holds at $(0,y)$ for any $C>0$.

- *Case 2:* There is no cell $B\subset Z$ of maximal dimension such that $O\subset\overline{B}$. In this case by Lemma 2.1.12 we can find a definable set $W_O\subset O$ with $\dim(W_O)<m$ and such that for all $(0,y)\in O\setminus W_O$ we can find $r_y>0$, $\varepsilon_y>0$, and $C_y>0$, such that $\|D_y f\|\leqslant C_y\,|f|$ on $]0,\varepsilon_y[\times B(y,r_y)$.

For cells like in case.1 we define $W_O:=\emptyset$. So finally the set

$$W := \left(\bigcup_{O,\ with\ \dim(O)=m} W_O\right)\bigcup\left(\bigcup_{\dim(O)<m} O\right)$$

satisfies the required properties.

□

Definition 2.1.14. Let X be a definable subset of a definable manifold N, and Ω a definable open subset of $\mathbb{R}^m$. Take $f: X\times\Omega\to\mathbb{R}$, $(x,y)\mapsto f(x,y)$ a definable function and C^1 with respect to y. We say that f is X-rectifiable with respect to y if we can find a definable partition $\mathcal{P}$ of X, and $c>0$ such that for every $D\in\mathcal{P}$ there is a box $B_D\subset\Omega$ such that

$$\forall x\in D \text{ and } \forall y\in B_D : \|D_y f(x,y)\|\leqslant c\,|f(x,y)|.$$

Remark 2.1.15. It is obvious that if there is a finite definable cover $(X_i)_i$ of X such that for each i $f_{|X_i\times\Omega}$ is X_i-rectifiable with respect to y, then f is X-rectifiable with respect to y.

Lemma 2.1.16. *Let X be a definable subset of $\mathbb{S}^n$, Ω be an open definable subset of $\mathbb{R}^m$, and $x_0\in\partial X$. Let $f: X\times\Omega\to\mathbb{R}$, $(x,y)\mapsto f(x,y)$ be a definable function, C^1 with respect to y. Then there is a definable neighborhood U of x_0 in $\mathbb{S}^n$ such that $f_{|U\cap X\times\Omega}$ is $U\cap X$-rectifiable with respect to y, recall that it means that we can find $\alpha>0$, $c>0$, $\mathcal{C}$ a definable partition of X, and $\mathcal{B}$ a finite collection of boxes in Ω such that for every $C\in\mathcal{C}$ with $x_0\in\overline{C}$, there is $B_C\in\mathcal{B}$ such that*

$$\forall x\in C \text{ with } d(x,\ x_0)<\alpha,\ \forall y\in B_C : \|D_y f(x,y)\|\leqslant c\,|f(x,y)|.$$

Proof. Induction on m.

- **The case of $m = 1$:**

We will proceed by contradiction. So assume that $\Omega =]a, b[$ and there is no neighborhood U of x_0 in $\mathbb{S}^n$ such that $f_{|U\cap X\times\Omega}$ is $U \cap X$-rectifiable with respect to y. We will discuss the next two cases:

- *Case 1:* Assume that $f^{-1}(0) = \emptyset$. We define the function

$$g : X \times \Omega \to \mathbb{R}_+$$

by

$$g(x, y) = \frac{\|D_y f(x, y)\|}{|f(x, y)|}.$$

Let's define the set

$$O = \{(x, y) \in X \times \Omega \ : \text{ such that } g(x, \cdot) \text{ is monotone near } y \}.$$

Since O can be expressed by a first order formula, O is a definable subset of $X \times \Omega$. Now take the set $\sum = (X \times \Omega) \setminus O$. By Lemma 1.0.6 we have for all $x \in X$

$$\dim(\textstyle\sum_x) = 0 < \dim(\Omega) = 1.$$

Hence by Lemma 2.1.4 we can find a definable partition $\mathcal{C}_1$ of X such that for all $C' \in \mathcal{C}_1$ there is a box $B_{C'} =]a_{c'}, b_{c'}[\subset \Omega$ such that

$$C' \times B_{C'} \cap \textstyle\sum = \emptyset.$$

Then this means that for every $x \in C'$, $g(x, \cdot)$ is monotone on $B_{C'}$, but we need that the monotonicity type does not depend on x but only on C'. Let's choose $\mathcal{C}_0$ to be a partition of X compatible with $\mathcal{C}_1$ and a collection of sets $\{M^+_{C'}, M^-_{C'}, M^0_{C'}\}_{C'\in\mathcal{C}_1}$, where $M^+_{C'}$, $M^-_{C'}$, and $M^0_{C'}$ are defined by

$$\begin{aligned} M^+_{C'} &= \{x \in C' \ : \ g(x, \cdot) \text{ is increasing}\}, \\ M^-_{C'} &= \{x \in C' \ : \ g(x, \cdot) \text{ is decreasing}\}, \\ M^0_{C'} &= \{x \in C' \ : \ g(x, \cdot) \text{ is constant}\}. \end{aligned}$$

Hence the monotonicity type of $g(x, \cdot)$ depends only on the elements of $\mathcal{C}_0$, it means that for any $C \in \mathcal{C}_0$ there is a box $B_C =]a_C, b_C[$ in Ω such that for any $x \in C$, $g(x, .)$ is monotone on B_C, with the same monotonicity type for all $x \in C$. We now apply our assumption to $\mathcal{C}_0$. Then we can replace X by an element $C \in \mathcal{C}_0$, with $x_0 \in \overline{C}$ such that for every box B in Ω, g is not bounded on $C \times B$. Take $D_C = [d_C, e_C] \subset B_C$ and let's consider the graph of g on $C \times D_C$

$$\Gamma_g = \{(x, y, g(x, y)) \ : \ y \in D_C \ , x \in C\} \subset \overline{C} \times D_C \times (\mathbb{R}_+ \cup \{+\infty\}).$$

Consider $\overline{\Gamma}_g$ the closure of Γ_g in $\overline{C} \times D_C \times (\mathbb{R}_+ \cup \{+\infty\})$. But, since for any box B in D_C, g is not bounded on $C \times B$, we can find $y_0 \in D_C$ such that

$$(x_0, y_0, +\infty) \in \overline{\Gamma_g} \setminus \Gamma_g.$$

By the curve selection Lemma there is a definable continuous curve $\gamma : [0, a[\to \overline{\Gamma_g}$, with $\gamma(t) = (x(t), y(t), g(x(t), y(t)))$ and such that

$$\gamma(0) = (x_0, y_0, \infty) \text{ and } \gamma(]0, a[) \subset \Gamma_g.$$

Assume that $g(x, \cdot)$ is increasing for all $x \in C$ (the other cases are similar). In this case by Lemma 2.1.13, we can shrink $]0, a[$ and we can find a box B in $]e_C, b_C[$ (it is not empty because $e_C < b_C$) and $L > 0$ such that for all $y \in B$, and for all $t \in]0, a[$ we have

$$g(x(t), y) \leqslant L.$$

But we have also for all $y \in B$, and for all $t \in]0, a[$

$$g(x(t), y(t)) \leqslant g(x(t), y) \leqslant L.$$

And this contradicts the fact that

$$\lim_{t \to 0} g(x(t), y(t)) = +\infty.$$

- *Case 2:* We assume that $Z = f^{-1}(0) \neq \emptyset$. Take $\varepsilon > 0$, define the set $A = \{x \in B(x_0, \varepsilon) \cap X \ : \ \dim(Z_x) = 1\}$. We have the following cases:
  - *Case.A:* Assume that $x_0 \in \overline{A}$, take $\mathcal{D}$ a definable partition of $B(x_0, \varepsilon) \cap X$ compatible with A. Since f is not $B(x_0, \varepsilon) \cap X$-rectifiable with respect to y near x_0, we can find $C \in \mathcal{D}$ with $x_0 \in \overline{C}$ and such that for any partition $\mathcal{C}_C$ of C there is $C' \in \mathcal{C}_C$ (with $x_0 \in \overline{C}'$) such that for any box $B \subset \Omega$ there is no $L > 0$ such that

$$\|D_y f(x, y)\| \leqslant L \, |f(x, y)|$$

  for $(x, y) \in C' \times B$ and x in some neighborhood of x_0.
  Here we discuss two subcases:
    - *Case.A.1:* Suppose $C \cap A = \emptyset$. Then by Lemma 2.1.4 we can find a partition $\mathcal{C}_C$ of C such that for every $C' \in \mathcal{C}_C$ there is a box $B_{C'}$ in Ω_2 such that $(C' \times B_{C'}) \cap Z = \emptyset$, hence g is well-defined on $C' \times B_{C'}$. Then we may apply the same argument as in the proof of case 1 by considering for every $C' \in \mathcal{C}$ the function

$$\begin{aligned} g : C' \times B_{C'} &\to \mathbb{R}_+ \\ (x, y) &\mapsto \tfrac{\|D_y f(x,y)\|}{|f(x,y)|}. \end{aligned}$$

    - *Case.A.2:* Suppose $C \subset A$. Then again by Lemma 2.1.4 we can find a partition $\mathcal{C}_C$ of C such that for every $C' \in \mathcal{C}_C$ there is a box $B_{C'}$ in Ω such that $(C' \times B_{C'}) \subset Z$ or $(C' \times B_{C'}) \cap Z = \emptyset$. If $(C' \times B_{C'}) \cap Z = \emptyset$, then we are again in the situation of case.A.1. If $(C' \times B_{C'}) \subset Z$, this gives that $f = 0$ and $D_y f = 0$ on $C' \times B_{C'}$ and this is a contradiction because we can choose any $L > 0$ such that $\|D_y f(x, y)\| \leqslant L \, |f(x, y)|$ for $(x, y) \in C' \times B_{C'}$, and x in some neighborhood of x_0.
  - *Case.B:* Assume that $x_0 \notin \overline{A}$. Then we can separate x_0 from A by some $B(x_0, \varepsilon')$ and proceed as in the case.A.1.

- **The case of** $m > 1$ **:** Assume that the statement of the lemma is true for any positive integer smaller than m. Fix $I = I_1 \times ... \times I_m$ a box in Ω, with $I_i =]a_i, b_i[$ and denote $I' = I_2 \times ... \times I_m$. Apply the induction hypothesis (the case of $m - 1$) to the function

$$\begin{aligned} f : (X \times I_1) \times I' &\to \mathbb{R} \\ ((x, y_1), y') &\mapsto f(x, (y_1, y')). \end{aligned}$$

Hence we can shrink I_1, and find a neighborhood U of x_0, such that f is $(X \cap U) \times I_1$-rectifiable with respect to y', it means that there is $L > 0$ and a partition $\mathcal{P}$ of $(X \cap U) \times I_1$ (we can choose it in a way that $\pi_n^{n+1}(\mathcal{P})$ is a partition of $X \cap U$) such that for any $C \in \mathcal{P}$ there is a box $B_C \subset I'$ and we have

$$\forall ((x, y_1), y') \in C \times B_C : \|D_{y'} f(x, y)\| \leqslant L\, |f(x, y)|.$$

So we can find a partition $\mathcal{D}$ of $X \cap U$ and definable maps $\phi_1, ..., \phi_{k_D} : D \to I_1$ for every $D \in \mathcal{D}$, such that every $C \in \mathcal{P}$ is either the graph of one of these maps or a band between two graphs. Now take the set $\sum$ defined by

$$\sum = \sqcup_{D \in \mathcal{D}} \bigcup_{i=1,\dots,k_D} \Gamma_{\phi_i} \subset (X \cap U) \times I_1.$$

Since for all $x \in X \cap U$ we have $\dim(\sum_x) = 0 < 1 = \dim(I_1)$. By Lemma 2.1.4 it suffices to consider the cells C of the form

$$C = D_C \times]a_C, b_C[,$$

where $\{D_C\}_{C \in \mathcal{P}}$ is a partition of $X \cap U$ and $]a_C, b_C[\subset I_1$. Hence to complete the proof we need to show that for such $C \in \mathcal{P}$ with $x_0 \in \overline{C}$, f is C-rectifiable with respect to y in some neighborhood of x_0. Now let's apply the case of $m = 1$ to the function

$$\begin{aligned} f : (D_C \times B_C) \times]a_C, b_C[&\to \mathbb{R} \\ ((x, y'), y_1) &\mapsto f(x, y). \end{aligned}$$

We can find a neighborhood U_C of x_0 in X and we can shrink B_C, so that f is $(U_C \cap D_C) \times B_C$-rectifiable with respect to y_1, that there is $L' > 0$ and a partition $\mathcal{P}_C$ of $(U_C \cap D_C) \times B_C$ such that $\pi_{n+1}^{n+m}(\mathcal{P}_C)$ is a partition of $U_C \cap D_C$ and for every $P \in \mathcal{P}_C$ there is a box $I_P \subset I_1$ and

$$\forall (x, y') \in P \text{ and } \forall y_1 \in I_P \text{ we have } \|D_{y_1} f(x, y)\| \leqslant L' \, |f(x, y)|\,.$$

Now by Lemma 2.1.4 it suffices to consider the elements $P \in \mathcal{P}_C$ of the form $P = M_P \times B_P$, where the collection $\{M_P\}_P$ is a partition of $U_C \cap D_C$ and $\{B_P\}_P$ are boxes in B_C. Therefore, finally, for any M_P there is a box $I_P \times B_P \subset I$ such that on this box we have

$$\|D_y f(x, y)\| \leqslant (L + L')\, |f(x, y)|.$$

This finishes the proof of the lemma.

$\square$

Theorem 2.1.17. *Let A be a definable subset of $\mathbb{R}^n$ such that $\dim(A) \leqslant n-1$ and take $p \in \mathbb{N}^*$. Then there are $\varepsilon > 0$, $C > 0$, and $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is $i \in \{1, ..., k\}$ such that π_{v_i} is (ε, C, p)-regular at x with respect to A.*

Proof. By Theorem 2.1.3, we can find a cell decomposition $\mathcal{C} = \{C_1, ..., C_k\}$ of $\mathbb{R}^n$, and $\varepsilon > 0$ such that for every $i \in \{1, ..., k\}$ there is $v_i \in \mathbb{R}^{n-1}$ such that π_{v_i} is (ε, p)-weak regular at every point $x \in C_i$ with respect to A. Hence, for every $i \in \{1, ..., k\}$ there are definable functions

$$\begin{aligned} f_{i,l} : C_i \times B(v_i, \varepsilon) &\to \mathbb{R}^* \\ (x, v) &\mapsto f_{i,l}(x, v), \end{aligned}$$

such that each $f_{i,l}$ is C^p with respect to v and we have

$$C_\varepsilon(x, v_i) \cap A = \bigsqcup_l \{x + f_{i,l}(x, v)(v, 1) \ : \ v \in B(v_i, \varepsilon)\}.$$

Now we will find a refinement of $\mathcal{C}$, constants $\varepsilon' < \varepsilon$ and $C > 0$, and a (ε', C, p)-regular projections of $\mathbb{R}^n$ with respect to A. For this, it is enough to prove that for every $C_i \in \mathcal{C}$ the maps $f_{i,l}$ are C_i-rectifiable with respect to v, it means that there is $c > 0$ and a definable partition $\mathcal{C}_i$ of C_i such that for every $D \in \mathcal{C}_i$ there is a box $B_D \subset B(v_i, \varepsilon)$ with

$$\frac{\|D_v f_{i,l}\|}{|f_{i,l}|} < c \text{ on } D \times B_D.$$

Take $C_i \in \mathcal{C}$. Then we have

• If C_i is of dimension 0, that is $C_i = \{x\}$ is a point, then by continuity of $\frac{\|D_v f_{i,l}\|}{|f_{i,l}|}$ we can find a closed ball $\overline{B}(v_i, \varepsilon') \subset B(v_i, \varepsilon)$ such that $\frac{\|D_v f_{i,l}\|}{|f_{i,l}|}$ is bounded by some $c > 0$ on this ball. Hence the maps $f_{i,l}$ are C_i-rectifiable with respect to v.

• Assume that C_i is of dimension $\dim(C_i) > 0$. We know that we have a natural definable embedding (it is the inverse map of the stereographic projection, and it is semi-algebraic) of $\mathbb{R}^n$ in $\mathbb{S}^n$.

$$E : \mathbb{R}^n \to \mathbb{S}^n.$$

We can replace C_i by $E(C_i)$ and $f_{i,l}$ by the maps $f_{i,l} \circ (E^{-1}, Id_{B(v_i,\varepsilon)})$. Indeed if the maps $f_{i,l} \circ (E^{-1}, Id_{B(v_i,\varepsilon)})$ are $E(C_i)$-rectifiable, then the maps $f_{i,l}$ are also C_i-rectifiable.

Take the closure $\overline{C}_i$ of C_i in $\mathbb{S}^n$ (it is a compact subset since $\mathbb{S}^n$ is compact). So we can find a definable cover $\mathcal{U} = \{U_1, ..., U_k\}$ of $\overline{C}_i$ such that for each $j \in \{1, ..., k\}$ we have

$$U_j = \overline{C}_i \cap B(x_j, r_j) \text{ for some } x_j \in \overline{C}_i \text{ and } r_j > 0.$$

Now take $x \in \partial C_i = \overline{C}_i \setminus C_i$, hence there is a $j_x \in \{1, ..., k\}$ such that $x \in U_{j_x}$. By applying Lemma 2.1.16 to the maps $f_{i,l} : (U_{j_x} \setminus \partial C_i) \times B(v_i, \varepsilon) \to \mathbb{R}$ (here $U_{j_x} \setminus \partial C_i = X$ and $B(v_i, \varepsilon) = \Omega$), we can find a neighborhood O_x of x in $\overline{C}_i$ such that $f_{i,l}$ are $O_x \cap C_i$-rectifiable with respect to v . Then by compactness of ∂C_i, we can choose a finite cover $(O_{x_1}, ..., O_{x_m})$ of an open neighborhood of ∂C_i in $\overline{C}_i$ such that $f_{i,l}$ are $(\bigcup_s O_{x_s}) \cap C_i$-rectifiable with respect to v. Now since $d(\partial C_i, C_i \setminus (\bigcup_s O_{x_s})) > 0$, we can find a compact subset $K \subset C_i$ with $C_i \setminus (\bigcup_s O_{x_s}) \subset K$, hence the functions $\frac{\|D_v f_{i,l}\|}{|f_{i,l}|}$ are bounded on $C_i \setminus (\bigcup_s O_{x_s})$, therefore $f_{i,l}$ are $C_i \setminus (\bigcup_s O_{x_s})$-rectifiable with respect to v. Finally, since $(\bigcup_s O_{x_s} \cap C_i, C_i \setminus \bigcup_s O_{x_s})$ is a definable cover of C_i, we deduce that the functions $f_{i,l}$ are C_i-rectifiable with respect to v.

