يوميات المخيم السادس للرياضيات

الوادي 29 – 24 مارس 2026



مقدمة

جمال ضو

ربما لا يوجد شيء تم ابتذاله في بلدان العالم الثالث مثلما تم ابتذال الملتقيات والندوات العلمية، فلقد باتت أغلب هذه اللقاءات العلمية إلى فرص يلتقي فيها المشاركون ليناقشوا خلالها كل شيء تقريبا إلا العلم الحقيقي، وإذا كانت الساحة لا تخلو من الاستثناءات، فإنها الاستثناءات هي التي تثبت القاعدة ولا تنفيها. وربما "مخيم الرياضيات" هو من بين هذه الاستثناءات في الجزائر، أو هكذا يريد القائمين عليه له أن يكون.

ولدت فكرة مخيم الرياضيات سنة 2016 من رحم لقاء صيفي بين مجموعة صغيرة من خيرة الطلبة الجزائريين في الرياضيات ببعض الأساتذة، في غرفة بسيطة في عنابة، ليشهد بعدها بشهور نقلة نوعية بتنظيم مخيم ثان خلال العطلة الربيعية سنة 2017 بمدينة الوادي، ليصبح بعدها الوادي المكان الدوري الذي ينظم فيه المخيم ربيع كل سنة بينما تتنقل النسخة الصيفية بين ولايات الشمال. وأصبح قبلة لخيرة طلبة الرياضيات يشدو إليها الرحال.

وهاهو المخيم يعود إلى مدينة الوادي بعد انقطاع دام لسنوات بسبب جائحة وكورنا. وكما جرت العادة فإن هذا اللقاء يمتد لأسبوع ، يجمع ما يربو من ثلاثين طالبا، أو يزيدون ، من خيرة طلبة الجامعات الجزائرية في الرياضيات، وحتى من بعض التخصصات العلمية الأخرى، ومن كل ربوع الوطن، فهو إلى حد كبير ملتقى النخبة. تقدم خلال المخيم محاضرات نوعية في الرياضيات وشيء من الفيزياء النظرية، وتعرض فيه كثير من المسائل الرياضية التي يناقشها الطلبة والأساتذة طلية اليوم. وإن كان برنامج المحاضرات يتوقف عند الرابعة أو الخامسة مساء، فإن النقاشات تستمر إلى موعد تناول العشاء وحتى بعده إلى العاشرة ليلا.

ومن التقاليد التي تم إرساؤها في مخيمات الرياضيات هو أن يتكفل الطلبة بإعداد تقارير كتابية عن كل محاضرات، هذه التقارير تجمع لاحقا بعد كتابتها وتنقيحها لتصدر في ملف يجمع هذه التقارير.

وهنا تجدر الإشارة إلى أن تقاليد التدوين تكاد تكون من التقاليد التي فقدتها كثير من المجتمعات، ومنها مجتمعنا، لتحل بدلها السفسطة الشفاهية. فإن كانت السفسطة متاحة للجميع من دون معايير، إلا أن التدوين العلمي بشكل خاص، يملي على صاحبه الدقة والصرامة والأمانة في

النقل، وهذا ما نحن بأمس الحاجة إليه اليوم ، وما يجب أن تتشربه الأجيال الصاعدة، علها تستعيد موقعا ومكانة فقدتها بلداننا منذ قرون.

وفي هذا الإطار يأتي هذا الملف الجامع لتقارير المخيم السابع، والذي يمثل عصارة مختصرة لأهم ما تم تقديمه من محاضرات طلية الأيام الست لهذا المخيم. فهذه التقارير تعطي من لم يحضر ومن حضر ملمحا عاما وصورة شاملة عما مجريات المخيم.

ومثلما سيكتشف القاريء من خلال هذه السلسة من التقارير، أن المواضيع التي تم التطرق إليها تنوعت بتنوع فروع الرياضيات، وخاصة الرياضيات البحتة، لتشمل:

نظرية الأعداد، نظرية الرسومات ((Graph theory)
هندسة السطوح، ونظرية الفئات (Category theory))، وفي الرياضيات التطبيقية نجد محاضرات في معادلة الحرارة والمعادلات التفاضلية الجزئية.

أما في الفيزياء النظرية، فلقد كانت هناك سلسلة من المحاضرات في ميكانيكا الكم، كان الهدف الرئيس منها إعطاء ملمحا عام عن أهم مسلمات ميكانيكا الكم والبنية الرياضياتية التي يقوم عليها هذا العلم، مع طرح بعض الإشكالية المحيطة بتفسير ميكانيكا الكم ومفارقة أينشتاين بودلسكي روزن.

إن هذه التقارير ليست بطبيعتها دروسا مفصلة أو مرجعا علميا، ولكنها تفتح أعين الطالب على فروع متعددة من الرياضيات ، وخاصة البحتة منها، ليرى تلاقي فروعها المختلفة وتطبيقاتها.

تقرير صبيحة اليوم الأول

بن حضرية إلياس جواد

8:00--9:20 Mini-course

The first class of the MathCamp6 event was presented by Mr. Billel Guelmane. Below are the notes taken during this class.

Mini-course title: Heat Equation

Retranscribing the course is unnecessary, as it is identical to the PDF file provided beforehand.

Some remarks were made during the class, such as the fact that t and x belong to $\mathbb{R}^+$ and $\mathbb{R}$, respectively.

At the end of the class, two questions were asked. The first concerned the Poisson equation and led to the correction of a typographical mistake made on the board. The second concerned the local existence in time of the PDEs under consideration.

9:20--10:00 Workshops

After the class ended, all the MathCamp members---students, teachers, and organizers---gathered outside the conference room, where mathematics-related workshops were presented at different tables.

While the students remained outside taking part in the workshops, most of the teachers, organizers, and administrative staff gathered in a small room, where they briefly discussed the MathCamp event.

10:00--11:00 Opening Ceremony

At 10:00 a.m., the conference room designated for the opening ceremony was opened. After everyone had taken their seats, the ceremony began with a speech by Inspector Selatna. This was followed by a brief pause during which all attendees stood for the national anthem.

Afterwards, the representative of the Wali of El Oued delivered a speech. This was followed by Professor Abdelghani Zeghib, who stated:

"Unfortunately, the Wilaya of El Oued used to be the leading wilaya in mathematics, but this is no longer the case, as its level in the discipline has declined significantly over recent years. However, the Wilaya of Batna could become the new leading wilaya in mathematics, as it is characterized by a strong concentration of talented students and teachers in the field."

Inspector Selatna then briefly took the floor again before giving it to Mr. Mesoud, who spoke about the importance of the Ministry of National Education's involvement in such events by providing the necessary resources for their organization.

At the end of the ceremony, all attendees gathered on the platform for a group photograph.

11:17--12:00 Seminar

A seminar was presented by Professor Abdelghani Zeghib. Various topics were discussed; the main points are summarized below:

- Professor Abdelghani Zeghib jokingly suggested renaming MathCamp to ``Ahmed Blidia."
- The University of Nice mobility grant was discussed, along with the Algerian students who had the opportunity to benefit from it.
- The seminar mainly focused on algebraic geometry. A parallel was drawn between algebraic geometry and cryptography. It was pointed out that it is relatively straightforward for a researcher specializing in algebraic geometry to move into cryptography, as the former provides many of the mathematical foundations required by the latter. The reverse transition, however, may prove more difficult.
- The discrete logarithm problem was briefly mentioned, as were tori, which can be defined as the quotient groups $R^n/\mathbb{Z}^n$.
- Surface theory was also discussed, together with its applications in computation and 3D modeling (for example, CAD software). The Kakeya conjecture was mentioned, having previously been discussed in the Bourbaki Seminar and in the ENS-Kouba magazine.
- Within the context of surface theory, developable surfaces, the Riemann curvature tensor, the Ricci tensor, and the scalar curvature were also discussed. Professor Abdelghani Zeghib stated that, while these notions are equivalent in dimension two, they become genuinely distinct starting from dimension three.

تقرير عشية اليوم الأول

مزيوة أبو بكر

After lunch and a short rest, we returned to the conference hall for an engaging session with ***Lamine***.

In the first part of the talk, an overview was given of the interactions between different areas of mathematics, namely geometry, group theory, and probability. These connections were illustrated through the dichotomy between Euclidean and hyperbolic geometry. From the perspective of group theory, this dichotomy was reflected in the fact that the group of transformations preserving Euclidean geometry is much ``smaller'' than the one preserving hyperbolic geometry. From the probabilistic point of view, it appeared in the behavior of random walks: in the Euclidean setting, a random walk may return to its starting point, whereas in the hyperbolic setting, it almost surely escapes toward the boundary of the space.