□

2.2 Application: Existence of regular covers.

Fix $\mathcal{D}$ an o-minimal structure on $(\mathbb{R}, +, \cdot)$. Let U be an open definable relatively compact subset of $\mathbb{R}^n$. By a regular cover of U, we mean a finite cover (U_i) by open definable sets, such that

(1) each U_i is homeomorphic to the open unit ball in $\mathbb{R}^n$ by a definable homeomorphism.

(2) there is a positive number C such that for all x in $\mathbb{R}^n$ we have

$$d(x, \mathbb{R}^n \setminus U) \leq C \max_i d(x, \mathbb{R}^n \setminus U_i).$$

Example 2.2.1. Take $U = \{(x, y) \in \mathbb{R}^2 \ ; \ 0 < x < 1 \text{ and } -1 < y < 1\}$. U is a definable set in any o-minimal structure on $\mathbb{R}$. The definable cover (U_1, U_2) of U defined by

$$U_1 = \{(x, y) \in U \ : \ y > -\tfrac{1}{2}\},$$
$$U_2 = \{(x, y) \in U \ : \ y < \tfrac{1}{2}\}.$$

is a regular cover. Indeed, take $p = (x, y) \in U$. Then we have

$$d(p, \mathbb{R}^2 \setminus U) < \max(d(p, \mathbb{R}^2 \setminus U_1), d(p, \mathbb{R}^2 \setminus U_2)).$$

Now take the definable cover (U_1', U_2') defined by

$$U_1' = \{(x, y) \in U \ : \ y < x^2\},$$
$$U_2' = \{(x, y) \in U \ : \ y > -x^2\}.$$

Then (U_1', U_2') is not a regular cover. Assume that this is a regular cover, take a point $p_a = (a, 0) \in U$. Then we have

$$d(p_a, \mathbb{R}^2 \setminus U) = a \text{ for } a < \tfrac{1}{2},$$
$$d(p_a, \mathbb{R}^2 \setminus U_1') = d(p_a, \mathbb{R}^2 \setminus U_2') < a^2.$$

Hence we have for $a < \frac{1}{2}$

$$\frac{d(p_a, \mathbb{R}^2 \setminus U)}{\max\left(d(p_a, \mathbb{R}^2 \setminus U_1'), d(p_a, \mathbb{R}^2 \setminus U_2')\right)} > \frac{1}{a} \to +\infty \ when \ a \to 0.$$

Hence this is a contradiction with the definition of a regular cover.

Theorem 2.2.2. *For any open relatively compact definable subset U of $\mathbb{R}^n$ there exists a regular cover.*

Proof. Take $X = \partial U = \overline{U} \setminus U$. X is compact and definable of dimension $n-1$. Take $Z = Sing^1(X)$ the set of the points where X is not a C^1-submanifold of $\mathbb{R}^n$ of dimension $n-1$ near these points (i.e that $Z = (Reg^1(X))^c$). For any linear projection $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ (i.e, a projection with respect to a vector in $\mathbb{R}^n$), we define the discriminant Δ_P by

$$\Delta_P = P(Z) \cup CV(P_{|Reg^1(X)}),$$

where $CV(\pi_{|Reg^1(X)})$ is the set of critical values of P on $Reg^1(X)$.
It is obvious that Δ_P is definable subset of $\mathbb{R}^{n-1}$, because P is a definable map and $CV(P_{|Reg^1(X)})$ can be described by a first order formula. And we have also

$$\overline{P(U)} = P(U) \cup \Delta_P.$$

Take $\Lambda = \{\pi_1, ..., \pi_k\}$ a set of $(\varepsilon, 1)$-weak regular projection with respect to X, and $v_1, ..., v_k \in \mathbb{R}^{n-1}$ such that $\pi_j = \pi_{v_j}$. For $x \in \mathbb{R}^n$ we denote by $C^j_\varepsilon(x)$ the cone

$$C^j_\varepsilon(x) = C_\varepsilon(x, v_j) = \{x + t(v, 1) \ : \ t \in \mathbb{R}^*, \ v \in B(\varepsilon, v_j)\}.$$

Lemma 2.2.3. *Take $\pi_j \in \Lambda$, and define*

$$R(\pi_j) = \{x \in U \ : \ \pi_j \textit{ is } (\varepsilon, 1)\textit{-weak regular at } x \textit{ with respect to } X\}.$$

Then we have $\pi_j(R(\pi_j)) \cap \Delta_{\pi_j} = \emptyset$ and there is some $C > 0$ such that for all $x \in R(\pi_j)$ we have

$$d(x, X \setminus C^j_\varepsilon(x)) \leq Cd\left(\pi_j(x), \pi_j(X \setminus C^j_\varepsilon(x))\right) \leq Cd(\pi_j(x), \Delta_{\pi_j}). \tag{3.1}$$

Proof. It is enough to assume that π_j is the standard projection $\pi_j = \pi_0 = \pi$. If $\pi(R(\pi)) \cap \Delta_\pi \neq \emptyset$, then there is $x' \in \pi(Z) \cup CV(\pi_{|Reg^1(X)})$ such that $x' = \pi(x)$ and $x \in R(\pi)$, but since π is ε-weak regular at x, it follows that $X \cap \pi^{-1}(\pi(x)) \subset (Reg^1(X))$. Hence the contradiction. So $\pi(R(\pi)) \cap \Delta_\pi = \emptyset$.
Since $\Delta_\pi \subset \pi(X \setminus C^0_\varepsilon(x))$, we have $d\left(\pi(x), \pi(X \setminus C^0_\varepsilon(x))\right) \leq d(\pi(x), \Delta_\pi)$, and to show the first inequality it is enough to find $C > 0$ such that

$$d(x, X \setminus C^0_\varepsilon(x)) \leq Cd\left(\pi(x), \pi(X \setminus C^0_\varepsilon(x))\right).$$

But for any $x' \notin C^0_\varepsilon(x)$ we have

$$\frac{d(x, x')}{d(\pi(x), \pi(x'))} \leqslant 1 + \frac{1}{\varepsilon}.$$

Indeed, take $x' = x + t(v, 1) \notin C^0_\varepsilon(x)$, with $t \in \mathbb{R}^*$ and $\|v\| \geqslant \varepsilon$. Then $\pi(x') = \pi(x) + tv$. Hence we have

$$\frac{d(x, x')}{d(\pi(x), \pi(x'))} = \frac{|\ t\ | \ \|(v, 1)\|}{|\ t\ | \ \|v\|} \leqslant 1 + \frac{1}{\varepsilon}.$$

Now for any $x' \in X \setminus C_\varepsilon(x)$ we have

$$d(x, x') \leqslant (1 + \frac{1}{\varepsilon})d(\pi(x), \pi(x')).$$

But since this is for any $x' \in X \setminus C_\varepsilon(x)$, by the definition of the infimum we deduce that

$$d(x, X \setminus C_\varepsilon(x)) \leq (1 + \frac{1}{\varepsilon}) d(\pi(x), \pi(X \setminus C_\varepsilon(x))).$$

□

Remark 2.2.4. For the first inequality in (3.1), we don't need x to be in $R(\pi_j)$.

For the proof of Theorem 2.2.2 we proceed by induction on n. Assume that Theorem 2.2.2 is true in $\mathbb{R}^{n-1}$. Fix $j \in \{1, ..., k\}$. Then by the induction assumption and by Lemma 2.2.3 there is a $C_j > 1$ and a finite definable cover $(U_{j,i})_{i \in I_j}$ of $\pi_j(U) \setminus \Delta_{\pi_j}$ such that for all $x' \in \mathbb{R}^{n-1}$ we have

$$d(x', \mathbb{R}^{n-1} \setminus (\pi_j(U) \setminus \Delta_{\pi_j})) \leq C_j \max_i d(x', \mathbb{R}^{n-1} \setminus (U_{j,i}))$$

and for all $x \in R(\pi_j)$ we have by Lemma 2.2.3

$$d(x, X \setminus C^j_\varepsilon(x)) \leq C_j d(\pi_j(x), \pi_j(X \setminus C^j_\varepsilon(x))) \leq C_j d(\pi_j(x), \Delta_{\pi_j}).$$

Now take a cell decomposition of $\mathbb{R}^n$ compatible with U, X, Z, $CP((\pi_j)_{|X})$, and $\pi_j(U) \setminus \Delta_{\pi_j}$. Then for each $i \in I_j$ there are definable functions

$$\phi_1 < \phi_2 < ... < \phi_{l_i} : U_{j,i} \to \mathbb{R},$$

such that $X \cap \pi_j^{-1}(U_{j,i})$ is the disjoint union of graphs of these functions, and $U \cap \pi_j^{-1}(U_{j,i})$ is the disjoint union of the open sets bounded by the graphs of these functions. So for every $j \in \{1, ..., k\}$ and $i \in I_j$ we have:

$$U \cap \pi_j^{-1}(U_{i,j}) = \bigcup_{m \in M_{j,i}} U_{j,i,m},$$

where $U_{j,i,m}$ are open definable subsets of U, given by

$$U_{j,i,m} = \{(x', x_n) \ : \ x' \in U_{j,i} \text{ and } \phi_{m_1}(x') < x_n < \phi_{m_2}(x') \}, \text{ with } \Gamma_{\phi_{m_1}} \subset X \text{ and } \Gamma_{\phi_{m_2}} \subset X.$$

Now take $x \in U$. Hence by the weak projection theorem there is a projection $\pi_j \in \Lambda$ such that $x \in R(\pi_j)$. Let's consider $i \in I_j$ such that $\pi_j(x) \in U_{j,i}$ and

$$d(\pi_j(x), \mathbb{R}^{n-1} \setminus (\pi_j(U) \setminus \Delta_{\pi_j})) \leq C_j d(\pi_j(x), \mathbb{R}^{n-1} \setminus (U_{j,i})). \tag{3.2}$$

Since $\partial(\pi_j(U) \setminus \Delta_{\pi_j}) \subset \Delta_{\pi_j}$, we have

$$d(\pi_j(x), \Delta_{\pi_j}) \leq C_j d(\pi_j(x), \mathbb{R}^{n-1} \setminus (U_{j,i})). \tag{3.3}$$

Now take $m \in M_{j,i}$ such that $x \in U_{j,i,m}$. Then we claim that

$$d(x, X) \leqslant (C_j)^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}). \tag{3.4}$$

To prove (3.4) we discuss two cases (the first is obvious)

(1) $d(x, \mathbb{R}^n \setminus U_{j,i,m}) \geqslant d(x, X)$.

(2) $d(x, \mathbb{R}^n \setminus U_{j,i,m}) < d(x, X)$. In this case let $V = \partial U_{j,i,m} \cap \pi_j^{-1}(\partial U_{j,i})$ (the vertical part of $\partial U_{j,i,m}$). We have then

$$d(x, \mathbb{R}^n \setminus U_{j,i,m}) = d(x, V),$$

because $d(x, \mathbb{R}^n \setminus U_{j,i,m}) = \min\{\ d(x, V),\ d(x, \partial U_{j,i,m} \cap X)\}$. Therefore by (3.1) and (3.3) we have

$$\begin{aligned} d(x, X) &\leqslant d(x, X \setminus C_\varepsilon^j(x)) \\ &\leqslant C_j d(\pi_j(x), \Delta_{\pi_j}) \\ &\leqslant C_j^2 d(\pi_j(x), \mathbb{R}^{n-1} \setminus U_{j,i}) \\ &\leqslant C_j^2 d(x, V) \\ &\leqslant C_j^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}). \end{aligned}$$

And this proves (3.4).
Finally, we have a finite cover $(U_{j,i,m})_{m \in M_{j,i}, i \in I_j, j}$ of U. Take

$$C = \max_j C_j^2.$$

Hence for $x \in U$ we have

$$d(x, \mathbb{R}^n \setminus U) = d(x, X) \leqslant (C_j)^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}) \leqslant C \max_{j,i,m} d(x, \mathbb{R}^n \setminus U_{j,i,m}).$$

□

Comments 2.2.5. The existence of regular covers was essentially needed in [6] for the construction of Sheaves on the subanalytic sites, and so Theorem 2.2.2 implies that the results in [6] works also on definable sites for arbitrary o-minimal structures. Also this implies the existence of Sobolev sheaves (see Chapter 3) proven by G.Lebeau [12] on any definable site.

Chapter 3

Sobolev sheaves on the definable site

Sheaves of functional spaces on the subanalytic topology (in the sense of Grothendieck) are important objects in algebraic analysis. We focus in this chapter on sheaves that are made of Sobolev functions. for $s \in \mathbb{R}$, the presheaf of $\mathbb{C}$-vector spaces

$$U \subset \mathbb{R}^n \to W^{s,2}(U) = \{F_{|U} \ : \ F \in W^{s,2}(\mathbb{R}^n)\},$$

is not always a sheaf for regularity reasons. This is related to the fact that if $U \subset \mathbb{R}^n$ is open subanalytic with (non Lipschitz) singularity in ∂U, then the space $W^{s,2}(U)$ doesn't have good properties. The aim of this part is to find for $s > 0$ an optimal sheafification of Sobolev spaces $W^{s,2}$ on the definable site (in a fixed o-minimal structure), optimal in the sense that for $U \subset \mathbb{R}^n$ the space $W^{s,2}(U)$ will be modified only if it is necessary.

In [12], G.Lebeau proved that for any $s < 0$, there is an object $\mathcal{F}^s$ in the derived category of sheaves on the subanalytic topology of $\mathbb{R}^n$ such that for any open subanalytic Lipschitz set $U \subset \mathbb{R}^n$ the complex $\mathcal{F}^s(U)$ is concentrated in degree 0 and equal to the classical Soblev space $W^{s,2}(U)$. The proof is based on the results of Guillermou and P. Schapira in [6] and the existence of good subanalytic covers in [22].

For $s \in \mathbb{N}$, we construct a sheaf $\mathcal{F}^s$ of distributions on the definable topology such that if $U \subset \mathbb{R}^2$ is a small open set then

$$\mathcal{F}^s(U) = W^{s,2}(U).$$

This sheaf is unique (thanks to existence L-regular decomposition in [10], [11], [23], and [24]) and agrees with $W^{s,2}$ on Lipschitz domains. The idea of the construction is based on understanding the local obstructions for $W^{s,2}$ to be a sheaf. Note that thanks to L-regular decomposition (see [23]), for $s \in]-\frac{1}{2}, \frac{1}{2}[$ the presheaf $U \mapsto W^{s,2}(U)$ is a sheaf. The obstructions are present for $s > 0$ big enough to have embedding of $W^{s,2}$ into at least the space of continuous functions. In the 2 dimensional case, the construction is easy and explicit because the Lipschitz structure of definable open subsets in $\mathbb{R}^2$ has an explicit classification.

3.1 Hilbert Sobolev spaces revisited.

Let $n \in \mathbb{N}$, we denote by:

- $\mathcal{S}(\mathbb{R}^n)$ the space of Schwartz functions (C^∞-functions that go to zero (with all the derivatives) at infinity faster than any polynomial).
- $\mathcal{S}'(\mathbb{R}^n)$ the topological dual of $\mathcal{S}(\mathbb{R}^n)$.

And we have natural continuous injections

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

We recall the Fourier Transform

$$u \in \mathcal{S}(\mathbb{R}^n) \mapsto \widehat{u} \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\widehat{u}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-iy.x} u(x) dx. \tag{3.1.1}$$

By duality, the Fourier transform extends in a canonical way to $\mathcal{S}'(\mathbb{R}^n)$. Finally for $s \in \mathbb{R}$ we recall the Sobolev space

$$W^{s,2}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \ : \ \|u\|_{W^{s,2}(\mathbb{R}^n)} = \sqrt{\int_{\mathbb{R}^n} (1+|y|^2)^s \left|\widehat{u}(y)\right|^2 dy} < +\infty\},$$

with the natural dense inclusions (for $s \geqslant 0$)

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset W^{s,2}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

An equivalent way to define $W^{s,2}(\mathbb{R}^n)$ is :

- For $k \in \mathbb{N}$

  $$W^{k,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \ : \ \forall \mid \alpha \mid \leqslant k, \ \partial^\alpha f \in L^2(\mathbb{R}^n)\},$$

  where $\partial^\alpha f$ is the distributional derivative of f for $\alpha \in \mathbb{N}^n$.
- For $s \in]k, k+1[$ for some $k \in \mathbb{N}$, then $W^{s,2}$ is the interpolation space

  $$W^{s,2}(\mathbb{R}^n) = [W^{k,2}(\mathbb{R}^n), W^{k+1,2}(\mathbb{R}^n)]_{s-k}.$$
- For $s < 0$, $W^{s,2}(\mathbb{R}^n)$ is the topological dual

  $$W^{s,2}(\mathbb{R}^n) := (W^{-s,2}(\mathbb{R}^n))'.$$

For open $U \subset \mathbb{R}^n$ and a closed $F \subset \mathbb{R}^n$ we define the spaces $W^{s,2}_F(\mathbb{R}^n)$ to be the closed subspace of distributions supported in F, with the induced norm.