In the second part of the talk, we considered the example of a random walk on $\mathbb{Z}$ and proved its recurrence by exploiting the connection between random walks and harmonic functions.

After a short break, we resumed with the final session of the day, led by Dr. Samir. This was the first exercisecise session, during which we began working on the first problem sheet.

The goal of this problem sheet was to establish several inequalities and properties of the function

$$\pi(x) = |\{\, p \text{ prime} : p \leq x \,\}|\,. \tag{1}$$

First of all, Dr. Samir gave us an overview of the material to be covered, explaining that these results would constitute part of the proof of the *Prime Number Theorem*.

He then introduced several functions that play an essential role in the proof.

Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{2}$$

Chebyshev functions:

$$\vartheta(x) = \sum_{p \leq x} \ln p, \qquad x \in \mathbb{R}, \tag{3}$$

and

$$\phi(s) = \sum_{p \text{ prime}} \frac{\ln p}{p^s}, \qquad \mathrm{Re}(s) > 1. \tag{4}$$

We then began proving some of their fundamental properties.

1. Show that $\zeta(s)$ and $\phi(s)$ converge normally for $\mathrm{Re}(s) > 1$.
2. If $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, then

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}} \qquad \text{(Euler's product formula).}$$

Hint: Show that

$$\zeta(s) \prod_{p} \left(1 - \frac{1}{p^s}\right) = 1.$$

To do so, consider

$$\zeta(s) \prod_{j=1}^{m} \left(1 - \frac{1}{p_j^s}\right),$$

and estimate

$$\left| \zeta(s) \prod_{j=1}^{m} \left(1 - \frac{1}{p_j^s}\right) - 1 \right|, \qquad m \geq N.$$

The next result establishes the order of growth of the Chebyshev function $\vartheta(x)$. More precisely, we proved that

$$\vartheta(x) = O(x), \qquad x \to \infty.$$

To this end, we considered the following exercisecises.

a- Show that the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

is divisible by every prime number in the interval $[n+1, 2n]$.

b- Using the binomial coefficient and part (a), prove that

$$2^{2n} \geq e^{\vartheta(2n) - \vartheta(n)}.$$

c- Let $x \geq 2$. Show that there exists a unique integer k such that

$$2^k \leq x < 2^{k+1}.$$

d- Deduce that

$$\vartheta(x) \leq 4x, \qquad x \geq 2.$$

This concluded another excellent session. Before returning to the youth hostel for dinner, we enjoyed some refreshments, including tea, juice, peanuts, and sweets, kindly provided by the volunteers.

تقرير صبيحة اليوم الثاني

بصديق نور الإسلام

Quantum Mechanics

Lesson Objective: To provide general concepts of quantum mechanics as a practical application of abstract mathematical structures.

Topics Covered

- Introduction to Quantum Mechanics
- Heisenberg's Uncertainty Principle
- The EPR Paradox
- Bell's Inequality
- Fundamental Principles of Classical Physics

Session Details

On Wednesday, March 25, 2026, at 8:00 AM, Professor Dr. Jamal Daou began his lecture with a general introduction to quantum mechanics, covering the following:

- **Tools:** Linear algebra and Hilbert spaces are used as fundamental tools.
- **Learning Method:** Quantum mechanics can best be understood by solving exercisecises together with studying lectures and definitions; it is unlike most other subjects. In this regard, Richard Feynman famously said: "*No one understands quantum mechanics*."
- **Black-body Experiment:** An iron rod was heated, and its color was observed to change, illustrating the phenomenon of black-body radiation.
- **Planck's Discovery:** Max Planck discovered the quantization of energy. During his study of black-body radiation, he found that matter does not absorb or emit energy continuously, but in discrete packets (quanta). He formulated the famous relation:

$$E = \bar{h}\,\nu$$

Heisenberg's Uncertainty Principle (1927)

It states that it is impossible to measure a pair of physical quantities, such as position and momentum, simultaneously with arbitrary precision:

$$\Delta x \, \Delta p \gtrsim \bar{h}$$

This revealed the limitations of Newton's second law, which assumes a well-defined trajectory. In quantum mechanics, particles are described by a wave function (Ψ) rather than by a single deterministic path.

The EPR Paradox (1935)

Named after Einstein, Podolsky, and Rosen, its purpose was not to prove quantum mechanics wrong, but rather to argue that it is an "incomplete" theory requiring "hidden variables" to provide a complete description of reality. This question remained open until John Bell proposed his famous inequality.

Bell's Inequality

If hidden variables exist (as Einstein suggested), the degree of correlation between measurement outcomes at different angles cannot exceed a certain limit.

Principle of Least Action

While Newton's laws describe how an object moves, the principle of least action explains why a system follows a particular path among all possible ones. The physical trajectory is the one that makes the action, defined as the time integral of the Lagrangian $L = T - V$, stationary:

$$S = \int_{t_i}^{t_f} L \, dt.$$

This leads to the Euler--Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

These equations are equivalent to Newton's second law, but they are formulated in terms of energy rather than forces, making many problems easier to solve.

Conclusion & Q&A

The session concluded with a discussion on the impossibility of determining a particle's exact position and momentum simultaneously, followed by a brief overview of the properties of Hilbert spaces.

- **Q:** In a stable system, is the uncertainty in measuring position and velocity relatively small?
- **A:** At atomic scales, the uncertainty associated with such measurements is inherently significant.

تقرير صبيحة اليوم الثالث

محمد الشريف

First thing in the morning, we had a quantum mechanics mini course given by Dr. Djamel Dho. I and Boumedienne were assigned to write the reviews. As I remember, it was a bit warm that day, at least hotter than the other days. After the initiation for quantum mechanics we had a day before, we moved straight ahead to the core concept of the course, which is the mathematical foundation of QM. We enjoyed a high quality lecture about the first postulates of QM.

I remember too that a fourth year applied math student at NHSM asked Dr. Dho about the Hamel basis and why Dr. Dho made a statement about the non-necessity of limiting Hilbert spaces used in QM formalism to finite dimensions (unless it was about quantum information theory, and this is a note beyond the scope of that day's discussion), and Pr. Zeghib did interfere in that discussion too, clarifying some concepts to the students, especially about the choice of $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$ spaces.

Not many questions have been asked during that session, mostly because nearly all the answers would be given during the next session.

After the lecture ended, we had a ten-minute pause outside, and I personally seized the opportunity to ask Dr. Dho some questions about quantum computing (i.e., book recommendations) as well as a small talk about the suitable logical system to describe quantum information theory, which is a very recent research topic, and also because we had a category theory course just after then. And in fact, that discussion was very informative and helpful for both of us, since it did indeed raise some truly core fundamentals of quantum information and Von Neumann's work.

تقرير عشية اليوم الثالث

محمد عميروش عادل

C'est sous une chaleur ardente que s'est déroulée la session de l'après-midi. Après un talk passionnant de Souheib sur les systèmes dynamiques, que j'ai malheureusement partiellement raté à cause d'une mésaventure au centre-ville d'El Oued, Alaa a demandé un volontaire pour faire le rapport de la séance de Lamine, et j'ai levé la main.

À 15 h 13, Lamine commence son cours sur *"the geometric point of view of groups"*.

$$X = \{a, b, c, \dots\}, \quad M(X) = \{\text{words on } X\} = \{ab, cde, billel, \dots\}$$

$(M(X), *)$ avec $ab * cd = abcd$ (concaténation). Soit id le mot vide :

$$ab * id = id * ab = ab$$

On ajoute $X \cup \{a^{-1}, b^{-1}, \dots\}$ avec $a * a^{-1} \sim id$ et on considère la relation d'équivalence $\sim$. Alors

$$(M(X)/\sim, *) \text{ est un groupe.}$$

Example :

$$X = \{a, b\}, \quad F(X) = \{ab, a^2, ab^{-1}a^2, \dots\}$$

Remark : for any group G, there exists a set X and a surjective homomorphism

$$(F(X), +) \to (G, \cdot)$$

Definition : G is finitely generated if $\exists g_1, \dots, g_n \in G$ such that

$$G = \langle g_1, \dots, g_n \rangle$$

Group presentation :

$$G = \langle X \mid R \rangle = F(X)/\langle\langle R \rangle\rangle$$

où $\langle\langle R \rangle\rangle$ est le plus petit sous-groupe normal contenant R.