Take $s \geqslant 0$ and $r = s - [s]$. It's classical that $f \in W^{s,2}(\mathbb{R}^n)$ if and only if $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for all $\mid \alpha \mid \leqslant [s]$ and (if $r > 0$)

$$\frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{\frac{n}{2}+r}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $\mid \alpha \mid = [s]$, and the norm of $W^{s,2}$ is given by

$$\|f\|_{W^{s,2}(\mathbb{R}^n)} = \sum_{|\alpha| \leqslant [s]} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} + 1_{r>0} \sum_{|\alpha|=[s]} \|\frac{\partial^\alpha f(x) - \partial^\alpha f(y)}{|x-y|^{\frac{n}{2}+r}}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}. \tag{3.1.2}$$

For $s \in \mathbb{R}$ and $U \subset \mathbb{R}^n$ open, we define the space

$$W^{s,2}(U) := \{f \in \mathcal{D}'(U) \ : \ \exists F \in W^{s,2}(\mathbb{R}^n) \text{ such that } F_{|U} = f\}. \tag{3.1.3}$$

With the norm

$$\| \ f \ \|_{W^{s,2}(U)} := \inf\{\| \ F \ \|_{W^{s,2}(\mathbb{R}^n)} \ : \ F_{|U} = f\}.$$

We have on $W^{s,2}(U)$ the quotient Hilbert structure induced by the natural isomorphism between $W^{s,2}(U)$ and

$$W^{s,2}(\mathbb{R}^n) \Big/ W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n).$$

Since $W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n)$ is a closed subspace of the Hilbert space $W^{s,2}(\mathbb{R}^n)$, it's complemented by the orthogonal and

$$W^{s,2}(\mathbb{R}^n) = W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n) \oplus (W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n))^\perp.$$

This induces an extension operator $\mathcal{T} : W^{s,2}(U) \longrightarrow W^{s,2}(\mathbb{R}^n)$ given by

$$\mathcal{T}(f) = Proj_{(W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n))^\perp}(F)$$

for any choice of $F \in W^{s,2}(\mathbb{R}^n)$ such that $F \mid_U = f$, where $Proj_{(W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n))^\perp} : W^{s,2}(\mathbb{R}^n) \to (W^{s,2}_{\mathbb{R}^n \setminus U}(\mathbb{R}^n))^\perp$ is the orthogonal projection.

The usual definition of Sobolev spaces: in our definition we follow [12]. Note that the usual Sobolev spaces $W^{s,2}_\star$ (see Lions and Magenes [13]) are defined as follows:

- If $k \in \mathbb{N}$, then

$$W^{k,2}_\star(U) := \{f \in L^2(U) \ : \ \partial^\alpha f \in L^2(U) \ \forall \mid \alpha \mid \leqslant k\}.$$

- If $s \in]k, k+1[$, then

$$W^{s,2}_\star(U) := [W^{k,2}_\star(U), W^{k+1,2}_\star(U)]_{s-k}.$$

And we have

$$W^{s,2}_\star(U) = \{f \in L^2(U) \ : \ \partial^\alpha f \in W^{s-k,2}_\star(U) \ \forall \mid \alpha \mid \leqslant k\}.$$

- For $s < 0$, $W_\star^{s,2}(U)$ is defined to be the topological dual space of $W_\star^{-s,2}(U)$.

Definition 3.1.1. An open set $U \subset \mathbb{R}^n$ is said to be Lipschitz if and only if for any $q \in \overline{U}\backslash U$ there are an orthogonal transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(q) = 0$, a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, and $r > 0$ such that

$$\phi(U \cap B(q,r)) = \{(y', y_n) \in B(0,r) \ : \ y_n > f(y')\}.$$

Thanks to the Stein extension Theorem (with the functoriality of interpolations (see Section 3.6)), for $U \subset \mathbb{R}^n$ Lipschitz bounded and $s \geqslant 0$ we have

$$W^{s,2}(U) = W_\star^{s,2}(U). \tag{3.1.4}$$

In fact the Stein extension Theorem gives more:

Theorem 3.1.2. *Take $U \subset \mathbb{R}^n$ open Lipschitz bounded. Then there is a linear continuous extension operator $Ext : L^2(U) \mapsto L^2(\mathbb{R}^n)$ such that for $k \in \mathbb{N}$ the restriction of Ext to $W_\star^{k,2}(U)$ induces a linear continuous operator*

$$Ext_{W_\star^{k,2}(U)} : W_\star^{k,2}(U) \mapsto W^{k,2}(\mathbb{R}^n).$$

Proposition 3.1.3. *Let $U \subset \mathbb{R}^n$ be open bounded Lipschitz and $s \geqslant 0$. Let $k = [s]$ and $r = s - [s]$. Then $f \in W^{s,2}(U)$ if and only if:*

(1) *$\forall\, |\alpha| \leqslant k$ we have $\partial^\alpha f \in L^2(U)$.*

(2) *If $r > 0$ we have*

$$\int\int_{U\times U} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^2}{|x-y|^{n+2r}} dxdy < +\infty. \tag{3.1.5}$$

Proof. It's a direct consequence of (3.1.2) and (3.1.4). □

3.2 The definable site and the main problem.

Let $X_\mathcal{A}(\mathbb{R}^n)$ be the category of open bounded definable sets in $\mathbb{R}^n$ (the morphisms are the inclusions, or the empty set), we endow $X_\mathcal{A}(\mathbb{R}^n)$ with the Grothendick topology (note that this definitions work for more general categories):

$S \subset X_\mathcal{A}(\mathbb{R}^n)$ is a covering for $U \in X_\mathcal{A}(\mathbb{R}^n)$ if and only if S is finite and $U = \bigcup_{O \in S} O$.

And we call it the definable site (associated to $\mathcal{A}$).

Definition 3.2.1. A sheaf of $\mathbb{C}$-vector spaces on the site $X_\mathcal{A}(\mathbb{R}^n)$ is a contravariant functor

$$\mathcal{F} : X_\mathcal{A}(\mathbb{R}^n) \to \mathbb{C}\text{-vector spaces},$$

such that for any $U, V \in X_\mathcal{A}(\mathbb{R}^n)$, the sequence

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$$

is exact.
This is equivalent to say that if $S = \{O_1, ..., O_l\} \subset X_{\mathcal{A}}(\mathbb{R}^n)$ is a cover of $O \in X_{\mathcal{A}}(\mathbb{R}^n)$, and $f_i \in \mathcal{F}(O_i)$ such that

$$f_i \mid_{O_i \cap O_j} = f_j \mid_{O_i \cap O_j} \text{ for all } i \neq j \; with \; O_i \cap O_j \neq \emptyset, \tag{3.2.1}$$

then there is a unique $f \in \mathcal{F}(O)$ such that $f \mid_{O_i} = f_i$ for $i = 1, ..., l$.
If in addition we have that for any $U, V \in X_{\mathcal{A}}(\mathbb{R}^n)$ the sequence

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

is exact, then we say that $\mathcal{F}$ is an *acyclic* sheaf.

The following example was introduced by Kashiwara in [8] to prove the Riemann-Hilbert correspondence:

Example 3.2.2. We denote by $(\mathbb{R}^n)_{an}$ the site associated the o-minimal structure of globally subanalytic sets. We define the trace of distributions on relatively compact subanalytic sets

$$\mathcal{T} : (\mathbb{R}^n)_{an} \to \mathbb{R}\text{-vector spaces},$$

such that for $U \subset \mathbb{R}^n$ we have

$$\mathcal{T}(U) = \{f \in \mathcal{D}'(U) \; : \; \exists F \in \mathcal{D}'(\mathbb{R}^n) \text{ such that } F_{|U} = f\}.$$

One can show that $f \in \mathcal{T}(U)$ if and only if there are $C > 0$, $m \in \mathbb{N}$, and $r \in \mathbb{N}$ such that for any $\phi \in C_c^\infty(U)$ we have

$$|< f, \phi >| \leqslant C \sum_{|\alpha| \leqslant m} \sup_{x \in U} \left(\frac{|\partial^\alpha \phi(x)|}{d(x, \partial U)^r} \right).$$

Then $\mathcal{T}$ is an acyclic sheaf on the subanalytic site $(\mathbb{R}^n)_{an}$, that means for any bounded subanalytic $U_1, U_2 \subset \mathbb{R}^n$ the sequence

$$0 \to \mathcal{T}(U_1 \cup U_2) \to \mathcal{T}(U_1) \oplus \mathcal{T}(U_2) \to \mathcal{T}(U_1 \cap U_2) \to 0$$

is exact. Indeed take $U_1, U_2 \subset \mathbb{R}^n$ bounded subanalytic, and take $f \in \mathcal{D}'(U_1 \cup U_2)$ such that $f_{|U_1} \in \mathcal{T}(U_1)$ and $f_{|U_2} \in \mathcal{T}(U_2)$. So there are $C_1 > 0$, $C_2 > 0$, $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$, $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$ such that for any $\phi \in C_c^\infty(U_i)$ we have

$$\left|< f_{|U_i}, \phi >\right| \leqslant C_i \sum_{|\alpha| \leqslant m_i} \sup_{x \in U_i} \left(\frac{|\partial^\alpha \phi(x)|}{d(x, \partial U_i)^{r_i}} \right).$$

By the Łojasiewicz's inequality there are $C > 0$ and $m \in \mathbb{N}$ such that

$$d(x, U_1) + d(x, U_2) \geqslant C(d(x, \partial(U_1 \cup U_2)))^m \text{ for all } x \in U_1 \cup U_2.$$

Take (φ_1, φ_2) a partition of unity associated to (U_1, U_2). Hence, for $\phi \in C_c^\infty(U_1 \cup U_2)$ we have

$$\begin{aligned}
|< f, \phi >| &= |< f, \varphi_1\phi + \varphi_2\phi >| \\
&\leqslant \left|< f_{|U_1}, \varphi_1\phi >\right| + \left|< f_{|U_2}, \varphi_2\phi >\right| \\
&\leqslant C_1 \sum_{|\alpha|\leqslant m_1} \sup_{x\in U_1} \left(\frac{|\partial^\alpha \varphi_1\phi(x)|}{d(x,\partial U_1)^{r_1}}\right) + C_2 \sum_{|\alpha|\leqslant m_2} \sup_{x\in U_2} \left(\frac{|\partial^\alpha \varphi_2\phi(x)|}{d(x,\partial U_2)^{r_2}}\right) \\
&\leqslant \frac{\max(C_1,C_2)}{C} \sum_{|\alpha|\leqslant \max(m_1,m_2)} \sup_{x\in U_1\cup U_2} \left(\frac{|\partial^\alpha \phi(x)|}{d(x,\partial(U_1\cup U_2))^{m\min(r_1,r_2)}}\right).
\end{aligned}$$

Therefore, $f \in \mathcal{T}(U_1 \cup U_2)$.

□

Problem: For a given $s > 0$, is there a sheaf $\mathcal{F}^s$ on the definable site $X_{\mathcal{A}}(\mathbb{R}^n)$ such that for any $U \in X_{\mathcal{A}}(\mathbb{R}^n)$ with Lipschitz boundary, we have

$$\mathcal{F}^s(U) = W^{s,2}(U) \text{ and } H^j(U, \mathcal{F}^s) = 0 \text{ for } j > 0.$$

Recall that for any contravariant functor (a presheaf) $\mathcal{F} : X_{\mathcal{A}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$-vector spaces, and $x \in \mathbb{R}^n$ we denote by $\mathcal{F}_x$ the germ at x of sections

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U) = \sqcup_{x\in U}\mathcal{F}(U)\,/\backsim,$$

such that for two sections $f_1 \in \mathcal{F}(U_1)$ and $f_2 \in \mathcal{F}(U_2)$ with $x \in U_1 \cap U_2$, we have $f_1 \backsim f_2$ if and only if there is a neighborhood $V \subset U_1 \cap U_2$ of x such that $f_1 |_V = f_2 |_V$. There is a canonical sheaf $\mathcal{F}_+$ associated to $\mathcal{F}$ defined by

$$U \in X_{\mathcal{A}}(\mathbb{R}^n) \mapsto \mathcal{F}_+(U) \subset F(U, \sqcup_{x\in U}\mathcal{F}_x),$$

with $f \in \mathcal{F}_+(U)$ if for any $x \in U$, $f(x) \in \mathcal{F}_x$ and there is a neighborhood $V \subset U$ of x and $\phi \in \mathcal{F}(V)$ such that for every $y \in V$, $f(y)$ is a representative of ϕ in $\mathcal{F}_y$.

For $s > 0$, let's consider $W^{s,2}_+$ the sheaf associated to $W^{s,2}$ on the site $X_{\mathcal{A}}(\mathbb{R}^n)$. Let $U \in X_{\mathcal{A}}(\mathbb{R}^n)$ be Lipschitz. Then one can prove that there is no way to identify $W^{s,2}_+(U)$ with $W^{s,2}(U)$, which make the sheafification method not good for our purpose. We want to get a sheaf out of Sobolev spaces but keep the best of it, because Sobolev spaces behave perfectly on Lipschitz domains. For $s < 0$, a sheafification in the derived category $D^+(X_{\mathbb{R}_{\text{an}}}(\mathbb{R}^n))$ of sheaves on the subanalytic site was given by G.Lebeau [12], but since the subanalytic argument involved in the proof was only the existence of regular covers, then this means that Theorem 2.2.2 implies directly a generalization of this to any definable site on $\mathbb{R}^n$:

Theorem 3.2.3. *Take $\mathcal{A}$ an o-minimal structure on $(\mathbb{R}, +, \cdot)$ and $s < 0$. Then there is an object $\mathcal{F}^s \in D^+(X_{\mathcal{A}}(\mathbb{R}^n))$ such that if $U \subset \mathbb{R}^n$ is a bounded open definable set with a Lipschitz boundary, the complex $\mathcal{F}^s(U)$ is concentrated in degree 0 and equal to $W^{s,2}(U)$.*

3.3 The spaces $W^{s,2}$ for $s \in]-\frac{1}{2}, \frac{1}{2}[$.

Using the results of Parusiński in [22], it was noticed in [12] that for $s \in]-\frac{1}{2}, \frac{1}{2}[$ the presheaf $U \mapsto W^{s,2}(U)$ is an acyclic sheaf on the subanalytic topology. We explain here by details why this is true in the o-minimal case. Fix $\mathcal{A}$ an o-minimal structure on $\mathbb{R}^n$. Let's first recall a classical result on fractional Sobolev spaces (see Theorem11.2 in [13]). Take $s \in]0, \frac{1}{2}[$ and $U \subset \mathbb{R}^n$ an open Lipschitz domain. Then there is $C > 0$ such that for any $f \in W^{s,2}(U)$ we have

$$\|\frac{f(x)}{d(x,U)^s}\|_{L^2(U)} \leqslant C\|f\|_{W^{s,2}(U)}. \tag{3.3.1}$$

Lemma 3.3.1. *Fix $s \in]-\frac{1}{2}, \frac{1}{2}[$ and let $U \in X_{\mathcal{A}}(\mathbb{R}^n)$ be Lipschitz. Then the linear operator*

$$\begin{aligned} 1_U : W^{s,2}(\mathbb{R}^n) &\longrightarrow W^{s,2}(\mathbb{R}^n) \\ f &\mapsto 1_U f \end{aligned}$$

is well defined.

Proof. The case of $s = 0$ is obvious. First assume $0 < s < \frac{1}{2}$. Let $f \in W^{s,2}(\mathbb{R}^n)$ and $U \in X_{\mathcal{A}}(\mathbb{R}^n)$ be Lipschitz. It is clear that $1_U f \in L^2(\mathbb{R}^n)$, so by (3.1.2) we need to prove that

$$L = \int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|1_U f(x) - 1_U f(y)|^2}{|x-y|^{n+2s}} dxdy < +\infty. \tag{3.3.2}$$

But

$$L = \int\int_{U\times U} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dxdy + 2\int_U |f(x)|^2 \left(\int_{U^c} \frac{1}{|x-y|^{n+2s}} dy\right) dx.$$

Since $f \in W^{s,2}(\mathbb{R}^n)$, by (3.3.1) it's enough to prove that

$$d(x,U)^{-2s} \lesssim \int_U \frac{1}{|x-y|^{n+2s}} dy \lesssim d(x,U)^{-2s}, \tag{3.3.3}$$

where $U \in X_{\mathcal{A}}(\mathbb{R}^n)$ is Lipschitz. Since ∂U is bounded we can assume that

$$U = \{(y', y_n) \in \mathbb{R}^n \ : \ y_n > 0\}.$$

And a simple computation shows that

$$d(x,U)^{-2s} = \frac{1}{|x_n|^{2s}} \lesssim \int_U \frac{1}{|x-y|^{n+2s}} dy \lesssim \frac{1}{|x_n|^{2s}} = d(x,U)^{-2s}.$$

Now take $s \in]-\frac{1}{2}, 0[$. For $T \in W^{s,2}(\mathbb{R}^n)$ we have

$$\begin{aligned} 1_U T : W^{-s,2}(\mathbb{R}^n) &\longrightarrow \mathbb{C} \\ f &\mapsto < 1_U T, f > := < T, 1_U f >. \end{aligned}$$

By the case of $s \in]0, \frac{1}{2}[$, $1_U T$ is well defined and in $W^{-s,2}(\mathbb{R}^n)$. $\square$

Denote $\mathcal{A}(\mathbb{R}^n)$ the algebra generated by the characteristic functions of open bounded definable sets in $\mathbb{R}^n$, that is

$$\mathcal{A}(\mathbb{R}^n) = \{\sum_{i\in I} m_i 1_{U_i} \ : \ I \text{ finite}, \ m_i \in \mathbb{Z}, \text{ and } U_i \in X_{\mathcal{A}}(\mathbb{R}^n)\}.$$

Then we have:

Lemma 3.3.2. *The algebra $\mathcal{A}(\mathbb{R}^n)$ is generated by the characteristic functions of Lipschitz definable domains.*

Proof. Take $U \subset \mathbb{R}^n$. Thanks to the existence of L-regular decomposition, it's enough to assume that U is L-regular with respect to the standard coordinates of $\mathbb{R}^n$.