Examples :

$$F = \langle a, b \rangle, \quad \mathbb{Z} = \langle a \rangle, \quad \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$$
$$\mathbb{Z}^2 = \langle a, b \mid [a, b] = aba^{-1}b^{-1} \rangle, \quad BS(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$$

Cette construction permet d'étudier l'existence d'un morphisme surjectif propre $G \to G$ non bijectif.

Monsieur Zeghib a posé la question : quel est le groupe de présentation

$$\langle a, b \mid a^n b^m \rangle$$

Nous avons réfléchi quelques minutes sans trouver de réponse.

Puis j'ai demandé pourquoi introduire les groupes *finitely generated* s'ils n'étaient pas utilisés immédiatement, et Lamine a répondu que, désormais, nous ne considérerions que ceux-ci.

Another example :

Heisenberg group :

$$H_3 = \langle x, y, z \in \mathrm{SL}_n(\mathbb{Z}) \mid [x, y] = z,\ [x, z] = [y, z] = 1 \rangle$$

Puis M. Zeghib a donné trois matrices 3×3 appartenant au groupe d'Heisenberg et a fait une analogie avec le cours de M. Djamel Daou sur la physique quantique, une analogie que je n'ai pas comprise et que je ne pourrais donc pas expliquer.

Cayley Graph :

Let $G = \langle S \mid R \rangle$ be finitely generated. Then

$$\mathrm{Cay}(G, S) = (V, E), \quad V = G,\ (g_1, g_2) \in E \iff g_2 = g_1 s$$

Exemples : $F = \langle a, b \rangle$, $\mathbb{Z}^2$

Pour conclure, Amira a partagé un *fun fact* sur les mots homophones en français, concernant des mots ayant la même prononciation, mais je n'ai pas très bien compris cette remarque.

La séance s'est terminée sur cette note.

تقرير عشية اليوم الثالث

تاربينت عبد الرحمان و سلامة براء محفوظ

After the lunch meal at 14:34, Mr.Alaa announced to begin the mini course of Mr.Souheib (replacing the sessions of homeworks for some reasons) at 3pm ,the chairman Mr.Samir introduced the presentation of Mr.Souheib about "Dynamical Systems",

The field of Dynamical Systems is a core branch of applied mathematics and physics. At its heart, it is the study of how a system evolves over time according to a set of fixed mathematical rules.

0.1 courses

- Examples on S^1 and $\mathbb{T}^2$.
- Symbolic dynamics.
- Horseshoe.

0.2 Basic Definitions

Let X be a topological space (compact metric space) and $f : X \to X$ be a continuous map. (Most of the time f is a homeomorphism).

Definition 0.2.1 (Orbit)**.** For $p \in X$, the orbit of p is defined as:

$$O_f(p) = \{f^n(p), n \in \mathbb{N}\}$$

If f is a homeomorphism, then:

$$O_f(p) = \{f^n(p), n \in \mathbb{Z}\}$$

Definition 0.2.2 (Periodic Point)**.** We say that p is a **periodic point** if $O_f(p)$ is finite.

Definition 0.2.3 (Topologically Transitive)**.** If $\exists\, p \in X$ s.t. $\overline{O_f(p)} = X$, we say that f is **topologically transitive**.

Definition 0.2.4 (Minimal)**.** We say that f is **minimal** if all orbits are dense.

Definition 0.2.5 (Topologically Mixing)**.** We say that f is **topologically mixing** if $\forall\, U, V \subset X$ non-empty open sets, $\exists\, N \in \mathbb{N}$ s.t.

$$f^k(U) \cap V \neq \emptyset, \quad \forall k \geq N$$

Example 0.2.1. Consider the map $R_\alpha : S^1 \to S^1$ defined by $z \mapsto e^{2\pi i\alpha}z$, where

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Exercise 0.2.1. Show that R_α is periodic iff $\alpha \in \mathbb{Q}$.
If $\alpha \in [0,1] \setminus \mathbb{Q}$, then R_α is minimal.

0.3 Propositions and Proofs

Proposition 0.3.1. Let $f : X \to X$ be a homeomorphism. Then f is topologically transitive iff $\forall\, U, V \subset X$ non-empty open sets, $\exists\, k \in \mathbb{Z}$ s.t. $f^k(U) \cap V \neq \emptyset$.

Corollary 0.3.1. Topological mixing $\implies$ Topological transitivity.

Exercise 0.3.1. Let $E_k : S^1 \to S^1$ be defined by $z \mapsto z^k$ with $|k| \neq 1$.

- Show that E_k is topologically mixing.
- Show that the set of all periodic points of E_k is countable and dense.

Remark 0.3.1. If f is topologically transitive $\implies$ "most" orbits are dense.
Here, "most" means the set of all points with dense orbits is a G_δ set.

Example 0.3.1. $\mathbb{Q}$ is not a G_δ set.

0.4 Conjugacy

In the category of dynamical systems, what are the morphisms?
Let (X, f) and (Y, g) be two systems. A continuous map $h : X \to Y$ is a morphism if:

$$h \circ f = g \circ h$$

If h is a homeomorphism, then (X, f) and (Y, g) are said to be **conjugate**.

Exercise 0.4.1. Let $f, g : [0,1] \to [0,1]$ be two increasing homeomorphisms without fixed points except $\{0,1\}$. Show that f is conjugate to g.

Exercise 0.4.2. Let $\alpha, \beta \in [0,1]$
$R_\alpha \overset{\text{conj}}{\sim} R_\beta$ iff $\alpha = \beta$ or $\alpha = 1 - \beta$

0.5 The Main Example: Linear Automorphisms

Consider the group $SL(2,\mathbb{Z})$:

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

This group acts on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. If $A \in SL(2,\mathbb{Z})$ and $u + \mathbb{Z}^2 \in \mathbb{T}^2$, then:

$$A(u + \mathbb{Z}^2) = A(u) + \mathbb{Z}^2$$

In what follows, we study:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The map $A : \mathbb{T}^2 \to \mathbb{T}^2$ is one of the best examples of **chaos**.

Periodic Orbits of A

$[u] \in \mathbb{T}^2$ is periodic if $\exists\, k$ such that:

$$A^k([u]) - [u] = [0] \iff (A^k - I)u \in \mathbb{Z}^2$$

If $\Gamma_k = \{[u] : A^k([u]) = [u]\}$, then Γ_k is finite and $\Gamma = \bigcup_{k\in\mathbb{N}} \Gamma_k$ is dense in $\mathbb{T}^2$.

To show that Γ is dense, it's enough to show that if:

$$\Lambda_k = (A^k - I)^{-1}(\mathbb{Z}^2)$$

then $\bigcup \Lambda_k$ is dense in $\mathbb{R}^2$.

Exercise 0.5.1.

- Show that A is topologically mixing.
- A is diagonalizable with eigenvalues $0 < \frac{1}{\lambda} < 1 < \lambda$.

0.6 Eigenlines and Dynamics

Let:

- l^s be the line associated to $1/\lambda$ (Stable line).
- l^u be the line associated to λ (Unstable line).

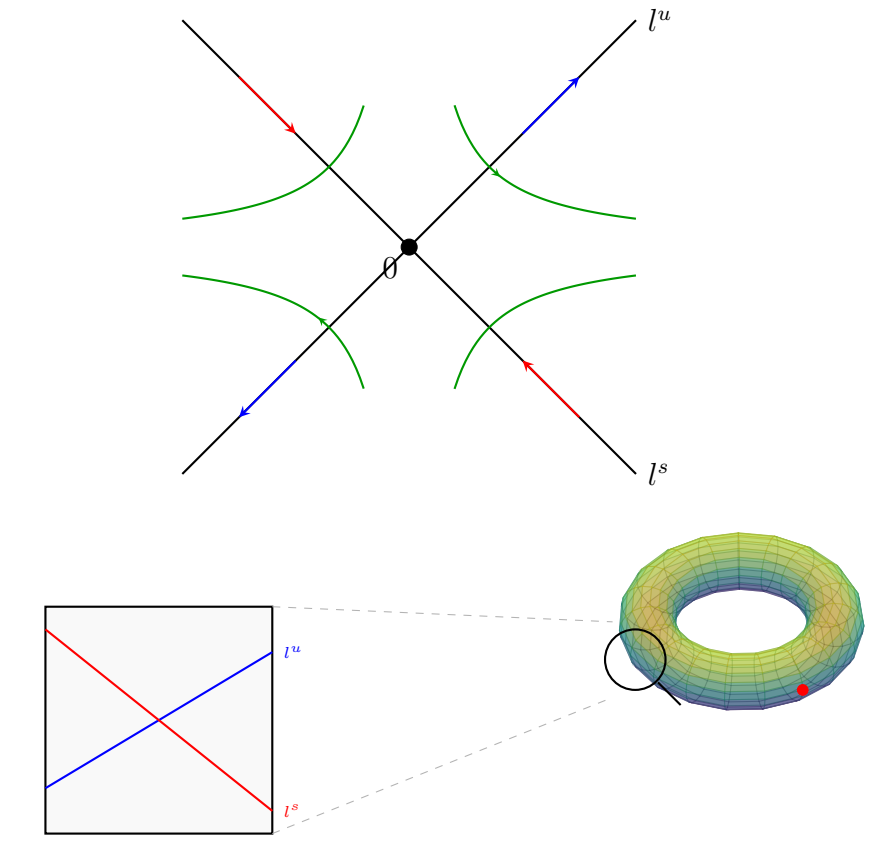


Figure 1: The square as a local chart (zoom) of the torus manifold.

Exercise 0.6.1. Show that the slopes of l^s and l^u are irrational.
Consequently, both l^s and l^u are dense in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

0.7 Homoclinic Points

Definition 0.7.1 (Homoclinic Point)**.** $x \in \mathbb{T}^2$ is a **homoclinic point** iff both positive and negative iterates of x converge to a fixed point.

Proposition 0.7.1. The set of Homoclinic points is countable and dense in $\mathbb{T}^2$.

0.8 Question for Further Study

Q: What can we do if we have higher dimensions?

the chairman Mr.Samir declared the end of the first session (after 90min).

A warm round of applause from the audience.

0.1 Second session

After around 10min the chairman Mr.Samir introduced the seminar of Mr.Abdennacir and Mr.Sami the presentation was made by Mr.Abdennacir about Intrinsic Geometry of Surfaces and the Gauss-Bonnet Theorem .

0.2 Regular Surfaces

Definition (Regular Surface): A surface $S \subset \mathbb{R}^3$ is called a *regular surface* if at each point $p \in S$, S can be parametrized by a smooth map $\varphi : U \subset \mathbb{R}^2 \to V \cap S$ such that:

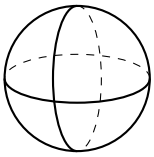
1. φ is a homeomorphism.
2. $D\varphi_p$ has a rank = 2.

Examples of Regular Surfaces



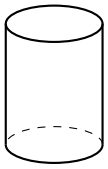
1. The Plane

$\varphi(u,v) = u\vec{w}_1 + v\vec{w}_2 + q$



2. The Sphere

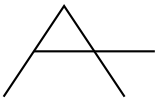
$x^2 + y^2 + z^2 = R^2$



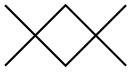
3. The Cylinder

$x^2 + y^2 = 1$

Non-Regular Surfaces (Counter-examples)



1. Folded Sheet

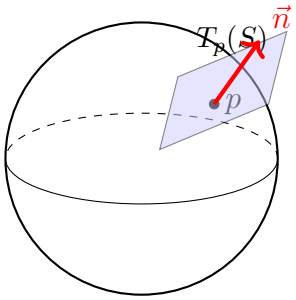


2. Self-intersecting

0.3 Tangent Space and Normal Vector

Definition (Tangent Space): The tangent space at a point $p \in S$, denoted by $T_p(S)$, is the subspace generated by the partial derivatives:

$$T_p(S) = \left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right\rangle$$



0.4 First Fundamental Form

Definition: Let $S \subset \mathbb{R}^3$ be a regular surface and $\varphi : U \to V \cap S$ a parametrization. The vectors:

$$w_1 = \frac{\partial \varphi}{\partial u}, \quad w_2 = \frac{\partial \varphi}{\partial v}$$

form a basis of the tangent space $T_p(S)$.

The **First Fundamental Form** at p is represented by the matrix:

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

where:

$$E = \langle w_1, w_1 \rangle, \quad F = \langle w_1, w_2 \rangle, \quad G = \langle w_2, w_2 \rangle$$

Examples

1. **The Plane:** $\varphi(u, v) = u\vec{w}_1 + v\vec{w}_2 + q$ with $\vec{w}_1$ and $\vec{w}_2$ orthogonal

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. **The Cylinder:** $\varphi(u, v) = (\cos u, \sin u, v)$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. **Sphere:** $\varphi(u, v) = (\cos u \cos v, \cos u \sin v, \sin v)$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 v \end{pmatrix}$$

(*Note*: *Circles shrink to a point at the north pole as* $\cos v \to 0$)

Length and Distance

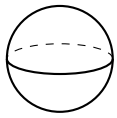
Length of a curve γ**:** $L(\gamma) = \int_I \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt$

Distance on surface: $d_s(p, q) = \inf\{L(\gamma) \mid \gamma(0) = p, \gamma(1) = q\}$

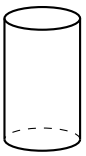
Next: *Local Isometries and the Gauss-Bonnet Theorem*

Gaussian Curvature K

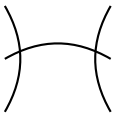
Intuition: $K = \kappa_1 \cdot \kappa_2$ (Product of principal curvatures).



$K = 1/r^2 > 0$



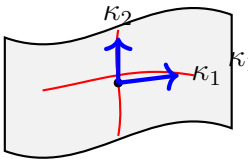
$K = 0$



$K < 0$

Theorema Egregium: Gaussian curvature depends **only** on the F.F.F. (Intrinsic property).

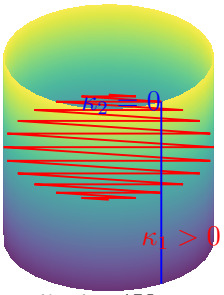
Intuition of Gaussian Curvature



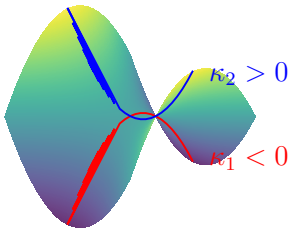
κ_1, κ_2: curvatures

The Gaussian curvature at a point p is given by:

$$K = \kappa_1 \cdot \kappa_2$$



linder (K



Saddle ($K < 0$)

Theorema Egregium

Gauss's "Theorema Egregium" states that K depends **only** on the First Fundamental Form.

> This theorem is foundational to understanding why projections cannot preserve all measurements simultaneously (e.g., area and angles). A classic example demonstrating this is the projection of the Earth onto a flat piece of paper. Since a sphere has non-zero constant Gaussian curvature ($\frac{1}{r^2}$) and a flat plane has zero curvature, no map can be created that perfectly represents the spherical Earth without distorting distance, area, or shape.

Question: How does one choose the directions of κ_1 and κ_2?

Important Information

The discussion on Gaussian curvature assumes several conditions:

- The surface should be *orientable*.
- Implicit surfaces $f(x, y, z) = 0$ require specific formulas.

Comment from Mr. Alaa about $K_{sphere} = 1/r^2$:

- r big $\rightarrow$ K small $\rightarrow$ surface is *more flat*.
- r small $\rightarrow$ K big $\rightarrow$ surface is *more curved*.

the chairman Mr.Samir declared the end of the first session (after 90min).

A warm round of applause from the audience.

--- End of Third day ---

تقرير صبيحة اليوم الرابع

حلفاوي محمد أيمن

0.5 Overview of the Day

The day started with three morning courses, each followed by a discussion and question session. The program of the day was as follows:

Time	Activity
08:00 -- 09:00	Quantum Mechanics with Djamel
09:00 -- 09:30	Questions and discussion
09:30 -- 10:30	Heat Equation with Billel
10:30 -- 11:00	Questions and discussion
11:00 -- 12:00	Category Theory with Amira
12:00 -- 12:30	Questions and discussion
Afterwards	Salat al-Jumu'ah
14:30	Meeting for the excursion

0.6 Courses

0.6.1 Quantum Mechanics --- Djamel

In this course, we worked on the basic formalism of quantum mechanics.

- A state vector is written $|\psi\rangle \in H$, where H is a Hilbert space.
- Physical states are represented by normalized vectors:

$$\langle \psi, \psi \rangle = 1.$$

- A state is determined only up to an overall phase. If

$$|\psi'\rangle = e^{i\theta}|\psi\rangle,$$

then $|\psi\rangle$ and $|\psi'\rangle$ represent the same physical state.