- *Case* $\dim(U) = n$*:* So take $U' = \pi(U) \subset \mathbb{R}^{n-1}$ L-regular. Assume that there are Lipschitz open definable $O_1, O_2, ..., O_N \in X_{\mathcal{A}}(\mathbb{R}^n)$ such that

$$1_{U'} = \sum_i m_i 1_{O_i} \text{ for some } m_i \in \mathbb{Z}.$$

By definition of L-regular cells, we have that

$$U = \Gamma(U', f, g) = \{(x', x_n) \in U' \times \mathbb{R} \ : \ f(x') < x_n < g(x')\},$$

where $f < g : U' \longrightarrow \mathbb{R}$ are Lipschitz definable functions with extensions $f \leqslant g : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$. For each $i \in \{1, ..., N\}$ it's clear that

$$1_{\Gamma(O_i,f,g,)} = 1_{\Gamma(O_i,f-1,g)} + 1_{\Gamma(O_i,f,g+1)} - 1_{\Gamma(O_i,f-1,g+1)}.$$

Hence

$$1_U = \sum_i m_i 1_{\Gamma(O_i,f-1,g)} + \sum_i m_i 1_{\Gamma(O_i,f,g+1)} + \sum_i (-) m_i 1_{\Gamma(O_i,f-1,g+1)}.$$

And we have bi-Lipschitz maps

$$\begin{aligned} \varphi_i^- : \Gamma(O_i, f-1, g) &\longrightarrow O_i \times]0,1[\\ x = (x', x_n) &\mapsto \varphi_i^-(x) = (x', \tfrac{x_n+1-f(x')}{g(x')-f(x')+1}), \end{aligned}$$

$$\begin{aligned} \varphi_i^+ : \Gamma(O_i, f, g+1) &\longrightarrow B(0,1) \times]0,1[\\ x = (x', x_n) &\mapsto \varphi_i^+(x) = (x', \tfrac{x_n-f(x')}{g(x')-f(x')-1}), \end{aligned}$$

$$\begin{aligned} \varphi_i : \Gamma(O_i, f+1, g+1) &\longrightarrow B(0,1) \times]0,1[\\ x = (x', x_n) &\mapsto \varphi_i(x) = (\phi_i(x'), \tfrac{x_n+1-f(x')}{g(x')-f(x')+2}). \end{aligned}$$

Hence $\Gamma(O_i, f+1, g)$, $\Gamma(O_i, f, g+1)$ and $\Gamma(O_i, f+1, g+1)$ are Lipschitz domains.

- *Case* $\dim(U) = l < n$*:* In this case there are l-dimensional L-regular $U_l \subset \mathbb{R}^l$ and a definable Lipschitz map $\phi = (\phi_1, ..., \phi_{n-l}) : \mathbb{R}^l \longrightarrow \mathbb{R}^{n-l}$ such that

$$U = \Gamma(U_l, \phi) = \{(x, \phi(x)) \ : \ x \in U_l\}.$$

For each $I \subset \{1, ..., n-l\}$, define $U_I \in X_{\mathcal{A}}(\mathbb{R}^n)$ as follows

$$U_I = \{(x,y) \in U_l \times \mathbb{R}^l \ : \ \phi_j(x) - 1 < y_j < \phi_j(x) + 1 \ \forall j \in \{1, ..., n-l\} \text{ and } y_i \neq \phi_i(x) \ for \ i \in I\}.$$

U_I is a disjoint union of open Lipschitz definable sets, and we have

$$1_U = \sum_I (-1)^{|I|} 1_{U_I}.$$

□

Proposition 3.3.3. *For $s \in]-\frac{1}{2}, \frac{1}{2}[$, the presheaf $W^{s,2}$ is an acyclic sheaf on the definable site $X_{\mathcal{A}}(\mathbb{R}^n)$, that is for any $U, V \in X_{\mathcal{A}}(\mathbb{R}^n)$ the sequence*

$$0 \to W^{s,2}(U \cup V) \to W^{s,2}(U) \oplus W^{s,2}(V) \to W^{s,2}(U \cap V) \to 0$$

is exact.

Proof. By the definition of $W^{s,2}$ we have the surjectivity of the map $W^{s,2}(U \cap V) \to 0$. Take $(f, g) \in W^{s,2}(U) \oplus W^{s,2}(V)$ such that $f_{|U\cap V} = g_{|U\cap V}$. Take $(\widehat{f}, \widehat{g}) \in (W^{s,2}(\mathbb{R}^n))^2$ such that

$$\widehat{f}_{|U} = f \text{ and } \widehat{g}_{|V} = g.$$

By Lemma 3.3.1 and Lemma 3.3.2 we have $h = 1_U \widehat{f} + 1_V \widehat{g} - 1_{U\cap V} \widehat{f} \in W^{s,2}(\mathbb{R}^n)$. Then $h_{|U\cup V} \in W^{s,2}(U \cup V)$, $(h_{|U\cup V})_{|U} = f$, and $(h_{|U\cup V})_{|V} = g$.

□

3.4 Construction of the sheaf $\mathcal{F}^k$ on $\mathbb{R}^2$ for $k \in \mathbb{N}$.

Given two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$, and $r > 0$ such that $\gamma_1(0) = \gamma_2(0) = p_0$. We denote by $R(r, \gamma_1, \gamma_2)$ the open definable subset (see Figure 3.1)

$$R(r, \gamma_1, \gamma_2) = \{P \in \mathbb{R}^2 \ : \ P \in B(p_0, r) \text{ and } P \text{ is between } \gamma_1 \text{ and } \gamma_2\}.$$

Formally

$$P \in R(r, \gamma_1, \gamma_2) \text{ if and only if}$$
$$Angl(\gamma_1 \cap C(p_0, \|P\|), \overrightarrow{e_1}(1,0)) < Angl(\overrightarrow{p_0P}, \overrightarrow{e_1}(1,0)) < Angl(\gamma_2 \cap C(p_0, \|P\|), \overrightarrow{e_1}(1,0))$$

Here,

$$C(p_0, \|P\|) = \{x \in \mathbb{R}^2 \ : \ \|x - p_0\| = \|P\|\}.$$

If we write γ_1 and γ_2 by the parametrisations (and assume that $p_0 = 0$, which is always possible up to a translation)

$$\gamma_1(t) = te^{i\theta_1(t)} \text{ and } \gamma_2(t) = te^{i\theta_2(t)} \text{ with } t \in [0, r[\text{ and } 0 < \theta_1(t) - \theta_2(t) < 2\pi.$$

Then

$$R(r, \gamma_1, \gamma_2) = \{te^{i\theta} \ : \ t \in]0, r[\text{ and } \theta_1(t) < \theta < \theta_2(t)\}.$$

Remark 3.4.1. We can always choose r small enough such that $R(r, \gamma_1, \gamma_2)$ is connected and the circle $C(p_0, r')$ is transverse to γ_1 and γ_2 at the intersection points (that consists of only two points).

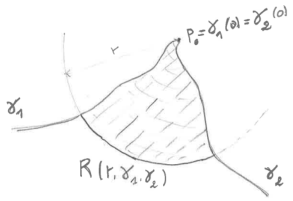

Figure 3.1: The domain $R(r, \gamma_1, \gamma_2)$.

3.4.1 The local nature of open definable sets in $\mathbb{R}^2$.

Let U be a bounded connected open definable subset of $\mathbb{R}^2$. By choosing a cell decomposition of $\mathbb{R}^2$ compatible with U and ∂U, we can prove that for any $p_0 \in \partial U$ there is $r > 0$ such that we have one of the following cases (see Figure 3.2):

(C_1): **Punctured disk.** $B_r(p_0) \cap U = B_r(p_0) \setminus \{p_0\}$.

(C_2): **Sector.** There exist two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $Angl(\gamma_1'(0), \gamma_2'(0)) \neq 0, 2\pi$, and

$$B_r(p_0) \cap U = R(r, \gamma_1, \gamma_2).$$

(C_3): **Cusp.** There exist two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $Angl(\gamma_1'(0), \gamma_2'(0)) = 0$, and

$$B_r(p_0) \cap U = R(r, \gamma_1, \gamma_2).$$

(C_4): **Cusp complement.** There exist two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $Angl(\gamma_1'(0), \gamma_2'(0)) = 2\pi$, and

$$B_r(p_0) \cap U = R(r, \gamma_1, \gamma_2).$$

(C_5): **Arc complement.** There exist a definable C^1-curve $\gamma : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma(0) = p_0$ and

$$B_r(p_0) \cap U = B_r(p_0) \setminus Im(\gamma).$$

(C_6) $B_r(p_0) \cap U$ is a disjoint union of copies of open sets like C_2, C_3, and C_4.

3.4.2 Local description (or definition) of the sheaf $\mathcal{F}^k$.

Lemma 3.4.2. *Let U, V be two Lipschitz definable bounded open subsets of $\mathbb{R}^2$ such that $U \cup V$ and $U \cap V$ are Lipschitz. Then for any $s \in \mathbb{R}_+$, the following sequence of Hilbert spaces*

$$0 \to W^{s,2}(U \cup V) \to W^{s,2}(U) \oplus W^{s,2}(V) \to W^{s,2}(U \cap V) \to 0$$

is exact.

Proof. See [12] for the proof (or see Section 3.6 for a categorical proof). □

Remark 3.4.3. For $s \in \mathbb{N}$, for the statement of Lemma 3.4.2 we don't need $U \cap V$ to be Lipschitz.

Proof. Take $s = k \in \mathbb{N}$. Then by (3.1.4), for $\Omega = U \cup V \subset \mathbb{R}^n$ we have

$$W^{k,2}(\Omega) = \{ f \in L^2(\Omega) \ : \ \forall \alpha \in \mathbb{N}^n \ : |\alpha| \leqslant k \Longrightarrow \partial^\alpha f \in L^2(\Omega) \ \},$$

Where $\partial^\alpha f$ is the distributional derivative of f. The Hilbert structure of $W^{k,2}(\Omega)$ is given by

$$\|f\|^2_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|^2_{L^2(\Omega)}.$$

Now let $(f, g) \in W^{k,2}(U) \oplus W^{k,2}(V)$ such that $f\ |_{U \cap V} = g\ |_{U \cap V}$. So there is $H \in L^2(U \cup V)$ such that $H\ |_U = f \in W^{k,2}(U)$ and $H\ |_V = g \in W^{k,2}(V)$. We want to show that for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leqslant k$ there is $h_\alpha \in L^2(U \cup V)$ such that $\partial^\alpha H = h_\alpha$ (in the distributional sense). Take (φ_U, φ_V) a partition of unity associated to (U, V). For any $\phi \in C_c^\infty(U \cup V)$ we have

$$\begin{aligned}
< \partial^\alpha H, \phi > &= < \partial^\alpha H, \varphi_U \phi > + < \partial^\alpha H, \varphi_V \phi > \\
&= (-1)^{|\alpha|} \int_U H \partial^\alpha (\varphi_U \phi) + (-1)^{|\alpha|} \int_V H \partial^\alpha (\varphi_V \phi) \\
&= (-1)^{|\alpha|} \int_U f \partial^\alpha (\varphi_U \phi) + (-1)^{|\alpha|} \int_V g \partial^\alpha (\varphi_V \phi) \\
&= (-1)^{|\alpha|} \int_U \partial^\alpha f (\varphi_U \phi) + (-1)^{|\alpha|} \int_V \partial^\alpha g (\varphi_V \phi) \\
&= (-1)^{|\alpha|} \int_{U \cup V} (\varphi_U \partial^\alpha f + \varphi_V \partial^\alpha g) \phi \\
&= (-1)^{|\alpha|} \int_{U \cup V} h_\alpha \phi.
\end{aligned}$$

Here $h_\alpha := \varphi_U \partial^\alpha f + \varphi_V \partial^\alpha g \in L^2(U \cup V)$, and that completes the proof. □

From now on we consider $k \in \mathbb{N}$. Let U be a connected open definable bounded subset of $\mathbb{R}^2$. We define the $\mathbb{C}$-vector space $\widehat{\mathcal{F}}^k(U)$ in the following special cases:

(C_1) If $U = B_r(p_0) \setminus \{p_0\}$. We can assume $p_0 = (0, 0)$ and $r = 1$. In this case we can decompose $U = U_1 \cup U_2$, where

$$U_1 = \{(x, y) \in U : \ y > x \ or \ y < -x\} \text{ and } U_2 = \{(x, y) \in U : \ y > -x \ or \ y < x\}.$$

We have the sequence

$$0 \longrightarrow W^{k,2}(U) \xrightarrow{d_0} W^{k,2}(U_1) \oplus W^{k,2}(U_2) \xrightarrow{d_1} W^{k,2}(U_1 \cap U_2)$$

And it is natural to choose $\widehat{\mathcal{F}}^k(U) := Ker(d_1)$.
It follows from Lemma 3.4.2 that

$$\widehat{\mathcal{F}}^k(U) = \{f \in L^2(U) \ : \ f\mid_L \in W^{k,2}(L) \text{ for any } L \text{ Lipschitz in } U\} = W^{k,2}_\star(U).$$

But we have a classical result on Sobolev spaces:
Fact: Take $\Omega \subset \mathbb{R}^n$ open and $W \subset \Omega$ such that $\mathcal{H}^{n-1}(W) = 0$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-Hausdorff measure on $\mathbb{R}^n$. Then we have

$$W^{k,2}_\star(\Omega \setminus W) = W^{k,2}_\star(\Omega).$$

That gives

$$W^{k,2}_\star(U) = W^{k,2}_\star(B_r(p_0)) = W^{k,2}(B_r(p_0)).$$

So finally this means that we can take

$$\widehat{\mathcal{F}}^k(U) = W^{k,2}(U).$$

(C_2) If U is connected with Lipschitz boundary, then we define $\widehat{\mathcal{F}}^k(U) := W^{k,2}(U)$.

(C_3) If U is a cusp, means that there are $r > 0$ and two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0)$, $Angl(\gamma_1'(0), \gamma_2'(0)) = 0$, and

$$U = R(r, \gamma_1, \gamma_2).$$

Then we define: $\widehat{\mathcal{F}}^k(U) := W^{k,2}(U)$.

(C_4) If U is a complement of a cusp, means that there are $r > 0$ and two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $Angl(\gamma_1'(0), \gamma_2'(0)) = 2\pi$, and

$$U = R(r, \gamma_1, \gamma_2).$$

Take $\gamma_3, \gamma_4 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_3(0) = \gamma_4(0) = p_0$, $Angl(\gamma_1'(0), \gamma_3'(0)) > 0$, and $Angl(\gamma_4'(0), \gamma_2'(0)) > 0$.
In this case the sequence

$$0 \to W^{k,2}(U) \to W^{k,2}(R(r, \gamma_1, \gamma_4)) \oplus W^{k,2}(R(r, \gamma_3, \gamma_2)) \to W^{k,2}(R(r, \gamma_3, \gamma_4)) \to 0$$

is not exact in general.

Example 3.4.4. Assume that $k > 2$, then we have the continuous embedding $W^{k,2}(\mathbb{R}^2) \hookrightarrow C^1(\mathbb{R}^2)$. Take $U, V \in X_{\mathcal{A}}(\mathbb{R}^2)$ defined by

$$U = (]-1,1[\times]-1,0[) \cup (]-1,0[\times]-1,1[),$$

and

$$V = (]-1,0[\times]-1,1[) \cup \{(x,y)\ :\ 0 \leqslant x < 1 \text{ and } x^{k+1} < y < 1\}.$$

Define $F \in L^2(U \cup V)$ by $F\ |_U = 0$, $F(x,y) = x^{k+1}$ for $x \in [0,1[$ and $x^{k+1} < y < 1$. It's clear that $F\ |_U \in W^{k,2}(U)$ and $F\ |_V \in W^{k,2}(V)$ but $F \notin W^{k,2}(U \cup V)$, because if $F \in W^{k,2}(U \cup V)$ then there will be a C^1 extension $\widehat{F}$ of F to $\mathbb{R}^2$, but this can not be true because

$$\lim_{x \to 0} \frac{\widehat{F}(x, x^{k+1}) - \widehat{F}(x, 0)}{x^{k+1} - 0} = 1.$$

□

Question 3. What happens in this case if we replace k by $s \in [\frac{1}{2}, 2]$? is the sequence

$$0 \to W^{s,2}(U) \to W^{s,2}(R(r,\gamma_1,\gamma_4)) \oplus W^{s,2}(R(r,\gamma_3,\gamma_2)) \to W^{s,2}(R(r,\gamma_3,\gamma_4)) \to 0$$

exact?

Now we define $\widehat{\mathcal{F}}^k(U = R(r,\gamma_1,\gamma_2))$ to be the kernal of the map

$$J : W^{k,2}(R(r,\gamma_1,\gamma_4)) \oplus W^{k,2}(R(r,\gamma_3,\gamma_2)) \to W^{k,2}(R(r,\gamma_3,\gamma_4)).$$

We use the notation

$$\widehat{\mathcal{F}}^k(U) = Ker(J) := K(\gamma_3,\gamma_4).$$

We need to prove that $K(\gamma_3,\gamma_4)$ doesn't depend on γ_3 and γ_4, but only on U. Take $\alpha, \beta : [0,a[\longrightarrow \mathbb{R}^2$ two definable curves that satisfy the same conditions as γ_3 and γ_4. Let's prove that

$$K(\gamma_3,\gamma_4) = K(\alpha,\beta).$$

We can identify $K(\gamma_3,\gamma_4)$ and $K(\alpha,\beta)$ to the spaces

$$K(\gamma_3,\gamma_4) = \{f \in \mathcal{D}'(U)\ :\ f_{|R(r,\gamma_1,\gamma_4)} \in W^{k,2}(R(r,\gamma_1,\gamma_4)) \text{ and } f_{|R(r,\gamma_3,\gamma_2)} \in W^{k,2}(R(r,\gamma_3,\gamma_2))\}$$

$$K(\alpha,\beta) = \{f \in \mathcal{D}'(U)\ :\ f_{|R(r,\gamma_1,\beta)} \in W^{k,2}(R(r,\gamma_1,\beta)) \text{ and } f_{|R(r,\alpha,\gamma_2)} \in W^{k,2}(R(r,\alpha,\gamma_2))\}.$$

We can distinguish four possible cases:

case1: $Im(\alpha) \subset R(r,\gamma_3,\gamma_4)$ and $Im(\beta) \subset R(r,\gamma_3,\gamma_4)$.

case2: $Im(\alpha) \subset R(r,\gamma_3,\gamma_4)$ and $Im(\beta) \subset R(r,\gamma_4,\gamma_2)$.

case3: $Im(\alpha) \subset R(r,\gamma_1,\gamma_3)$ and $Im(\beta) \subset R(r,\gamma_3,\gamma_4)$.

case4: $Im(\alpha) \subset R(r,\gamma_1,\gamma_3)$ and $Im(\beta) \subset R(r,\gamma_1,\gamma_3)$.

The first case is obvious, because in this case we have $R(r,\gamma_1,\beta) \subset R(r,\gamma_1,\gamma_4)$ and $R(r,\alpha,\gamma_2) \subset R(r,\gamma_3,\gamma_2)$. The cases 3 and 4 can be proven using the same computation as case2.

Proof in case2: We will prove that $K(\gamma_3,\gamma_4) \subset K(\alpha,\beta)$ (the other inclusion follows from the other cases). Take $f \in K(\gamma_3,\gamma_4)$, since in this case $R(r,\alpha,\gamma_2) \subset R(r,\gamma_3,\gamma_2)$, we have $f_{|R(r,\alpha,\gamma_2)} \in W^{k,2}(R(r,\alpha,\gamma_2))$. Now let's prove that

$$f_{|R(r,\gamma_1,\beta)} \in W^{k,2}(R(r,\gamma_1,\beta)).$$

Take $c : [0,a[\longrightarrow \mathbb{R}^2$ a definable curve such that $c(0) = p_0$, $Angl(\gamma_1'(0), c'(0)) > 0$, $Angl(c'(0), \gamma_2'(0)) > 0$, $Angl(\beta'(0), c'(0)) > 0$, and $Im(c) \subset R(r,\beta,\gamma_2)$. We can see that $f_{|R(r,\gamma_1,\gamma_4)} \in W^{k,2}(R(r,\gamma_1,\gamma_4))$ and $f_{|R(r,\gamma_3,c)} \in W^{k,2}(R(r,\gamma_3,c))$ (note that $R(r,\gamma_3,c) \subset R(r,\gamma_3,\gamma_2)$). Now by Lemma3.4.2 the short sequence

$$0 \to W^{k,2}(R(r,\gamma_1,c)) \to W^{k,2}(R(r,\gamma_1,\gamma_4)) \oplus W^{k,2}(R(r,\gamma_3,c)) \to W^{k,2}(R(r,\gamma_3,\gamma_4))$$

is exact.
Hence $f_{|R(r,\gamma_1,c)} \in W^{k,2}(R(r,\gamma_1,c))$, therefore $f_{|R(r,\gamma_1,\beta)} \in W^{k,2}(R(r,\gamma_1,\beta))$.

□

(C_5) If there exists a definable C^1-curve $\gamma : [0,a[\longrightarrow \mathbb{R}^2$ such that $\gamma(0) = p_0$ and

$$U = B_r(p_0) \setminus Im(\gamma).$$

Take $\gamma_1, \gamma_2 : [0,a[\longrightarrow \mathbb{R}^2$ two C^1 definable curves such that $Angl(\gamma_1'(0), \gamma_2'(0)) \neq 2\pi$. By the Sobolev embedding and continuity reasons we can find an example such that the short sequence

$$0 \to W^{k,2}(U) \to W^{k,2}(R(r,\gamma,\gamma_2)) \oplus W^{k,2}(R(r,\gamma_1,\gamma)) \to W^{k,2}(R(r,\gamma_1,\gamma_2))$$

is not exact.

Example 3.4.5. Assume that $k > 1$. So we have an embedding $W^{k,2}(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$. Take $U, V \in X_{\mathcal{A}}(\mathbb{R}^2)$ defined by

$$U = (]-1,1[\times]-1,0[) \cup (]-1,0[\times]-1,1[),$$

and

$$V = (]-1,0[\times]-1,1[) \cup (]-1,1[\times]0,1[).$$

Define $F \in L^2(U \cup V)$ by $F\,|_U = 0$ and $F(x,y) = e^{-\frac{1}{x^2}}$ for $0 < x < 1$ and $0 < y < 1$. It's obvious that $F\,|_U \in W^{k,2}(U)$ and $F\,|_V \in W^{k,2}(V)$ but $F \notin W^{k,2}(U \cup V)$, because it can not be extended to a continuous function on $\mathbb{R}^2$.

□

Question 4. What happens in this case if we replace k by $s \in [\frac{1}{2}, 1]$? is the sequence

$$0 \to W^{s,2}(U) \to W^{s,2}(R(r,\gamma,\gamma_2)) \oplus W^{s,2}(R(r,\gamma_1,\gamma)) \to W^{s,2}(R(r,\gamma_1,\gamma_2))$$

exact?

So this motivate us to define $\widehat{\mathcal{F}}^k(U)$ to be the kernal of the map

$$J : W^{k,2}(R(r,\gamma,\gamma_2)) \oplus W^{k,2}(R(r,\gamma_1,\gamma)) \to W^{k,2}(R(r,\gamma_1,\gamma_2)).$$

That is

$$\widehat{\mathcal{F}}^k(U) = Ker(J) = K(\gamma_1, \gamma_2).$$

Applying the same technics we did with the previous case, we can show that $K(\gamma_1, \gamma_2)$ doesn't depend by γ_1 and γ_2.

Remark 3.4.6. Note that this is a special case of the previous case.

(C_6) If U is like in the case (C_6). Then we take $\widehat{\mathcal{F}}^k(U)$ to be direct sum of $\widehat{\mathcal{F}}^k$ of the connected components of $U \cap B_r(p_0)$.

3.4.3 The global definition of $\mathcal{F}^k$ on the site $X_{\mathcal{A}}(\mathbb{R}^2)$.

Take $k \in \mathbb{N}$. For every $U \in X_{\mathcal{A}}(\mathbb{R}^2)$, we define $\mathcal{F}^k(U)$ by

$$\mathcal{F}^k(U) := \{ f \in W^{k,2}_{loc}(U) \ : \text{ for each } x \in \partial U,\ \exists r > 0 \text{ such that } B(x,r) \cap U \in \{C_1, ..., C_6\} \text{ and } f\,|_{B(x,r)\cap U} \in \widehat{\mathcal{F}}^k(B(x,r) \cap U)\ \}.$$

Claim: $\mathcal{F}^k$ is a sheaf on the site $X_{\mathcal{A}}(\mathbb{R}^2)$.

Proof. We need to prove that for $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ and $V \in X_{\mathcal{A}}(\mathbb{R}^2)$, the sequence

$$0 \to \mathcal{F}^k(U \cup V) \to \mathcal{F}^k(U) \oplus \mathcal{F}^k(V) \to \mathcal{F}^k(U \cap V)$$

is exact. It is enough to prove that if $f \in W^{k,2}_{loc}(U \cup V)$ (or even $\mathcal{D}'(U \cup V)$) such that $f\,|_U \in \mathcal{F}^k(U)$ and $f\,|_V \in \mathcal{F}^k(V)$, then one has $f \in \mathcal{F}^k(U \cup V)$. It is also enough to assume that f is supported at a small neighborhood of a given point $x \in \overline{U \cup V}$ (if (ϕ_i) is a partition of unity such that $\sum_i \phi_i = 1$ near $\overline{U \cup V}$, then clearly $f = \sum_i \phi_i f$), and more precisely of a given singular point $x \in \partial U \cap \partial V$ such that ∂U and ∂V has different germs at x.

So take $x \in \partial U \cap \partial V$ such that $(\partial U, x) \neq (\partial V, x)$ (and also no inclusion between the two germs).

- **Case(A):** Assume here that none of U, V, and $U \cup V$ are like the case C_1.

Step1: we assume that U, V, and $U \cup V$ are locally connected near x.
There is $r > 0$ such that $B(x,r) \cap U$, $B(x,r) \cap V$, $B(x,r) \cap (U \cup V)$, $B(x,r) \cap (U \cap V) \in \{C_2, ..., C_6\}$, hence there are definable curves $\gamma_i : [0, a[\longrightarrow \mathbb{R}^2$ $(i = 1, ..., 4)$ such that $\gamma_i(0) = x$ and

$$U \cap B(x,r) = R(r, \gamma_1, \gamma_2),\ (U \cap V) \cap B(x,r) = R(r, \gamma_2, \gamma_3),\ V \cap B(x,r) = R(r, \gamma_3, \gamma_4), \text{ and } (U \cup V) \cap B(x,r) = R(r, \gamma_1, \gamma_4).$$

By the definition of $\mathcal{F}^k$ and assuming that f is supported in $(U \cup V) \cap B(x,r)$, it's enough to prove that $f \mid_{(U \cup V) \cap B(x,r)} \in \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = \widehat{\mathcal{F}}^k(R(r, \gamma_1, \gamma_4))$ knowing that $f \mid_{U \cap B(x,r)} \in \widehat{\mathcal{F}}^k(U \cap B(x,r))$ and $f \mid_{V \cap B(x,r)} \in \widehat{\mathcal{F}}^k(V \cap B(x,r))$. We will discuss several cases for this:

- **case(1)** $Angl(\gamma_1'(0), \gamma_4'(0)) = 0$ **:** in this case everything is a cusp near x. So we can find U' and V' Lipschitz such that $U' \cup V'$ is Lipschitz, $U \cap B(r,x) \subset U'$, $V \cap B(r,x) \subset V'$, and $U' \cap V' = (U \cap V) \cap B(r,x)$. In this case we have

$$\widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = W^{k,2}((U \cup V) \cap B(x,r))$$
$$\widehat{\mathcal{F}}^k(U \cap B(x,r)) = W^{k,2}(U \cap B(x,r))$$
$$\widehat{\mathcal{F}}^k(V \cap B(x,r)) = W^{k,2}(V \cap B(x,r))$$

Take $f_{U'} \in W^{k,2}(U')$ an extension of $f \mid_{U \cap B(x,r)}$ and $f_{V'} \in W^{k,2}(V')$ an extension of $f \mid_{V \cap B(x,r)}$, and define $F \in \mathcal{D}'(U' \cup V')$ by gluing $f_{U'}$ and $f_{V'}$. By Lemma 3.4.2 we have that $F \in W^{k,2}(U' \cup V')$ and since $F \mid_{(U \cup V) \cap B(x,r)} = f \mid_{(U \cup V) \cap B(x,r)}$, $f \in W^{k,2}((U \cup V) \cap B(x,r)) = \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$.

- **case(2)** $Angl(\gamma_1'(0), \gamma_4'(0)) \neq 0, 2\pi$ **:** In this case either U is Lipschitz or V is Lipschitz. If both are Lipschitz then the prove follow from Lemma 3.4.2. The case if there's one not Lipschitz, let's assume it's U. In this case we can find U' Lipschitz such that $U' \cup V$ is Lipschitz, $U \cap B(r,x) \subset U'$, and $U' \cap V = (U \cap V) \cap B(r,x)$. Same as the previous case we have

$$\widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = W^{k,2}((U \cup V) \cap B(x,r))$$
$$\widehat{\mathcal{F}}^k(U \cap B(x,r)) = W^{k,2}(U \cap B(x,r))$$
$$\widehat{\mathcal{F}}^k(V \cap B(x,r)) = W^{k,2}(V \cap B(x,r))$$

Take $f_{U'} \in W^{k,2}(U')$ an extension of $f \mid_{U \cap B(x,r)}$, and define $F \in \mathcal{D}'(U' \cup V)$ by gluing $f_{U'}$ and $f \mid_V$. By Lemma 3.4.2 we have that $F \in W^{k,2}(U' \cup V)$ and since $F \mid_{(U \cup V) \cap B(x,r)} = f \mid_{(U \cup V) \cap B(x,r)}$, $f \in W^{k,2}((U \cup V) \cap B(x,r)) = \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$.

- **case(3)** $Angl(\gamma_1'(0), \gamma_4'(0)) = 2\pi$ **:**

  - **Subcase3.1:** If $U \cap B(x,r)$ and $V \cap B(x,r)$ are lipschitz then we have by definition that

$$\widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = K(\gamma_2, \gamma_3)$$

And this give that $f \mid_{(U \cup V) \cap B(x,r)} \in \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$.

- **Subcase3.2:** if $Angl(\gamma_1'(0), \gamma_3'(0)) = 2\pi$ and $Angl(\gamma_2'(0), \gamma_4'(0)) = 2\pi$, then in this case we can find α and β in $(U \cap V) \cap B(x,r)$ with a starting point x, such that
$$\widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = K(\alpha, \beta)$$
And since $f\mid_{R(r,\gamma_1,\beta)} \in W^{k,2}(R(r,\gamma_1,\beta))$ and $f\mid_{R(r,\alpha,\gamma_4)} \in W^{k,2}(R(r,\alpha,\gamma_4))$, we have $f\mid_{(U\cup V)\cap B(x,r)} \in \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$.
- **Subcase3.3:** If $Angl(\gamma_1'(0), \gamma_3'(0)) = 0$ and $Angl(\gamma_2'(0), \gamma_4'(0)) = 2\pi$, then in this case we can find $\alpha,\ \beta : [0,a[\longrightarrow \mathbb{R}^2$ such that $Im(\beta),\ Im(\alpha) \subset V \cap B(x,r)$ and
$$\widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r)) = K(\alpha, \beta)$$
we have that $f\mid_{R(r,\alpha,\gamma_4)} \in W^{k,2}(R(r,\alpha,\gamma_4))$, and by applying **case (2)** on $R(r,\gamma_1,\gamma_3)$, $R(r,\gamma_2,\beta)$, we deduce that also $f\mid_{R(r,\gamma_1,\beta)} \in W^{k,2}(R(r,\gamma_1,\beta))$, hence we have that $f\mid_{(U\cup V)\cap B(x,r)} \in \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$.
- **Subcase3.4:** If $Angl(\gamma_1'(0), \gamma_3'(0)) = 2\pi$ and $Angl(\gamma_2'(0), \gamma_4'(0)) = 0$, then it's the symmetry statement of subcase3.

Remark 3.4.7. Note that the case where $\gamma_1(t) = \gamma_4(t)$ is included in the **case(3)**.

Step2: We don't assume here the local connectivity of U, V, and $U \cup V$.

In this case there's a finite number of definable curves (with beginning point x) $\gamma_1, \lambda_1 ..., \gamma_m, \lambda_m :$ $[0,a[\longrightarrow \mathbb{R}^2$, $\alpha_1, \beta_1 ..., \alpha_l, \beta_l : [0,a[\longrightarrow \mathbb{R}^2$ such that

$$B(x,r) \cap U = \sqcup_i R(r,\gamma_i,\lambda_i) \text{ and } B(x,r) \cap V = \sqcup_i R(r,\alpha_i,\beta_i).$$

Take $f \in \mathcal{D}'((U \cup V) \cap B(x,r))$ such that $f\mid_{U\cap B(x,r)} \in \widehat{\mathcal{F}}^k((U) \cap B(x,r))$ and $f\mid_{V\cap B(x,r)} \in$ $\widehat{\mathcal{F}}^k((V) \cap B(x,r))$, clearly this implies that

$$f\mid_{R(r,\gamma_i,\lambda_i)} \in \widehat{\mathcal{F}}^k(R(r,\gamma_i,\lambda_i)) \text{ and } f\mid_{R(r,\alpha_j,\beta_j)} \in \widehat{\mathcal{F}}^k(R(r,\alpha_j,\beta_j)) \text{ for all } i \text{ and } j.$$

We want to prove that $f\mid_{(U\cup V)\cap B(x,r)} \in \widehat{\mathcal{F}}^k((U \cup V) \cap B(x,r))$. By the local definition of $\widehat{\mathcal{F}}^k$, it's enough to prove that $f\mid_C \in \widehat{\mathcal{F}}^k(C)$ for every connected component C of $(U \cup V) \cap B(x,r)$. So take C' a connected component of C, we can reorder the curves $\gamma_1, \lambda_1 ..., \gamma_m, \lambda_m, \alpha_1, \beta_1 ..., \alpha_l, \beta_l$ to find a definable curves $c_1, ..., c_n$ such that

$$C' = \cup_i R(r,c_i,c_{i+1}),\ f\mid_{R(r,c_i,c_{i+1})} \in \widehat{\mathcal{F}}^k(R(r,c_i,c_{i+1})), \text{ and}$$
$$R(r,c_i,c_{i+1}) \cap R(r,c_{i+2},c_{i+3}) \neq \emptyset \text{ for any } i \in \{1,...,n-3\}.$$

Using induction and **Step1** we deduce that $f\mid_{C'} \in \widehat{\mathcal{F}}^k(C')$.

- **Case(B):** Let's be out of the assumption of **Case(A)** .Since we assumed that the germs $(\partial U, x)$ and $(\partial V, x)$ are not comparable, the only non trivial case is when U,$V \in \{C_2, ..., C_6\}$ and $U \cup V$ is like C_1. Let L be Lipschitz open subset in $U \cup V$. If $x \notin \overline{L}$, then $f\mid_L \in W^{k,2}(L)$ because for any $p \in \overline{L}$ there is a neighborhood O_p of p in U or V such that $f\mid_{O_p} \in W^{k,2}(O_p)$. Now, if $x \in \overline{L}$ then in this case near x, L is like C_2 and covered by two open sets $U_L \in$ $\{C_2, ..., C_6\}$ and $V_L \in \{C_2, ..., C_6\}$ such that $f\mid_{U_L} \in \widehat{\mathcal{F}}^k(U_L)$ and $f\mid_{V_L} \in \widehat{\mathcal{F}}^k(V_L)$, and by the discussion of the **Case(A)**, it follows that $f\mid_L \in \widehat{\mathcal{F}}^k(L) = W^{k,2}(L)$.

□

Remark 3.4.8. Take $k \in \mathbb{N}$. By analyzing each case, we can show that

(1) Let $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ such that U is of type C_1,...,C_6. Then we have

$$\mathcal{F}^k(U) = \widehat{\mathcal{F}}^k(U).$$

(2) If $W \in X_{\mathcal{A}}(\mathbb{R}^2)$ has only cuspidal singularities (singularities in the boundary of W are Lipschitz or of type C_3) then

$$\mathcal{F}^k(W) = W^{k,2}(W).$$

Hence if $U, V \in X_{\mathcal{A}}(\mathbb{R}^2)$ such that U, V, $U \cap V$, and $U \cup V$ have only cuspidal singularities, then the sequence:

$$0 \to W^{k,2}(U \cup V) \to W^{k,2}(U) \oplus W^{k,2}(V) \to W^{k,2}(U \cap V) \to 0.$$

is exact.

(3) For each $U \in X_{\mathcal{A}}(\mathbb{R}^n)$, $\mathcal{F}^k(U)$ has a natural Hilbert structure. Take $\mathcal{L} = (L_1, L_2, ..., L_m)$ an L-regular decomposition of U. Since each open L-regular set in $\mathbb{R}^2$ has only cuspidal singularities, the following map

$$\mathcal{N}_{\mathcal{L}} : \mathcal{F}^k(U) \longrightarrow \mathbb{R} \quad f \mapsto \mathcal{N}_{\mathcal{L}}(f) = \sum_{\dim(L_i)=2} \|f_{|L_i}\|_{W^{k,2}(L_i)},$$

is a well defined Hilbert structure on $\mathcal{F}^k(U)$, and independent of $\mathcal{L}$. Moreover if U has only cuspidal singularities, then this norm coincide with the Sobolev norm $\| \|_{W^{k,2}(U)}$.