- The evolution of the state vector is governed by the Schrödinger equation
$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle,$$
where the Hamiltonian satisfies $\hat{H}^* = \hat{H}$.
- The time evolution is unitary. The course introduced an operator $U(t, t_0)$ such that
$$U^*U = I, \qquad U(t, t_0)|\psi(t_0)\rangle = |\psi(t)\rangle.$$

Then we studied observables. If $\hat{A}$ is an observable, the possible outcomes of the measurement are the eigenvalues a_n of $\hat{A}$, with eigenvectors $|a_n\rangle$ such that
$$\hat{A}|a_n\rangle = a_n|a_n\rangle.$$
Since $\hat{A}$ is self-adjoint, the eigenvectors form a basis of H.

If the system is in the state $|\psi\rangle$, the probability of obtaining the result a_n is
$$P_n = |\langle a_n|\psi\rangle|^2.$$
Thus, the state vector provides statistical information about the result of a measurement.

Another point in the course was the collapse of the state vector. Before measurement, one can have a superposition
$$|\psi\rangle = \alpha|a_1\rangle + \beta|a_2\rangle.$$
After measuring $\hat{A}$, the state collapses to one of the eigenstates. For example, if the result is a_1, then the new state is $|a_1\rangle$.

The course contained an example in $H = \mathbb{C}^2$ with
$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Its eigenvalues are
$$\lambda = \pm 1,$$
and the corresponding eigenvectors are
$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
For the state
$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big),$$
we get
$$P(+1) = |\langle +|\psi\rangle|^2 = \frac{1}{2}, \qquad P(-1) = |\langle -|\psi\rangle|^2 = \frac{1}{2}.$$
This example clearly illustrates the probabilistic aspect of measurement.

0.6.2 Heat Equation --- Billel

In this course, we worked on the heat equation together with Fourier series, the Fourier transform, and convolution.

We first used the periodic setting

$$\mathbb{T} = \mathbb{R}/T\mathbb{Z} \simeq \left[-\frac{T}{2}, \frac{T}{2}\right],$$

and the space

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \; ; \; \frac{1}{T}\int_T |f|^2 < +\infty \right\}.$$

The Fourier coefficients of a T-periodic function were defined by

$$\widehat{f}(n) = \frac{1}{T}\int_T f(y) e^{-2\pi i n y/T}\, dy,$$

and the Fourier series was written as

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x/T}.$$

The exponential family $\{e_n\}_{n\in\mathbb{Z}}$ with

$$e_n(x) = e^{2\pi i n x/T}$$

was recalled, together with the Plancherel formula

$$\|f\|_{L^2(\mathbb{T})} = \|\widehat{f}\|_{\ell^2(\mathbb{Z})}.$$

The course also mentioned symmetry properties of Fourier coefficients for real-valued periodic functions.

Then we applied these tools to the heat equation

$$\partial_t u - \partial_x^2 u = 0,$$

with periodic boundary conditions and initial data

$$u(0,x) = u_0(x), \qquad u(t,0) = u(t,T).$$

To search for solutions, we used separation of variables:

$$u(t,x) = \tau(t)X(x).$$

This gives

$$\frac{\tau'}{\tau} = \frac{X''}{X} = -\lambda^2,$$

so we obtain

$$X'' = -\lambda^2 X, \qquad \tau' = -\lambda^2 \tau.$$

Hence

$$\tau(t) = e^{-\lambda^2 t}, \qquad X(x) = a\cos(\eta x) + b\sin(\eta x).$$

Because of periodicity, the frequencies are

$$\eta_n = \frac{2\pi n}{T}.$$

So the solution can be written as

$$u(t,x) = \sum_{n\geq 0} e^{-\eta_n^2 t}\Big(a_n \cos(\eta_n x) + b_n \sin(\eta_n x)\Big).$$

This shows how the heat equation smooths the solution as time increases.

After that, the course moved to the Fourier transform on $\mathbb{R}$. For $f \in L^1(\mathbb{R})$, it was written as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-i2\pi\xi y}\, dy.$$

The inversion formula was also mentioned through Riemann sums. Among the properties presented in the course, we had:

1. translation:
$$\mathcal{F}\Big(f(x-a)\Big)(\xi) = e^{-2\pi i a\xi}\, \widehat{f}(\xi),$$

2. differentiation:
$$\mathcal{F}(f')(\xi) = (2\pi i\xi)\widehat{f}(\xi),$$

3. the convolution--product relation.

Finally, the convolution product was recalled:

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy,$$

with the identity

$$\mathcal{F}(f * g) = \mathcal{F}(f)\, \mathcal{F}(g).$$

Theorem 0.6.1 (BG's theorem)**.**

$$2\pi = 1.$$

Proof. Secret. □

0.6.3 Category Theory --- Amira

In this course, we started with a review of categories: objects, morphisms, and commutative diagrams. Then we moved to additive categories.

A category $\mathcal{C}$ is additive if it satisfies the following properties:

1. for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Mor}(A, B)$ is an abelian group;
2. composition is bilinear:

$$f \circ (g + h) = f \circ g + f \circ h, \qquad (f + g) \circ h = f \circ h + g \circ h;$$

3. $\mathcal{C}$ has a zero object;
4. finite products exist.

The main example in the course was the category $\mathbf{Ab}$ of abelian groups. In this case, the neutral element in $\mathrm{Mor}(A, B)$ is the zero morphism.

Then the course introduced kernels and cokernels. For a morphism

$$f : G \to H,$$

in $\mathbf{Ab}$, we have

$$\ker(f) = f^{-1}(0), \qquad \mathrm{coker}(f) = H/\mathrm{Im}(f).$$

After that, the abstract definition was given. A kernel of a morphism $f : A \to B$ is a morphism

$$i : K \to A$$

such that

$$f \circ i = 0,$$

and satisfying the universal property: for every morphism $g : X \to A$ satisfying $f \circ g = 0$, there exists a unique morphism $g' : X \to K$ such that

$$i \circ g' = g.$$

Dually, a cokernel of $f : A \to B$ is a morphism

$$p : B \to C$$

such that

$$p \circ f = 0,$$

and for every morphism $q : B \to Y$ with $q \circ f = 0$, there exists a unique morphism $q' : C \to Y$ such that

$$q' \circ p = q.$$

The course also emphasized that kernels behave like monomorphisms and cokernels behave like epimorphisms. Thus, this part connected the concrete example of abelian groups with the abstract definition in additive categories.

0.7 What Happened During the Day

The morning was entirely devoted to the three courses and their discussion sessions. After each lecture, there was time for questions, clarifications, and interaction with the teacher. This made the sessions more useful because the participants could review the definitions, formulas, and examples introduced during the lectures.

After the morning program, we left for Salat al-Jumu'ah. Then, in the afternoon, we met again at 14:30 in order to go on the excursion. Thus, the day combined serious mathematical work in the morning with a more relaxed and social activity later in the day.

0.8 Conclusion

Overall, Friday was a rich day at MathCamp. The three courses covered very different topics---quantum mechanics, the heat equation with Fourier methods, and category theory---and each of them opened a different perspective. The question sessions after each lecture were also important because they helped connect the written notes with the main ideas of the courses. The day ended with the excursion, which provided a pleasant conclusion to an already intense and interesting program.

تقرير صبيحة اليوم الرابع

برحال عبد الله

Conference 3: Pr. Djamal

Remark 1: We must take the state vector $|\phi\rangle$ such that $\langle\phi,\phi\rangle = 1$.

Remark 2: The state does not change if we multiply it by a non-zero scalar. Indeed, we take a Hilbert space $H \setminus \{0\}$ and define the equivalence relation:

$$\psi \sim \phi \Leftrightarrow \exists\lambda \in \mathbb{C} \ / \ \psi = \lambda\phi$$

We take the projective space:
$P(H) = (H\setminus\{0\})/\sim$.
So here the state is not $|\phi\rangle$ but $[\phi]$.

Example:
If we have a gas, we can know its volume, heat, … and they still remain unchanged, but we can't know **********
****
****

1. The Hamiltonian $\hat{H}$ in the Schrödinger equation is self-adjoint.

2. The state of a system $|\psi\rangle$ evolves in time in a way that preserves probabilities. This is written as:

$$|\psi\rangle = U(t)|\psi(0)\rangle$$

where $U(t)$ is a unitary operator.

0.9 Black Hole Information Paradox

The core idea of the problem:
In quantum mechanics, time evolution must be unitary, i.e., information is preserved.
But in black hole physics (from general relativity), it seems that information is actually lost.

Now, let's go to the fourth postulate:

4) $A =$ observable.
The outcome of any measurement of A is one of the eigenvalues of A.

Remark: These results are obtained in an ideal experiment.
For example, in real life we have no lines or points, but we work with them and obtain remarkable results. The same idea applies here.