Proof. • **(1)** We discuss case by case. The cases C_1 and C_2 are direct consequences of the fact that any $x \in \partial U$ (except for the center of the punctured disk) has a Lipschitz boundary in U. The case C_6 follows from the additive propriety of $\mathcal{F}^k$ and the other cases. So we prove C_3 and C_4 (C_5 is the same as C_4):

- C_3 **:** In this case $U = R(r, \alpha, \beta)$ is a cusp between α and β. If $f \in \mathcal{F}^k(U)$ then any $x \in \overline{U}$ has $r_x > 0$ such that $f\ |_{U_r(x)} \in \widehat{\mathcal{F}}^k(U_r(x)) = W^{k,2}(U_r(x))$ because locally in the boundary U is only of type C_2 and C_3, hence by a partition of unity argument $f \in W^{k,2}(U) = \widehat{\mathcal{F}}^k(U)$. Now if $f \in \widehat{\mathcal{F}}^k(U) = W^{k,2}(U)$, then clearly $f \in \mathcal{F}^k(U)$, because $W^{k,2}$ is always a subspace of $\mathcal{F}^k$.
- C_4 **:** In this case there exist two definable C^1-curves $\gamma_1, \gamma_2 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $Angl(\gamma_1'(0), \gamma_2'(0)) = 2\pi$, and

$$U = R(r, \gamma_1, \gamma_2).$$

Take $\gamma_3, \gamma_4 : [0, a[\longrightarrow \mathbb{R}^2$ such that $\gamma_3(0) = \gamma_4(0) = p_0 \in \mathbb{R}^2$, $Angl(\gamma_1'(0), \gamma_3'(0)) > 0$, and $Angl(\gamma_4'(0), \gamma_2'(0)) > 0$. So

$$\widehat{\mathcal{F}}^k(U) = \{f \in \mathcal{D}'(U) \ : \ f_{|R(r,\gamma_1,\gamma_4)} \in W^{k,2}(R(r,\gamma_1,\gamma_4)) \ and \ f_{|R(r,\gamma_3,\gamma_2)} \in W^{k,2}(R(r,\gamma_3,\gamma_2))\}.$$

Let $f \in \widehat{\mathcal{F}}^k(U)$ and $x \in \partial U$, then for $r > 0$ big enough we have that $U_r(x) = U$ and $f \mid U_r(x) \in \widehat{\mathcal{F}}^k(U_r(x))$, and so $f \in \mathcal{F}^k(U)$. Reciprocally, take $f \in \mathcal{F}^k(U)$. For p_0 we can find $r' > 0$ such that $U'_r(p_0) = R(r', \gamma_1, \gamma_2)$ and $f \mid U'_r(p_0) \in \widehat{\mathcal{F}}^k(U'_r(p_0))$, and by definition

$$\widehat{\mathcal{F}}^k(U'_r(p_0)) = \{f \in \mathcal{D}'(U) \ : \ f_{|R(r',\gamma_1,\gamma_4)} \in W^{k,2}(R(r',\gamma_1,\gamma_4)) \ and \ f_{|R(r',\gamma_3,\gamma_2)} \in W^{k,2}(R(r',\gamma_3,\gamma_2))\}.....(\star)$$

But we have also U is lipschitz near each point $x \in \partial U \setminus \{p_0\}$, so this implies that f is Sobolev near each of these points. So this together with $(\star)$ implies that $f_{|R(r,\gamma_1,\gamma_4)} \in W^{k,2}(R(r,\gamma_1,\gamma_4))$ *and* $f_{|R(r,\gamma_3,\gamma_2)} \in W^{k,2}(R(r,\gamma_3,\gamma_2))$, and therefore $f \in \widehat{\mathcal{F}}^k(U)$.

- **(2)** If $W \in X_{\mathcal{A}}(\mathbb{R}^2)$ has only cuspidal singularities, then for any $x \in \overline{W}$ there is $r_x > 0$ such that $W_{r_x}(x)$ is either Lipschitz or a standard cusp, and so $\widehat{\mathcal{F}}^k(W_{r_x}(x)) = W^{k,2}(W_{r_x}(x))$. If we take a partition of unity $(\phi_x)_{x\in W}$ of the covering $(W_{r_x}(x))_{x\in W}$, then clearly

$$f = \sum_x \phi_x f \mid_{W_{r_x}(x)} \in W^{k,2}(W).$$

And the exactness on cuspidal domains follows immediately.

- **(3)** Obvious from L-regular decomposition and the (2) in the remark.

□

Notation: For $k \in \mathbb{N}$ and $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ with only cuspidal singularities, we denote by E_U an extension operator

$$E_U : W^{k,2}(U) \longrightarrow W^{k,2}(\mathbb{R}^2)$$
$$f \mapsto E_U(f) \text{ with } (E_U(f))_{|U} = f.$$

3.5 Cohomology of the sheaf $\mathcal{F}^k$.

For the computation we need the notion of a good direction.

Good directions: Let $A \subset \mathbb{R}^n$ be a definable subset and $\lambda \in \mathbb{S}^{n-1}$. We say that λ is a *good direction* for A if there is $\varepsilon > 0$ such that for any $x \in A^{reg}$ we have

$$d(\lambda, T_x A^{reg}) > \varepsilon.$$

Given $\lambda \in \mathbb{S}^{n-1}$. We denote by $\pi_\lambda : \mathbb{R}^n \longrightarrow N_\lambda =< \lambda >^\perp$ the orthogonal projection, and by x_λ the coordinate of x along $< \lambda >$.
Let $A \subset \mathbb{R}^n$ and $A' \subset N_\lambda$ definable sets, $f : A' \longrightarrow \mathbb{R}$ a definable function. We say that A is the graph of the function f for λ if

$$A = \{x \in \mathbb{R}^n \ : \ \pi_\lambda(x) \in A' \text{ and } x_\lambda = f(\pi_\lambda(x))\}.$$

Note that $\lambda \in \mathbb{S}^{n-1}$ is a good direction for A if and only if A is the union of graphs of Lipschitz definable functions over some subsets of N_λ. One can see immediately that the sphere $\mathbb{S}^n$ has a no good direction, we need to decompose it into finitely many pieces such that each one has a one good direction (this can be proven using the regular projection theorem), but we have the beautiful theorem of G.Valette [31] which states that after a bi-Lipschitz deformation of the ambient space, we have always one good direction:

Theorem 3.5.1. *Let $A \subset \mathbb{R}^n$. There is a definable bi-Lipschitz $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $h(A)$ has one good direction $\lambda \in \mathbb{S}^{n-1}$.*

Definition 3.5.2. Take $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ and $\mathcal{U} = (U_i)_{i \in I}$ a cover of U in the definable site $X_{\mathcal{A}}(\mathbb{R}^2)$. An *adapted* cover of $\mathcal{U}$ is a definable cover $\mathcal{V} = \{V_j\}_{j \in J}$ of $\mathbb{R}^2$ such that the following properties are satisfied:

(1) $\mathcal{V}$ is compatible with $\mathcal{U}$, that is each element in $\mathcal{U}$ is a finite union of elements in $\mathcal{V}$.

(2) Every finite intersection of elements in $\mathcal{V}$ is either empty or a connected domain with only cuspidal singularities, and intersection of more than three elements is always empty.

(3) There are $m \in \mathbb{N}$, $r > 0$, and $(k_l, p_l) \in \mathbb{N}^2$ for each $l \in \{0, ..., m\}$ such that $\mathcal{V} = \{V_j\}_{j \in J}$ can be rearranged as follows

$$\begin{aligned}\mathcal{V} &= \{O_{l,p} \ : \ l \in \{0,1,...,m+1\} \text{ and } p \in \{0,...,p_l\}\} \\ &\cup \ \{\widehat{O}_{l,p} \ : \ l \in \{0,1,...,m+1\} \text{ and } p \in \{0,...,p_l - 1\}\} \\ &\cup \ \{V_{l,k} \ : \ l \in \{0,1,...,m\} \text{ and } k \in \{0,...,k_l+1\}\} \\ &\cup \ \{B(a_{l,k}, r) \ : \ a_{k,l} \in \mathbb{R}^2, \ l \in \{0,1,...,m\} \text{ and } k \in \{0,...,k_l\}\}\end{aligned}$$

(4) • For each $l \in \{1, ..., m\}$ and $p \in \{0, ..., p_l\}$ there is a unique $(L(l,p), R(l,p)) \in \mathbb{N}^2$ such that the only possible non-Lipschitz singularities of $O_{l,p}$ and $\widehat{O}_{l,p}$ (only the case of $p < p_l$) are $a_{l-1,L(l,p)}$ and $a_{l,R(l,p)}$.

• For each $p \in \{0, ..., p_0\}$ there is a unique $R(0,p) \in \mathbb{N}$ such that the only possible non-Lipschitz singularities of $O_{0,p}$ and $\widehat{O}_{0,p}$ (only the case of $p < p_l$) is $a_{l,R(0,p)}$.

• For each $p \in \{0, ..., p_{m+1}\}$ there is a unique $L(m+1,p) \in \mathbb{N}$ such that the only possible non-Lipschitz singularities of $O_{m+1,p}$ and $\widehat{O}_{m+1,p}$ (only the case of $p < p_l$) is $a_{l,L(m+1,p)}$.

(5) The only non empty intersections of two open sets in $\mathcal{V}$ are the open sets $O_{l,p} \cap \widehat{O}_{l,p}$, $\widehat{O}_{l,p} \cap O_{l,p+1}$, $O_{l,p} \cap V_{l,R(l,p)}$, $O_{l,p} \cap V_{l-1,L(l,p)}$, $B(a_{l,k}, r) \cap V_{l,k}$, $B(a_{l,k}, r) \cap V_{l,k+1}$, $B(a_{l-1,L(l,p)}, r) \cap \widehat{O}_{l,p}$, $B(a_{l,R(l,p)}, r) \cap \widehat{O}_{l,p}$, $B(a_{l-1,L(l,p)}, r) \cap O_{l,p}$, and $B(a_{l,R(l,p)}, r) \cap O_{l,p}$.

(6) The only non empty intersections of three open sets in $\mathcal{V}$ are the open sets $O_{l,p} \cap V_{l,R(l,p)} \cap B(a_{l,R(l,p)}, r)$, $O_{l,p} \cap V_{l-1,L(l,p)} \cap B(a_{l-1,L(l,p)}, r)$, $\widehat{O}_{l,p} \cap V_{l,R(l,p)} \cap B(a_{l,R(l,p)}, r)$, $\widehat{O}_{l,p} \cap V_{l-1,L(l,p)} \cap B(a_{l-1,L(l,p)}, r)$, $O_{l,p} \cap \widehat{O}_{l,p} \cap B(a_{l,R(l,p)}, r)$, $O_{l,p} \cap \widehat{O}_{l,p} \cap B(a_{l-1,L(l,p)}, r)$, $O_{l,p+1} \cap \widehat{O}_{l,p} \cap B(a_{l,R(l,p+1)}, r)$, and $O_{l,p+1} \cap \widehat{O}_{l,p} \cap B(a_{l-1,L(l,p+1)}, r)$.

This definition is motivated by the construction in Figure 3.2 and explained in detail in the proof of Proposition 3.5.3. These covers will be essential in the computation of the cohomology of Sobolev sheaves (see Theorem 3.5.5).

Čech cohomology: Recall that for a given sheaf $\mathcal{F}$ on a topological space M and a covering $\mathcal{U} = (U_i)_{i\in I}$ with I an ordered set, we have the Čech complex $\mathcal{C}^\star_{\mathcal{U}}(M,\mathcal{F})$ defined by:

$$\mathcal{C}^0_{\mathcal{U}}(M,\mathcal{F}) \xrightarrow{d_0} \mathcal{C}^1_{\mathcal{U}}(M,\mathcal{F}) \xrightarrow{d_1} \mathcal{C}^2_{\mathcal{U}}(M,\mathcal{F})....,$$

such that

$$\mathcal{C}^m_{\mathcal{U}}(M,\mathcal{F}) = \bigoplus_{J=(i_0<i_1<...<i_m)} \mathcal{F}(U_J)$$

and

$$(d_m\alpha)_{U_J} := (d_m\alpha)_{J=\{i_0<...<i_m\}} = \sum_{j=0,...,m} (-1)^j (\alpha_{J\setminus i_j})_{|U_J}.$$

Clearly if $\mathcal{V}$ is a refinement of $\mathcal{U}$ then there is a canonical morphism $\mathcal{C}^\star_{\mathcal{U}}(M,\mathcal{F}) \longrightarrow \mathcal{C}^\star_{\mathcal{V}}(M,\mathcal{F})$. So the Čech cohomology of degree j of M with respect to $\mathcal{F}$ is defined to be the colimite:

$$H^j(M,\mathcal{F}) = \lim_{\mathcal{U}} H^j(\mathcal{C}^\star_{\mathcal{U}}(M,\mathcal{F})).$$

It's well know that this cohomology coincide with the cohomology of the sheaf $\mathcal{F}$ on paracompact spaces, and so on definable sets. We prove in the following Proposition that any cover in the site $X_{\mathcal{A}}(\mathbb{R}^2)$ has an *adapted* cover and so we can use only adapted covers to compute the cohomology of $\mathcal{F}^k$.

Proposition 3.5.3. *Take $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ and $\mathcal{U} = (U_i)_{i\in I} \subset X_{\mathcal{A}}(\mathbb{R}^2)$ a cover of U. Then there is an adapted cover $\mathcal{V}$ of $\mathcal{U}$.*

Proof. Take $\mathcal{U} = (U_i)_{i\in I}$ a definable cover of U, with $I = \{1,...,m\}$. It's obvious that it is enough to find such a cover after a bi-Lipschitz definable homeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. So by Theorem 3.5.1 we can assume that $\bigcup_i \partial(U_i)$ is included in a finite union of graphs of definable Lipschitz functions $\xi_j : \mathbb{R} \longrightarrow \mathbb{R}$. We are going to construct an adapted cover $\mathcal{V}$ (see Figure 3.2).

Take

$$n = Max\{\sharp(\pi^{-1}(x) \cap (\bigcup_j \Gamma_{\xi_j}))\ :\ x \in \mathbb{R}\} < +\infty.$$

Take $\mathcal{C} = \{]-\infty, a_0[, \{a_0\},]a_0, a_1[, ...,]a_{m-1}, a_m[, \{a_m\},]a_m, +\infty[\}$ a cell decomposition of $\mathbb{R}$ compatible with the collection of sets

$$A_k = \{x \in \mathbb{R}\ :\ \sharp(\pi^{-1}(x) \cap (\textstyle\bigcup_j \Gamma_{\xi_j})) = k\} \text{ for } k \in \{1,...,n\}.$$

For $l \in \{0,...,m\}$, we have

$$\pi^{-1}(a_l) \cap (\bigcup_j \Gamma_{\xi_j}) = \{a_{l,0}, a_{l,1}, ..., a_{l,k_l}\}.$$

We denote $a_{-1} := -\infty$ and $a_{m+1} := +\infty$. For $l \in \{-1,0,...,m\}$ there are Lipschitz definable functions

$$\phi_{l,0} < ... < \phi_{l,p_l} :]a_l, a_{l+1}[\to \mathbb{R}$$

such that $\pi^{-1}\left(\pi(\bigcup_i \partial(U_i))\cap]a_l, a_{l+1}[\right) \cap \left(\bigcup_j \Gamma_{\xi_j}\right) = \bigcup_p \Gamma_{\phi_{l,p}}$. Now for each $l \in \{0, 1, ..., m\}$ and $p \in \{0, ..., p_l\}$ there are definable Lipschitz functions $\phi^-_{l,p} < \phi^+_{l,p} :]a_l, a_{l+1}[\mapsto \mathbb{R}$, such that we have

$$\phi^-_{l,0} < \phi_{l,0} < \phi^+_{l,0} < \phi^-_{l,1} < \phi_{l,1} < \phi^+_{l,p_l} ...... < \phi^+_{l,p_l},$$

$$\lim_{t\to a_l} \phi^-_{l,p} = \lim_{t\to a_l} \phi^+_{l,p} = \lim_{t\to a_l} \phi_{l,p},$$

and

$$\lim_{t\to a_{l+1}} \phi^-_{l,p} = \lim_{t\to a_{l+1}} \phi^+_{l,p} = \lim_{t\to a_{l+1}} \phi_{l,p}.$$

Denote by $a_{l,-1} := -\infty$ and $a_{l,k_l+1} := +\infty$. For each $l \in \{0, ..., m\}$ and $k \in \{-1, ..., l_k\}$ there are Lipschitz functions (with respect to the direction $\{(0, 1)\}$) $\varphi^-_{l,k} < a_l < \varphi^+_{l,k}$: $]a_{l,k}, a_{l,k+1}[\to \mathbb{R}$ such that the graphs of these functions do not intersect the graphs of the functions $\phi^s_{l,p}$ (for any l, p, and $s \in \{0, -, +\}$ with $\phi^0_{l,p} := \phi_{l,p}$), and

$$\lim_{t\to a_{l,k}} \varphi^s_{l,k} = a_l = \lim_{t\to a_{l,k+1}} \varphi^s_{l,k}.$$

For each (l, k) such that $a_{l,k}, \in U$, there is $r_{l,k} > 0$ such that $B(a_{l,k}, r_{l,k}) \subset U_i$ for all U_i that contain $a_{l,k}$. Take $r < \min_{l,k}(r_{l,k})$ such that $\partial B(a_{l,k}, r)$ is transverse to all the graphs oh the functions $\phi^s_{l,p}$ and $\varphi^s_{l,k}$ (here also $\varphi^0_{l,k} := a_{l,k}$), with

$$\overline{B}(a_{l',k'}, r) \cap \overline{B}(a_{l,k}, r) = \emptyset \text{ if } (l, k) \neq (l', k').$$

Take the collection of open definable sets

$$\mathcal{V} = \{\Gamma(\varphi^-_{l,k}, \varphi^+_{l,k}), \Gamma(\phi^-_{l,p}, \phi^+_{l,p}), \Gamma(\phi_{l,p}, \phi_{l,p+1}), B(a_{l,k}, r)\ \}_{l,k,p}.$$

Then clearly the collection $\mathcal{V}$ is an adapted cover of $\mathcal{U}$. □

Figure 3.2 represents an example of an adapted cover $\mathcal{V}$ following the notation in the proof of Proposition 3.5.3.

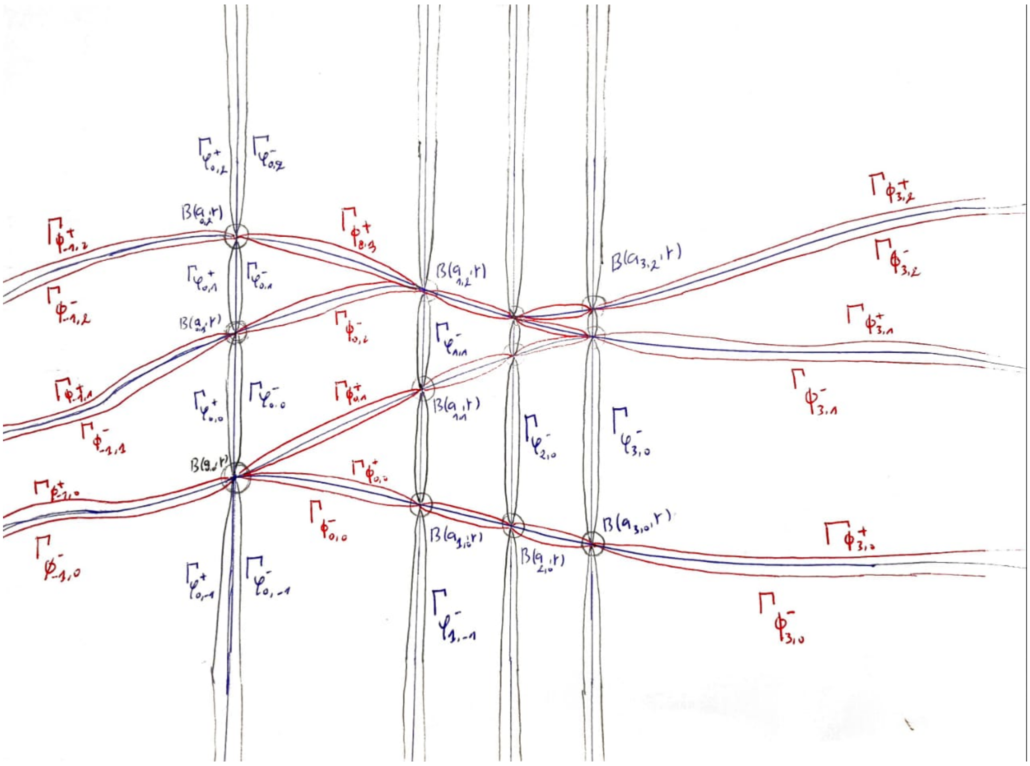

Figure 3.2: The cover $\mathcal{V}$.

Remark 3.5.4.(1) The sheaf $\mathcal{F}^k : X_{\mathcal{A}}(\mathbb{R}^2) \longrightarrow \mathbb{C}-$vector spaces is not acyclic for $k > 1$. In this case we have an inclusion $W^{k,2}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$. Take the punctured disk $W = B(0,1) \setminus \{0\} = U \cup V$, with:

$$U = \{(x,y) \in W : \ y > x \ or \ y < -x\} \text{ and } V = \{(x,y) \in W : \ y > -x \ or \ y < x\}.$$

And $U \cap V = O_1 \sqcup O_2$ with $\overline{O_1} \cap \overline{O_2} = \{0\}$ such that:

$$O_1 = \{(x,y) \in W : \ y >\mid x \mid\} \text{ and } O_2 = \{(x,y) \in W : \ y < - \mid x \mid\}.$$

If $H^1(W, \mathcal{F}^s) = 0$, then by the long Mayer-Vietoris theorem, the sequence

$$0 \mapsto \mathcal{F}^k(W) \mapsto W^{k,2}(U) \oplus W^{k,2}(V) \mapsto W^{k,2}(O_1) \oplus W^{k,2}(O_2) \mapsto 0$$

is exact. But this is not possible because for $(f \equiv 1, g \equiv 0) \in W^{k,2}(O_1) \oplus W^{k,2}(O_2)$ there is no continuous functions $(u,v) \in W^{k,2}(U) \oplus W^{k,2}(V)$ such that:

$$(u-v) \mid_{O_1} = 1 \text{ and } (u-v) \mid_{O_2} = 0.$$

Hence $H^1(W, \mathcal{F}^k) \neq 0$.

(2) In Theorem 3.5.5 we will compute the cohomology of $\mathcal{F}^k$. The proof of Theorem 3.5.5 will be based on the following observations: from the construction of *adapted* covers we

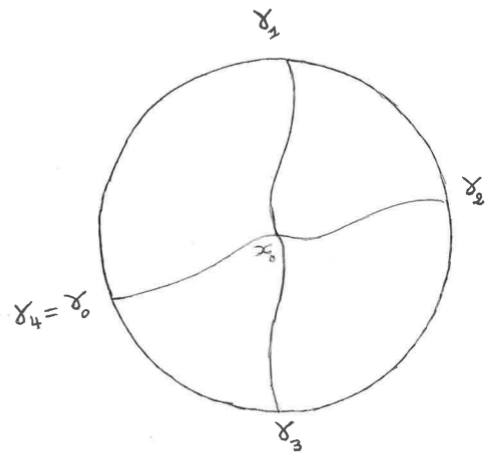

Figure 3.3: The curves γ_i around x_0.

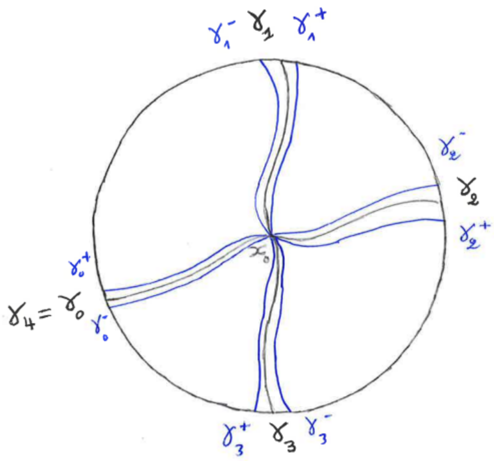

Figure 3.4: The curves γ_i^- and γ_i^+.

can deduce that $H^j(\cdot, \mathcal{F}^k) = 0$ for $j \geqslant 2$. For the first cohomology groups of $\mathcal{F}$ the only obstruction for $H^1(U, \mathcal{F}^k)$ to vanish is the existence of punctured disk in U. Indeed if we take U with no punctured disk singularity then locally gluing cocycles from $\mathcal{C}^1_{\mathcal{U}}(U, \mathcal{F}^k)$ to cochains in $\mathcal{C}^0_{\mathcal{U}}(U, \mathcal{F}^k)$ is summarized in the following simple example: take $x_0 \in \partial U$ and $\gamma_0, ..., \gamma_4 = \gamma_0 : [0, a[\to \overline{U}$ (see Figure 3.3) with $\gamma_0(0) = ... = \gamma_4(0) = x_0$ and $\gamma_i^- < \gamma_i^+ : [0, a[\to \mathbb{R}^2$ (see Figure 3.4), then locally we choose two situations (in fact they are the only situations that will show up locally in the proof of Theorem 3.5.5):

- **Situation 1:** we assume that $x_0 \in U$. In this situation for some $0 < r < r'$, we assume that (see Figure 3.4)

$$U = \bigcup_i R(r', \gamma_i, \gamma_{i+1}) \bigcup_i R(r', \gamma_i^-, \gamma_i^+) \bigcup B(x_0, r).$$

For each i we have functions $f_{i,+} \in W^{k,2}(R(r', \gamma_i \gamma_i^+))$, $f_{-,i} \in W^{k,2}(R(r', \gamma_i^- \gamma_i))$, $g_i \in W^{k,2}(R(r, \gamma_i, \gamma_{i+1}))$, and $h_i \in W^{k,2}(B(r, \gamma_i^-, \gamma_i^+))$ such that

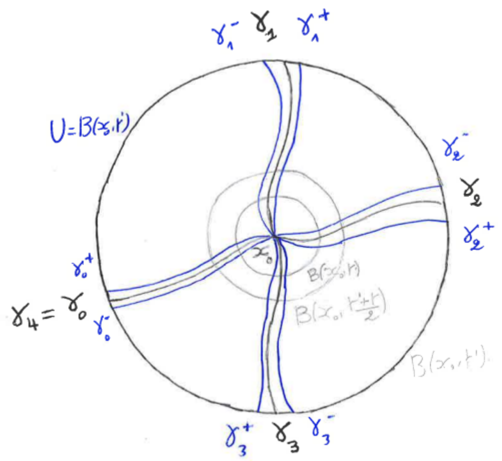

Figure 3.5: The covering of U in Situation 1.

$$
\begin{aligned}
(f_{i,+})_{|R(r,\gamma_i\gamma_i^+)} &= (g_i)_{|R(r,\gamma_i\gamma_i^+)},\\
(f_{-,i})_{|R(r,\gamma_i^-\gamma_i)} &= (g_{i-1})_{|R(r,\gamma_i^-\gamma_i)},\\
(h_i)_{|R(r,\gamma_i\gamma_i^+)} &= (g_i)_{|R(r,\gamma_i\gamma_i^+)}, \text{ and}\\
(h_i)_{|R(r,\gamma_i^-\gamma_i)} &= (g_{i-1})_{|R(r,\gamma_i^-\gamma_i)}.
\end{aligned}
$$

We want to glue these functions to functions in $W^{k,2}(R(r',\gamma_i,\gamma_{i+1}))$, $W^{k,2}(R(r',\gamma_i^-,\gamma_i^+))$, and $W^{k,2}(B(x_0,r))$. Take (ϕ',ϕ) a partition of unity associated to the covering $(C = B(x_0,r')\setminus B(x_0,r), B(\frac{r+r'}{2},x_0))$ (see Figure 3.5). Define $u \in W^{k,2}(B(x_0,r))$ by taking just the values of $g_i's$ and $h_i's$. On each $R(r',\gamma_i^-,\gamma_i^+)$ we choose the zero functions. We take smooth compactly supported functions $F_i : \mathbb{R}^2 \to [0,1]$ such that $F_i = 1$ on a neighborhood of $R(r',\gamma_i^-,\gamma_i^+)\cap C$ and $F_i = 0$ on the other sets of type $R(r',\gamma_j^-,\gamma_j^+)\cap C$. So in each $W^{k,2}(R(r',\gamma_i,\gamma_{i+1}))$ we define v_i by

$$
v_i := \Big(\phi'(F_i E_{R(r',\gamma_i,\gamma_i^+)}(f_{i,+}) + F_{i+1}E_{R(r',\gamma_{i+1}^-,\gamma_{i+1})}(f_{i+1,-})) + \phi(E_{B(x_0,r)}(u))\Big)_{|R(r',\gamma_i,\gamma_{i+1})}.
$$

Then clearly the functions u, 0 and v_i glue the functions $f_{i,+}$, $f_{-,i}$, g_i and h_i.

- **Situation 2:** we assume that $x_0 \notin U$. In this situation (and with the same notation as in the first case) we assume that (see Figure 3.6)

$$
U = R(r',\gamma_0,\gamma_1)\bigcup R(r',\gamma_1^-,\gamma_1^+)\bigcup R(r',\gamma_1,\gamma_2)\bigcup R(r',\gamma_2^-,\gamma_2^+)\bigcup R(r',\gamma_2,\gamma_3),
$$

with a given functions $f_{i,+} \in W^{k,2}(R(r',\gamma_i,\gamma_i^+))$ and $f_{-,i} \in W^{k,2}(R(r',\gamma_i^-,\gamma_i))$. To glue this functions to functions it is enough to take the functions

$$
\begin{aligned}
v_0 &:= 0 \in W^{k,2}(R(r',\gamma_0,\gamma_1)),\\
u_1 &:= \Big(E_{R(r',\gamma_1^-,\gamma_1)}(f_{-,1})\Big)_{|R(r',\gamma_1^-,\gamma_1^+)} \in W^{k,2}(R(r',\gamma_1^-,\gamma_1^+)),\\
v_1 &:= \Big(E_{R(r',\gamma_1^-,\gamma_1^+)}(f_{1,+}) + E_{R(r',\gamma_1^-,\gamma_1^+)}(u_1)\Big)_{|R(r',\gamma_1,\gamma_2)} \in W^{k,2}(R(r',\gamma_1,\gamma_2)),\\
u_2 &:= \Big(E_{R(r',\gamma_2^-,\gamma_2)}(f_{-,2}) + E_{R(r',\gamma_1,\gamma_2)}(v_1)\Big)_{|R(r',\gamma_2^-,\gamma_2^+)} \in W^{k,2}(R(r',\gamma_2^-,\gamma_2^+)),\\
v_2 &:= \Big(E_{R(r',\gamma_2,\gamma_2^+)}(f_{2,+}) + E_{R(r',\gamma_2^-,\gamma_2^+)}(u_2)\Big)_{|R(r',\gamma_2,\gamma_3)} \in W^{k,2}(R(r',\gamma_2,\gamma_3)).
\end{aligned}
$$

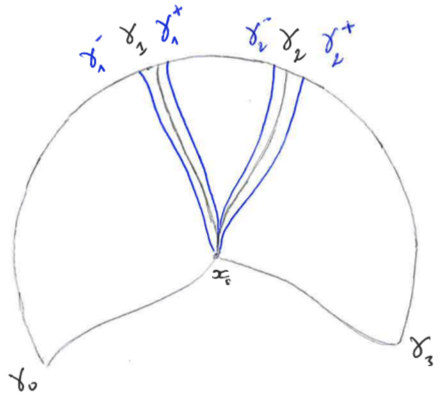

Figure 3.6: The covering of U in Situation 2.

Theorem 3.5.5. *Take $U \in X_{\mathcal{A}}(\mathbb{R}^2)$ and $\mathcal{F}$ a Sobolev sheaf on the definable site $X_{\mathcal{A}}(\mathbb{R}^2)$ (that is $\mathcal{F} = \mathcal{F}^s$ for some $s > 0$). Then for any $j > 1$ we have*

$$H^j(U, \mathcal{F}) = 0.$$

And if U has no singularities of type C_1. Then for any $j \in \mathbb{N}$

$$H^j(U, \mathcal{F}) = \begin{cases} \mathcal{F}(U) & \text{if } j = 0 \\ \{0\} & \text{if } j \geqslant 1. \end{cases}$$

Proof. By the definition of the Čech cohomology, it is enough to compute the Čech cohomology on an adapted cover. So take $\mathcal{V}$ an adapted cover of $\{U\}$ as given by Proposition 3.5.3 and take $\mathcal{W}$ the cover of U defined by

$$\mathcal{W} = \{O \in \mathcal{V} \ : \ O \subset U\}.$$

Then we have the Čech complex

$$\mathcal{C}^0_{\mathcal{W}}(U, \mathcal{F}) \overset{d_0}{\to} \mathcal{C}^1_{\mathcal{W}}(U, \mathcal{F}) \overset{d_1}{\to} \mathcal{C}^2_{\mathcal{W}}(U, \mathcal{F}) \to 0.$$

For $j > 2$ we have $\mathcal{C}^j_{\mathcal{W}}(U, \mathcal{F}) = 0$, because the intersection of four elements in $\mathcal{W}$ is always empty. Take $\omega \in \mathcal{C}^2_{\mathcal{U}}(U, \mathcal{F})$. So we can write ω as follows

$$\omega = \sum_{W \in \mathcal{W}_2} \omega(W),$$

where for $O \in \mathcal{W}_2$ we define

$$(\omega(W))_O = \begin{cases} (\omega)_W & \text{if } O = W \\ 0 & \text{if } O \neq W. \end{cases}$$

To show that $\omega = 0$ in $H^2(U, \mathcal{F})$ it is enough to find for each $W \in \mathcal{W}_2$ an element $\alpha(W) \in \mathcal{C}^1_{\mathcal{W}}(U, \mathcal{F})$ such that $d(\alpha(W)) = \omega(W)$. For each $a_{k,l} \in \mathbb{R}^2$ we take a smooth function $F_{k,l} \in C^\infty_c(\mathbb{R}^2)$ such that $F_{k,l} = 1$ on $B(a_{k,l}, r)$ and $F_{k,l} = 0$ on each other $B(a_{k',l'}, r)$. Take

$W \in \mathcal{W}_2$. Then $W = B(a_{l,k}, r) \cap Y$, where Y is one of the cases in (5) of Proposition 3.5.3. For any $O \in \mathcal{W}_2$ we define

$$(\alpha(W)_O = \begin{cases} (\sum_{k,l} F_{k,l} E_W((\omega)_W))_{|Y} & if\ O = Y \\ 0 & if\ not. \end{cases}$$

So clearly we have

$$d(\alpha(W)) = \omega(W),$$

and so $H^2(U, \mathcal{F}) = 0$.
Now assume that U has no punctured disk singularity and let's show that $H^1(U, \mathcal{F}) = 0$. Take $\alpha \in \mathcal{C}^1_{\mathcal{W}}(U, \mathcal{F})$ such that $d(\alpha) = 0$, so we need to find $u \in \mathcal{C}^0_{\mathcal{W}}(U, \mathcal{F})$ such that $d(u) = \alpha$. For $O \in \mathcal{W}$, we define $u \in \mathcal{C}^0_{\mathcal{W}}(U, \mathcal{F})$ by induction on l and p (see (3) of Proposition 3.5.3):

- **$O = O_{0,0}$:** in this case we define $u_O = 0 \in W^{s,2}(O)$.
- **$O = \widehat{O}_{0,p}$:** Assuming that we have constructed $u_{O_{0,p}}$, we define $u_O \in W^{s,2}(O)$ by

$$u_O = \Big(E_{O_{0,p}}(u_{O_{0,p}}) + E_{O_{0,p} \cap \widehat{O}_{0,p}}(\alpha_{O_{0,p} \cap \widehat{O}_{0,p}})\Big)_{|O}.$$