5) Measurement.
The result is always one of the eigenvalues with probability:

$$P_n = |\langle a_n|\psi\rangle|^2, \qquad n = 1, 2, \dots$$

6) Collapse.
Before: $|\psi\rangle = \alpha|a_1\rangle + \beta|a_2\rangle$.
After: $\dfrac{a_1}{|\psi\rangle} = |a_1\rangle$.

The collapse of the state vector $|\psi\rangle$ only gives statistical information.
Measurement affects the outcome, but so far we cannot explain the collapse.
But if we do not measure, how does the world function? Who does the measuring?
This is called the *measurement problem*.

Example:

$$H = \mathbb{C}^2,$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s = \pm 1.$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$P(1) = |\langle +|\psi\rangle|^2 = \left|(1 \;\; 0)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right|^2$$
$$= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$
$$P(-1) = \frac{1}{2}.$$

He talked about the Schrödinger's cat experiment (alive or dead), and he said that two matrices commute if and only if they have the same eigenvectors.

Conference 2: Pr. Billel

Today we will talk about:

1. Separation of variables.
2. Fourier series.
3. Fourier transform.
4. Convolution product.

$$T > 0, \quad \mathbb{T} = \mathbb{R}/T\mathbb{Z} \simeq \left[-\frac{T}{2}, \frac{T}{2}\right] \quad \text{(torus)}$$
$$L^2(\mathbb{T}) = \left\{f : \mathbb{T} \to \mathbb{C} \; ; \; \int_{\mathbb{T}} |f|^2 \, dx < \infty\right\}$$
$$\langle f, g\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\overline{g(x)}\, dx$$
$$\ell^2(\mathbb{Z}) = \left\{\alpha : \mathbb{Z} \to \mathbb{C} \; ; \; \sum_{n\in\mathbb{Z}} |\alpha_n|^2 < \infty\right\}$$
$$\langle \alpha, \beta\rangle = \sum_{n\in\mathbb{Z}} \alpha_n\overline{\beta_n}.$$

Note that ℓ^2 and L^2 are Hilbert spaces.

0.10 Separation of Variables

Recall the heat equation:

$$\begin{cases} U_t - \partial_x^2 U = 0, \\ U(0,x) = U_0(x), \\ U(t,0) = U(t,T). \end{cases}$$

$$U(t,x) = \tau(t)X(x), \tag{5}$$
$$\tau' X - \tau X'' = 0, \tag{6}$$
$$\frac{\tau'(t)}{\tau(t)} = \frac{X''(x)}{X(x)}. \tag{7}$$

The first term depends only on t and the second term depends only on x. Since x and t are independent, there exists a constant c such that:

$$\frac{\tau'(t)}{\tau(t)} = \frac{X''(x)}{X(x)} = c = -\lambda^2.$$

$$\begin{cases} X'' = -\lambda^2 X, \\ \tau' = -\lambda^2 \tau. \end{cases}$$

$$\tau(t) = e^{-\lambda^2 t}, \qquad X(x) = a\cos(\lambda x) + b\sin(\lambda x), \qquad \lambda_n = \frac{2\pi n}{T}.$$

$$U(t,x) = \sum_{n\geq 0} \exp(-\lambda_n^2 t)\left(a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x)\right),$$
$$U_0(x) = \sum a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x).$$

So let us take a T-periodic function and try to write it as a sum of cosines and sines.

If f is T-periodic:

$$\begin{aligned} f(x) &= \sum_{n\geq 0} a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right) \\ &= \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{\frac{2\pi i n x}{T}}, \\ B &= \{e_n,\ n \in \mathbb{Z}\} \subset L^2(\mathbb{T}). \end{aligned}$$

such that

$$e_n(x) = e^{\frac{2\pi i n x}{T}}.$$

$$\langle e_n, e_m \rangle = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

$$\langle e_n, e_m \rangle = \frac{1}{T} \int_0^T e_n(x) \overline{e_m(x)}\, dx.$$

Thus,

$$\langle f, e_n \rangle = \widehat{f}(n).$$

If $f \in C^2(\mathbb{T})$:

$$\int_0^T |f(x) - S_N(x)|^2\, dx \to 0.$$

So every function can be written as a Fourier series.

Parseval / Plancherel

$$\|f\| = \|\widehat{f}\|.$$

Derivative

$$f'(x) = \sum \widehat{f}(n) \frac{2\pi i n}{T} e_n(x),$$

$$\widehat{f'}(n) = \frac{2\pi i n}{T} \widehat{f}(n).$$

So we can transform ODEs into algebraic equations.

$$\begin{aligned} f(x) &= \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{\frac{2\pi i n}{T} x} \\ &= \sum_{n \in \mathbb{Z}} \left(\frac{1}{T} \int_{\mathbb{T}} f(y) e^{-\frac{i 2\pi n y}{T}}\, dy \right) e^{\frac{i 2\pi n x}{T}}. \end{aligned}$$

$$\xi_n = \frac{n}{T}, \qquad \Delta\xi = \xi_{n+1} - \xi_n = \frac{1}{T}.$$

$$\begin{aligned} f(x) &= \sum \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(y) e^{-2\pi i \xi_n y}\, dy \right) e^{2\pi i \xi_n x} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) e^{-2\pi i \xi y}\, dy\, e^{2\pi i \xi x}\, d\xi. \end{aligned}$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx.$$

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

$$\mathcal{F}(f(x-a)) = e^{-2\pi i \xi a} \mathcal{F}(f)(\xi).$$

$$\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}.$$

$$\widehat{\partial_x f} = (2\pi i \xi) \widehat{f}.$$

Scaling

We are going to do some things wrong (up to a constant, namely $2\pi = 1$), but it's okay (as Billel said):

$$\mathcal{F}(f(ax)) = \frac{1}{|a|} \widehat{f}\left(\frac{\xi}{a}\right).$$

Convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy.$$

If $f \in C^k$, then

$$f * g \in C^k.$$

$$\partial^k(f * g) = (\partial^k f) * g.$$

Remark: We can do the same with g because convolution is commutative:

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g).$$

تقرير صبيحة اليوم الخامس

مداح أحمد عدنان

Quantum Mechanics Session

Organisational note. This course was given by Mr. Djamel. The session opened with a concise review of the fundamental postulates covered in the preceding lecture, before moving on to the superposition principle, the interference term, the path-integral formulation, expectation values and variance, and finally the uncertainty principle.

Recap of Previous Session

The postulates recalled at the beginning of the session constitute the axiomatic backbone of quantum mechanics, and are summarised below.

State of a quantum system. A quantum system is fully characterised by a state vector $|\psi\rangle$ living in a complex Hilbert space $\mathcal{H}$. All the physical information about the system is encoded in this vector.

Measurement postulate and Born rule. Let $\hat{A}$ be an observable, that is, a Hermitian operator on $\mathcal{H}$, with eigenvalues $\{a_n\}$ and corresponding orthonormal eigenstates $\{|a_n\rangle\}$:

$$\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle. \tag{8}$$

Upon measuring the observable $\hat{A}$ on a system in the state $|\psi\rangle$, the probability of obtaining the eigenvalue a_n is given by the **Born rule**:

$$P(a_n) = |\langle a_n|\psi\rangle|^2\,. \tag{9}$$

More generally, the probability of finding the system in an arbitrary normalised state $|\phi\rangle$ is

$$P_{\psi\to\phi} = |\langle\phi|\psi\rangle|^2\,. \tag{10}$$

Remark (raised by Mr. Zeghib). If two state vectors both have unit norm and $|\langle\phi|\psi\rangle|^2 = 1$, one can show that $|\phi\rangle = e^{i\alpha}|\psi\rangle$ for some real phase $\alpha \in \mathbb{R}$. States differing only by a global phase are physically identical, since no observable quantity depends on that global phase. This is the reason why we identify rays in $\mathcal{H}$, that is, equivalence classes under global phase, with physical states.

The Superposition Principle

Statement. Suppose a system in the state $|\psi\rangle$ has a non-zero probability of being found in the state $|\phi_1\rangle$ and also in the state $|\phi_2\rangle$:

$$|\langle\phi_1|\psi\rangle|^2 \neq 0, \qquad |\langle\phi_2|\psi\rangle|^2 \neq 0. \tag{11}$$

Then, by the linearity of the Hilbert space, the system can also exist in *any* linear combination of these states:

$$|\phi\rangle = a\,|\phi_1\rangle + b\,|\phi_2\rangle, \qquad a, b \in \mathbb{C}. \tag{12}$$

Probability of a superposed state. We compute $P_\phi = |\langle\phi|\psi\rangle|^2$ explicitly:

$$\begin{aligned} P_\phi = |\langle\phi|\psi\rangle|^2 &= \Big|\Big(a^*\langle\phi_1| + b^*\langle\phi_2|\Big)|\psi\rangle\Big|^2 \\ &= |a^*\langle\phi_1|\psi\rangle + b^*\langle\phi_2|\psi\rangle|^2 \\ &= |a|^2\,|\langle\phi_1|\psi\rangle|^2 + |b|^2\,|\langle\phi_2|\psi\rangle|^2 + 2\,\mathrm{Re}\Big(a^*b\,\langle\phi_1|\psi\rangle^*\langle\phi_2|\psi\rangle\Big). \end{aligned} \tag{13}$$

When $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal, $\langle\phi_1|\phi_2\rangle = 0$, and the state space is spanned solely by them, we may identify

$$P(a) = |a|^2\,, \qquad P(b) = |b|^2\,, \tag{14}$$

so that the probability of finding the system in $|\phi\rangle$ decomposes as

$$\boxed{P_\phi = P(a)\,P_1 + P(b)\,P_2 + \underbrace{2\,\mathrm{Re}\Big(a^*b\,\langle\phi_1|\psi\rangle^*\langle\phi_2|\psi\rangle\Big)}_{\text{interference term}}} \tag{15}$$

The Interference Term

The term

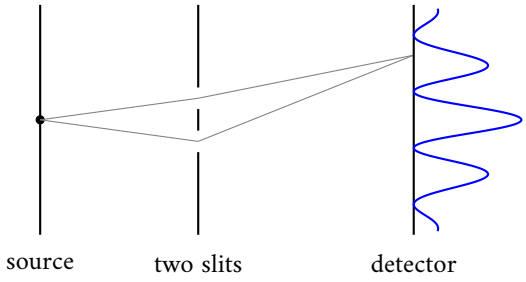
$$\mathcal{I} = 2\,\mathrm{Re}\Big(a^*b\,\langle\phi_1|\psi\rangle^*\langle\phi_2|\psi\rangle\Big) \tag{16}$$

has no classical analogue. It is the mathematical origin of wave-like interference in quantum mechanics. The interference term can be:

- **constructive** ($\mathcal{I} > 0$): the total probability is *greater* than the classical sum;
- **destructive** ($\mathcal{I} < 0$): the total probability is *less* than the classical sum, which leads to dark fringes;
- **zero**: when the two amplitudes are out of phase by $\pi/2$, the interference vanishes.

Physical Illustration: the Double-Slit Experiment

Consider sending 5×10^6 electrons through a barrier with two slits. Classically, one would expect the intensity pattern on the detector to be the sum of two individual single-slit patterns. However, the experiment shows a characteristic **fringe pattern**, with a central maximum in the middle.



The observed fringe pattern, with a central maximum, instead of the classical sum of two single-slit patterns.

This is explained directly by the interference term: the amplitude for reaching a detector point x is the sum of the amplitudes through each slit,

$$\mathcal{M}(x) = \mathcal{M}_1(x) + \mathcal{M}_2(x), \tag{17}$$

and the probability is $P(x) = |\mathcal{M}(x)|^2$, which contains the cross term $2\,\mathrm{Re}(\mathcal{M}_1^* \mathcal{M}_2)$ responsible for the fringes.

Path Integral Formulation (Feynman)

The double-slit discussion motivates the **path-integral**, or sum-over-histories, approach to quantum mechanics, due to Feynman. The transition amplitude from an initial state i to a final state f is

$$\mathcal{M}_{i\to f} \propto \sum_{\substack{\text{all paths} \\ i\to f}} e^{iS[\gamma]/\hbar}, \tag{18}$$

where the **action** along a path γ is

$$S[\gamma] = \int_{t_i}^{t_f} L(q, \dot{q}, t)\, dt, \tag{19}$$

L being the classical Lagrangian, and the sum running over a *non-countable* (functional) set of all conceivable paths between i and f. The measured transition probability is then

$$P_{i\to f} = |\mathcal{M}_{i\to f}|^2 . \tag{20}$$

Classical limit. When $\hbar \to 0$, or equivalently when the action S is very large compared with $\hbar$, the phase $e^{iS/\hbar}$ oscillates extremely rapidly for nearby paths and the contributions cancel by destructive interference, *except* near the path for which $\delta S = 0$, that is, the classical trajectory. This is precisely the **principle of stationary action**, and it shows that classical mechanics --- Newton's equation $m\ddot{x} = F$, or Hamilton's equations --- emerges from quantum mechanics in the limit $\hbar \to 0$. The Schrödinger equation plays the role for quantum mechanics that Newton's second law plays in classical mechanics.

Expectation Values and Variance of Observables

Expectation value. Treating the observable $\hat{A}$ as a quantum-mechanical random variable with outcomes $\{a_n\}$ and probabilities $P_n = |\langle a_n|\psi\rangle|^2$, the expectation value is

$$\begin{aligned}\langle \hat{A} \rangle = \sum_n a_n P_n = \sum_n a_n \, |\langle a_n|\psi\rangle|^2 = \sum_n a_n \langle\psi|a_n\rangle\langle a_n|\psi\rangle \\ = \langle\psi| \left(\sum_n a_n |a_n\rangle\langle a_n| \right) |\psi\rangle. \end{aligned} \tag{21}$$

Using the **spectral decomposition** $\hat{A} = \sum_n a_n |a_n\rangle\langle a_n|$, which follows from $\hat{A}$ being Hermitian with a complete eigenbasis, together with the completeness relation $\sum_n |a_n\rangle\langle a_n| = \mathbf{1}$, we obtain

$$\boxed{\langle \hat{A} \rangle = \langle\psi|\hat{A}|\psi\rangle.} \tag{22}$$

Variance. The variance of $\hat{A}$ in the state $|\psi\rangle$ is defined as

$$\mathrm{Var}(\hat{A}) = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \langle\psi|\hat{A}^2|\psi\rangle - \Big(\langle\psi|\hat{A}|\psi\rangle \Big)^2. \tag{23}$$

Special case: an eigenstate. If $|\psi\rangle = |a_n\rangle$, then

$$\langle \hat{A} \rangle = \langle a_n|\hat{A}|a_n\rangle = a_n \langle a_n|a_n\rangle = a_n, \tag{24}$$

$$\langle \hat{A}^2 \rangle = \langle a_n|\hat{A}^2|a_n\rangle = a_n^2, \tag{25}$$

and therefore $\mathrm{Var}(\hat{A}) = a_n^2 - a_n^2 = 0$. This reflects the fact that an eigenstate of $\hat{A}$ yields a *definite, deterministic* measurement outcome a_n with probability 1.

Interpretation of the mean. A large expectation value is not necessarily the most probable measurement outcome if the variance is also large. Consider two Gaussian probability distributions with identical means but very different widths: the peak, that is the most probable value, coincides with the mean in both cases, but the spread of possible outcomes is dramatically different. The expectation value alone is thus an incomplete characterisation of the measurement statistics; one should always consider the variance as well, or equivalently the standard deviation $\Delta A = \sqrt{\mathrm{Var}(\hat{A})}$.

The Uncertainty Principle

Incompatible observables and non-commutativity. Two observables $\hat{A}$ and $\hat{B}$ are said to be **incompatible** if their commutator is non-zero:

$$\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0. \tag{26}$$

Incompatible observables cannot in general be simultaneously diagonalised, that is, they do not share a common eigenbasis.

Physical consequences. Suppose the system is prepared in an eigenstate $|a_n\rangle$ of $\hat{A}$, so that measuring $\hat{A}$ yields a_n with certainty and $\mathrm{Var}(\hat{A}) = 0$. Consider now a second observable $\hat{B}$ with $[\hat{A}, \hat{B}] \neq 0$. Since $\hat{B}$ and $\hat{A}$ share no common eigenstates, $|a_n\rangle$ is not an eigenstate of $\hat{B}$:

$$\hat{B}|a_n\rangle \neq b\,|a_n\rangle \qquad \text{for any scalar } b \in \mathbb{C}. \tag{27}$$

We can expand $|a_n\rangle$ in the eigenbasis $\{|b_i\rangle\}$ of $\hat{B}$:

$$|a_n\rangle = \sum_i \alpha_i |b_i\rangle, \qquad \alpha_i = \langle b_i | a_n \rangle. \tag{28}$$

The probability of obtaining the eigenvalue b_i upon measuring $\hat{B}$ is

$$P_i = |\alpha_i|^2 = |\langle b_i | a_n \rangle|^2 . \tag{29}$$

Since $|a_n\rangle$ is *not* an eigenstate of $\hat{B}$, several values $\{b_i\}$ are obtained with non-zero probability, which means that the outcome of the $\hat{B}$-measurement is **irreducibly random**.