- **$O = O_{0,p+1}$:** Assuming that we have constructed $u_{\widehat{O}_{0,p}}$, we define $u_O \in W^{s,2}(O)$ by

$$u_O = \Big(E_{\widehat{O}_{0,p}}(u_{\widehat{O}_{0,p}}) + E_{\widehat{O}_{0,p} \cap O_{0,p+1}}(\alpha_{\widehat{O}_{0,p} \cap O_{0,p+1}})\Big)_{|O}.$$

This was induction on p with fixing $l = 0$. Now assume that for l fixed we have constructed $u_{O_{l,p}}$ and $u_{\widehat{O}_{l,p}}$ for each p. If $O = V_{l,k} \in \mathcal{W}$, then by (4) of Proposition 3.5.3 there is a unique p such that

$$O_{l,p} \cap V_{l,k} \neq \emptyset.$$

In this case we define u_O by

$$u_O = \big(E_{O_{l,p}}(u_{O_{l,p}}) + E_{O_{l,p} \cap V_{l,k}}(\alpha_{O_{l,p} \cap V_{l,k}})\big)_{|O}.$$

To finish we need to construct u on each $O = O_{l+1,p}$ and $O = \widehat{O}_{l+1,p}$ for each p. We discuss the following cases

- **$O = O_{l+1,0}$:** Assume that there is a unique k such that $O \cap V_{l,k} \neq \emptyset$ (if not we define u_O to be 0), so we define u_O by

$$u_O = \big(E_{V_{l,k}}(u_{V_{l,k}}) + E_{O \cap V_{l,k}}(\alpha_{O \cap V_{l,k}})\big)_{|O}.$$

- **$O = \widehat{O}_{l+1,p}$:** Assume that we've constructed $u_{O_{l+1,p}}$. We define u_O by

$$u_O := \Big(E_{O_{l+1,p}}(u_{O_{l+1,p}}) + E_{\widehat{O}_{l+1,p} \cap O_{l+1,p}}(\alpha_{\widehat{O}_{l+1,p} \cap O_{l+1,p}})\Big)_{|O}.$$

- **$O = O_{l+1,p+1}$):** we break it into two cases

- **Case(1):** For any k we have $V_{l+1,k} \cap O = \emptyset$. We define u_O by

$$u_O := \left(E_{\widehat{O}_{l+1,p}}(u_{\widehat{O}_{l+1,p}}) + E_{\widehat{O}_{l+1,p} \cap O_{l+1,p+1}}(\alpha_{\widehat{O}_{l+1,p} \cap O_{l+1,p+1}}) \right)_{|O}.$$

- **Case(2):** There exists k such that

$$V_{l+1,k} \cap O \neq \emptyset.$$

In this case $B(a_{l+1,k}, r) \in \mathcal{W}$ (because otherwise $a_{l+1,k}$ will be a punctured disk singularity for U), and we choose $u_{B(a_{l+1,k},r)}$ to take the values of α. Take $r' > r$ such that $B(a_{l+1,k}, r') \cap B(a_{l+1,k+1}, r') = \emptyset$ and (f, g) a partition of unity associated to the cover $(B(a_{l+1,k}, r'), \mathbb{R}^2 \setminus B(a_{l+1,k}, r))$. We take also $h, h' \in C^\infty(\mathbb{R}^2)$ such that

$$h_{|V_{l+1,k} \cap \mathbb{R}^2 \setminus B(a_{l+1,k},r))} = 0 \text{ and } h_{|\widehat{O}_{l+1,p} \cap \mathbb{R}^2 \setminus B(a_{l+1,k},r))} = 1$$
$$h'_{|V_{l+1,k} \cap \mathbb{R}^2 \setminus B(a_{l+1,k},r))} = 1 \text{ and } h'_{|\widehat{O}_{l+1,p} \cap \mathbb{R}^2 \setminus B(a_{l+1,k},r))} = 0.$$

So in this case we define u_O by

$$u_O := h\left(fE_{B(a_{l+1,k},r)}(u_{B(a_{l+1,k},r)}) + gE_{\widehat{O}_{l+1,p}}(u_{\widehat{O}_{l+1,p}}) \right)_{|O} +$$
$$h'\left(fE_{B(a_{l+1,k},r)}(u_{B(a_{l+1,k},r)}) + gE_{V_{l+1,k}}(u_{V_{l+1,k}}) \right)_{|O}.$$

And in this case for any O such that $a_{l+1,k} \in \overline{O}$ we need to modify the definition of u_O by (note here u'_O the old definition given in the previous stages of the induction)

$$u_O := \left(fE_{B(a_{l+1,k},r)}(u_{B(a_{l+1,k},r)}) + gE_O(u'_O) \right)_{|O}.$$

Finally by definition of u we have $d(u) = \alpha$.

□

3.6 $(W^{1,2}, W^{0,2})$-double extension is a sufficient condition for the sheafification of $W^{s,2}$.

In this section, for $U, V \in X_{\mathcal{A}}(\mathbb{R}^n)$ we give a categorical proof of Lemma 3.4.2, and we discuss the case where $U \cap V$ is not Lipschitz. The only assumption we require here is that U, V, and $U \cup V$ are Lipschitz. We use the fact that the sequences

$$0 \to W^{0,2}(U \cup V) \to W^{0,2}(U) \oplus W^{0,2}(V) \to W^{0,2}(U \cap V) \to 0$$

and

$$0 \to W^{1,2}(U \cup V) \to W^{1,2}(U) \oplus W^{1,2}(V) \to W^{1,2}(U \cap V) \to 0$$

are exact.

We assume that we have the following double extension:
Assumption: There exist a linear continuous operator

$$\mathcal{T} : W^{0,2}(U \cap V) \longrightarrow W^{0,2}(\mathbb{R}^n),$$

such that $\mathcal{T}$ induces a linear continous map from $W^{1,2}(U \cap V)$ to $W^{1,2}(\mathbb{R}^n)$.

Remark 3.6.1. Note that this assumption holds if $U \cap V$ is Lipschitz, due to the Stein extension Theorem.

Note that here $W^{0,2} = L^2$, and we need only sobolev spaces with regularity $s \in (0, 1)$. We will passe to our exact sequence for $s \in (0, 1)$ by a linear combination of the last two, this make us expect it to be exact. For that, we will use the notion of **exact category** (see [5]), it is a categories that is not abelian, but has a structure that allow us to do homological algebra.

Let $\mathcal{C}$ be an additive category. A pair of composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is said to be a **KC-pair** (Kernel-Cokernel pair) if f is the kernel of g and g is the Cokernel of f. Fix $\mathcal{E}$ a class of KC-pairs. An **admissible monomorphism** (with respect to $\mathcal{E}$) is a morphism f such that there is a morphism g with $(f, g) \in \mathcal{E}$. **Admissible epimorphisms** are defined dually.

Definition 3.6.2. An **exact structure** is a couple $(\mathcal{C}, \mathcal{E})$ where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a class of KC-pairs, closed under isomorphisms and satisfies the following proprieties:

(E_0) For any $X \in Obj(\mathcal{C})$, Id_X is an admissible monomorphism.

$(E_0)^c$ The dual statement of (E_0).

(E_1) The composition of admissible monomorphisms is an admissible monomorphism.

$(E_1)^c$ The dual statement of (E_1).

(E_2) If $f : X \to Y$ is an admissible monomorphism and $t : X \to T$ a morphism, then the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s_Y} \\ T & \xrightarrow[s_T]{} & S \end{array}$$

exists and s_T is an admissible monomorphism.

$(E_2)^c$ The dual statement of (E_2).

If $(\mathcal{C}, \mathcal{E})$ is an exact structure, a morphism $f : X \longrightarrow Y$ is said to be **$\mathcal{E}$-strict** if it can be decomposed into

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle e}\searrow \quad \circlearrowright \quad \nearrow{\scriptstyle m} & \\ & Z & \end{array}$$

where $e : X \longrightarrow Z$ is an admissible epimorphism (with respect to $\mathcal{E}$), and $m : Z \longrightarrow Y$ is an admissible monomorphism (with respect to $\mathcal{E}$).

Now fix $\mathcal{C}$ an additive category. It is well known (see [5]) that the following class of KC-pairs

$$\mathcal{E}_0 = \{(f, g) \ : \quad X \xrightarrow{f} Y \xrightarrow{g} Z \ \text{ split}\}$$

is an exact structure on $\mathcal{C}$ (it is the smallest one on $\mathcal{C}$).

Definition 3.6.3. Let $(\mathcal{C}, \mathcal{E})$ be an exact structure, $\mathcal{A}$ an **abelian** category, and $F : \mathcal{C} \longrightarrow \mathcal{A}$ an **additive functor**. F is said to be **injective** if for any pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$, the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in $\mathcal{A}$.

The following result is well known in the theory of exact categories:

Proposition 3.6.4. *F is injective if and only if it preserve the Kernel of every $\mathcal{E}$-strict morphism.*

Proof. See [5]. □

We will construct the category $\mathcal{C}$ to serve our case, and the category $\mathcal{A}$ will be just the category of $\mathbb{C}$-vector spaces. Let's recall the concept of Interpolation:

Definition 3.6.5. A good pair of Banach spaces (or **GB-pair**) is a pair (X, Y) of Banach spaces such that $X \subset Y$ with continuous inclusion, that is there is $C > 0$ such that for any $x \in X$ we have

$$\|x\|_Y \leqslant C\|x\|_X.$$

We recall the interpolation K-method. So fix (X, Y) a GB-pair and $t > 0$, and define the K-norm on Y by

$$u \mapsto K(t, u) = \inf\{\|x\|_X + t\|y\|_Y \ : \ u = x + y, \ x \in X, \ y \in Y\}.$$

For $s \in]0, 1[$, we define the interpolation space $[X, Y]_s$ by

$$[X, Y]_s = \{u \in Y \ : \ \int_0^{+\infty} \left(t^{-s} K(t, u)\right)^2 \frac{dt}{t} < +\infty\}.$$

it's a Banach space with the norm

$$\| u \|_{[X,Y]_s} = \left(\int_0^{+\infty} \left(t^{-s} K(t, u)\right)^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Recall the following theorem of interpolation spaces:

Theorem 3.6.6. *Let (X, Y) and (X', Y') be two GB-pairs and*

$$L : Y \longrightarrow Y'$$

a continuous linear map such that L induces a continuous linear map from X to X'. Then, for any $s \in]0, 1[$, L induced a linear continuous map from $[X, Y]_s$ to $[X', Y']_s$.

Proof. See [13]. □

Let $\mathcal{A}$ be the category of $\mathbb{C}-$vector spaces and $\mathcal{C}$ be the category where the object are GB-pairs, and for two $((X, Y), (X', Y')) \in (Obj(\mathcal{C}))^2$ we define the morphisms:

$$Hom_{\mathcal{C}}((X, Y), (X', Y')) = \{L \in \mathcal{L}(Y, Y') \ : \ L\mid_X \in \mathcal{L}(X, X')\ \}.$$

Clearly, $\mathcal{C}$ is an additive category. We consider the exact structure $\mathcal{E}_0$ on $\mathcal{C}$ of splitting KC-pairs. For any $s \in]0, 1[$ we define the functor $F_s : \mathcal{C} \longrightarrow \mathcal{A}$ by

$$F_s((X, Y)) = [X, Y]_s \text{ and for } f \in Hom_{\mathcal{C}}((X, Y), (X', Y'))$$
$$F_s(f) = f\mid_{[X,Y]_s}.$$

By Theorem 3.6.6 , F_s is well defined additive functor.

Lemma 3.6.7. *For $(X, Y), (X', Y') \in Ob(\mathcal{C})$ and for $s \in [0, 1]$, there is a natural isomorphism :*

$$[X \oplus X', Y \oplus Y']_s \simeq [X, Y]_s \oplus [X', Y']_s.$$

Proof. Take the projections

$$P : Y \oplus Y' \longrightarrow Y, \text{ and}$$

$$P' : Y \oplus Y' \longrightarrow Y'.$$

Since $P\mid_{X\oplus X'} \in \mathcal{L}(X \oplus X', X)$ and $P'\mid_{X\oplus X'} \in \mathcal{L}(X \oplus X', X')$, by Theorem 3.6.6 this induces a continuous linear map

$$(P, P') : [X \oplus X', Y \oplus Y']_s \longrightarrow [X, Y]_s \oplus [X', Y']_s,$$
$$(u) \mapsto (P(u), P'(u)).$$

The same way applying Theorem 3.6.6 on the injections

$$I : Y \longrightarrow Y \oplus Y' \text{ and } I' : Y' \longrightarrow Y \oplus Y',$$

We get a continuous linear map

$$(I, I') : [X, Y]_s \oplus [X', Y']_s \longrightarrow [X \oplus X', Y \oplus Y']_s,$$
$$(z, z') \mapsto z \oplus z'.$$

It's clear that $(I, I') \circ (P, P') = Id$ and $(P, P') \circ (I, I') = Id$. □

Lemma 3.6.8. *The functor $F_s : \mathcal{C} \longrightarrow \mathcal{A}$ is injective with respect to the exact structure $(\mathcal{C}, \mathcal{E}_0)$.*

Proof. By Proposition 3.6.4, it's enough to prove that F_s preserve the Kernel of every $\mathcal{E}_0$-strict morphism. Take $f : (X,Y) \longrightarrow (X',Y')$ a $\mathcal{E}_0$-strict morphism. Then there exists an admissible epimorphism $e : (X,Y) \longrightarrow (Z,W)$ and an admissible monomorphism $m : (Z,W) \longrightarrow (X',Y')$ such that we have a decomposition

$$\begin{array}{ccc} (X,Y) & \xrightarrow{\quad f \quad} & (X',Y') \\ & {\scriptstyle e}\searrow \quad \circlearrowleft \quad \nearrow{\scriptstyle m} & \\ & (Z,W) & \end{array}$$

By **Remark 3.28** in [5], if $k_f : K_f \longrightarrow (X,Y)$ is the Kernel of f, then $(k_f, e) \in \mathcal{E}_0$. Easy computation show that the kernel of f is the morphism

$$k_f : K_f = (X \cap Ker(f), Ker(f)) \longrightarrow (X,Y).$$
$$u \longrightarrow k_f(u) = u.$$

Here $Ker(f)$ is given the norm of Y, and $X \cap Ker(f)$ is given the norm

$$\|u\|_{X \cap Ker(f)} = \max\{\|u\|_X, \|u\|_{Ker(f)}\}.$$

By Lemma3.8 in [5], there exist a morphism $P : (X,Y) \longrightarrow K_f$ such that $P \circ k_f = Id_{K_f}$, and this means that $(X \cap Ker(f), Ker(f))$ is a complemented sub-couple of (X,Y), hence by Theorem.1 in section 1.17.1 of [26], we have

$$[X \cap Ker(f), Ker(f)]_s = Ker(f) \cap [X,Y]_s = Ker(F_s(f)).$$

□

Now we have the KC-pair in the category $\mathcal{C}$

$$(W^{1,2}(U \cup V), L^2(U \cup V)) \to (W^{1,2}(U) \oplus W^{1,2}(V), L^2(U) \oplus L^2(V)) \to$$
$$(W^{1,2}(U \cap V), L^2(U \cap V)).$$

And by the assumption of the existence of $(W^{1,2}, W^{0,2})$-double extension, this sequence split, so it is in the structure $\mathcal{E}_0$. Hence by Lemma 3.6.8 if we apply the functor F_{1-s} (for any $s \in]0,1[$) we get an exact sequence. Therefore, by (3.1.4) we get the exact sequence

$$0 \to W^{s,2}(U \cup V) \to [W^{1,2}(U) \oplus W^{s,2}(V), L^2(U) \oplus L^2(V)]_{1-s} \to [W^{s,2}(U \cap V), L^2(U \cap V)]_{1-s}.$$

By Lemma 3.6.7 and (3.1.4) we can write it the following way

$$0 \to W^{s,2}(U \cup V) \to W^{s,2}(U) \oplus W^{s,2}(V) \to [W^{1,2}(U \cap V), L^2(U \cap V)]_{1-s}.$$

Hence we have the exactness of the sequence

$$0 \to W^{s,2}(U \cup V) \to W^{s,2}(U) \oplus W^{s,2}(V) \to W^{s,2}(U \cap V) \to 0$$

□

Remark 3.6.9. So the answer to the exactness of the sequence

$$0 \to W^{s,2}(U \cup V) \to W^{s,2}(U) \oplus W^{s,2}(V) \to W^{s,2}(U \cap V) \to 0$$

is important, since a positive answer implies the possibility of sheafifying Sobolev spaces in the usual sense, and a negative answer will implies that there exist non degree-independent extension operator from $W^{i,2}(\Omega)$ to $W^{i,2}(\mathbb{R}^n)$ (for $i \in \{[s],[s]+1\}$) when Ω is a cuspidal domain.

Remark 3.6.10. It may be helpful to create a bigger exact structure $\mathcal{E}$ on the category $\mathcal{C}$ of GB-pairs so that the KC-pair

$$(W^{1,2}(U \cup V), L^2(U \cup V)) \mapsto (W^{1,2}(U) \oplus W^{1,2}(V), L^2(U) \oplus L^2(V)) \mapsto (W^{1,2}(U \cap V), L^2(U \cap V)) \text{.............} (\star)$$

is in $\mathcal{E}$. For example we can show that the maximal class of all the KC-pair is exact on $\mathcal{C}$ (this is not true in general (see [5])), but the problem is that making the class $\mathcal{E}$ bigger makes also the class of $\mathcal{E}$-strict morphisms bigger, for example in the case of taking $\mathcal{E}$ the maximal class, a morphism $f : (X,Y) \longrightarrow (X',Y')$ is $\mathcal{E}$-strict if and only if $f(Y)$ is closed in Y', $f(X)$ is closed X', f is open into $f(Y)$, and $f\mid_X$ is open into X'. But to the moment there is no result where it's proven that the interpolation is compatible with the Kernel of this kind of morphisms. There's in [14] and [7] some sufficient conditions on the morphism to have a Kernel which is compatible with the interpolation, but It's not clear how it can be connected with our situation. So maybe one could create an exact structure on the category of GB-pairs $\mathcal{C}$ that contains the KC-pair $(\star)$ and such that the class of strict morphisms satisfies the conditions given in [14] and [7] .

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