General formulation: the Robertson uncertainty relation. The quantitative statement of this incompatibility is the **Robertson uncertainty relation**: for any two observables $\hat{A}$ and $\hat{B}$ and any state $|\psi\rangle$,

$$\boxed{\Delta A \cdot \Delta B \;\geq\; \frac{1}{2}\left|\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle\right|,} \tag{30}$$

where $\Delta A = \sqrt{\mathrm{Var}(\hat{A})}$ and $\Delta B = \sqrt{\mathrm{Var}(\hat{B})}$ are the standard deviations.

Famous case: position and momentum. The canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$ immediately gives the **Heisenberg uncertainty principle**:

$$\Delta x \cdot \Delta p \;\geq\; \frac{\hbar}{2}.$$

This is not a statement about measurement disturbance, but a fundamental property of nature: a quantum state cannot have simultaneously well-defined position and momentum.

Physical interpretation.

- **Mathematical origin:** the uncertainty principle is a theorem of Hilbert-space mathematics, a consequence of the Cauchy--Schwarz inequality applied to the norms of $(\hat{A} - \langle\hat{A}\rangle)|\psi\rangle$ and $(\hat{B} - \langle\hat{B}\rangle)|\psi\rangle$.
- **Physical origin:** two non-commuting observables cannot be simultaneously assigned sharp values. The very act of projecting the state onto an eigenstate of $\hat{A}$ inevitably randomises the outcome of any subsequent $\hat{B}$-measurement.
- **Classical limit:** in the classical limit $\hbar \to 0$, all commutators scaled by $\hbar$ vanish, and simultaneous precise knowledge of all observables is recovered, which is consistent with the deterministic Newtonian picture.

Discussion note. Even though the system was prepared in a definite eigenstate of $\hat{A}$, so that measuring $\hat{A}$ gives a predictable result, measuring $\hat{B}$ immediately afterwards will in general produce a random result, distributed according to $P_i = |\langle b_i|a_n\rangle|^2$. It is fundamentally impossible to predict which value b_i will be obtained in a single run. This is not a limitation of our experimental apparatus; it is a statement about the structure of physical reality in the quantum regime.

Summary of Key Results

Concept	Mathematical expression	Remark
Born rule	$P(a_n) = \lvert\langle a_n \lvert \psi\rangle\rvert^2$	Measurement probability
Superposition	$\lvert\phi\rangle = a\lvert\phi_1\rangle + b\lvert\phi_2\rangle$	Linearity of $\mathcal{H}$
Interference term	$2\,\mathrm{Re}(a^* b\,\langle\phi_1\lvert\psi\rangle^*\langle\phi_2\lvert\psi\rangle)$	Quantum, no classical analogue
Expectation value	$\langle\hat{A}\rangle = \langle\psi\lvert\hat{A}\rvert\psi\rangle$	Spectral theorem
Variance	$\mathrm{Var}(\hat{A}) = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$	Zero if and only if eigenstate
Robertson relation	$\Delta A\,\Delta B \geq \frac{1}{2}\left\lvert\langle[\hat{A},\hat{B}]\rangle\right\rvert$	General uncertainty
Heisenberg	$\Delta x\,\Delta p \geq \hbar/2$	Canonical pair

Partial Differential Equations Session

Organisational note. This course was presented by Mr. Billel and introduced by Mr. Alaa. It opened with a short warm-up recalling the Fourier-based approach to the heat equation seen in the previous course, before moving on to the main topics of the morning: the heat kernel, well-posedness (existence, uniqueness, stability), the energy identity, the maximum principle, and decay estimates.

Warm-up: The Heat Equation via Fourier Series

Setting and Fourier expansion. Consider the heat equation on a periodic domain of period T:

$$\partial_t u - \partial_x^2 u = 0, \qquad u(0,\cdot) = u_0, \tag{31}$$

with initial data $u_0 \in L^2(\mathbb{R}/T\mathbb{Z})$. We expand u_0 in a Fourier series:

$$u_0(x) = \sum_{n\geq 0}\left(a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin(\lambda_n x)\right) = \sum_{n\in\mathbb{Z}} \widehat{u_0}(n)\, e^{i\lambda_n x}, \tag{32}$$

where $\lambda_n = \dfrac{2\pi n}{T}$ are the Fourier frequencies and $\widehat{u_0}(n)$ the Fourier coefficients of u_0.

Solution in frequency space. Applying the Fourier transform in x, each Fourier mode $\widehat{u}(t,n)$ satisfies the ordinary differential equation

$$\frac{d}{dt}\widehat{u}(t,n) + \lambda_n^2\,\widehat{u}(t,n) = 0, \tag{33}$$

whose unique solution is

$$\widehat{u}(t,n) = e^{-\lambda_n^2 t}\,\widehat{u_0}(n). \tag{34}$$

Since $u_0 \in L^2$, Parseval's theorem guarantees $\sum_n |\widehat{u_0}(n)|^2 < \infty$, and the decay factor $e^{-\lambda_n^2 t} \le 1$ ensures that the series converges for all $t \ge 0$. This gives **existence** of a solution.

The Heat Equation on $\mathbb{R}$: Heat Kernel and Duhamel's Formula

The inhomogeneous problem. We now work on the whole real line. Given a forcing term $f \in C(\mathbb{R}, L^2(\mathbb{R}))$, consider the inhomogeneous heat equation

$$\partial_t u - \partial_x^2 u = f(t,x), \qquad u(0,\cdot) = u_0. \tag{35}$$

Taking the Fourier transform in x, with frequency variable $\xi \in \mathbb{R}$, this becomes

$$\frac{d}{dt}\widehat{u}(t,\xi) + 4\pi^2\xi^2\,\widehat{u}(t,\xi) = \widehat{f}(t,\xi). \tag{36}$$

This is a first-order linear ODE in t. Multiplying through by the integrating factor $e^{4\pi^2\xi^2 t}$,

$$\frac{d}{dt}\Big(e^{4\pi^2\xi^2 t}\,\widehat{u}(t,\xi)\Big) = \widehat{f}(t,\xi)\,e^{4\pi^2\xi^2 t}, \tag{37}$$

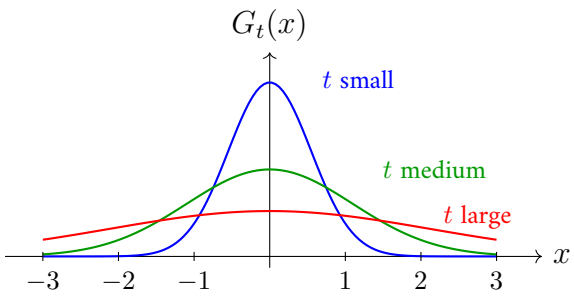
and integrating from 0 to t we obtain

$$\widehat{u}(t,\xi) = e^{-4\pi^2\xi^2 t}\,\widehat{u_0}(\xi) + \int_0^t e^{-4\pi^2\xi^2(t-s)}\,\widehat{f}(s,\xi)\,ds. \tag{38}$$

The Gaussian heat kernel. The key observation is that the Fourier multiplier $e^{-4\pi^2\xi^2 t}$ is itself the Fourier transform of a Gaussian. Specifically,

$$\mathcal{F}^{-1}\Big(e^{-4\pi^2\xi^2 t}\Big)(x) = \frac{1}{\sqrt{4\pi t}}\,e^{-x^2/(4t)} =: G_t(x). \tag{39}$$

The function G_t is called the **Gaussian heat kernel**, or **fundamental solution**, of the heat equation. It is a smooth, positive, bell-shaped function for every $t > 0$.



The heat kernel $G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ for three values of t. As t increases, the kernel flattens and spreads, modelling heat diffusion.

Duhamel's formula. Inverting the Fourier transform and using the convolution theorem $\mathcal{F}^{-1}(\widehat{f} \cdot \widehat{g}) = f * g$, we obtain

$$\boxed{u(t,x) = \Big(G_t * u_0\Big)(x) + \int_0^t \Big(G_{t-s} * f(s,\cdot)\Big)(x)\, ds.} \tag{40}$$

This is **Duhamel's formula**. It decomposes the solution into two parts:

- $G_t * u_0$: the *free evolution* of the initial data under the heat flow;
- $\int_0^t G_{t-s} * f(s,\cdot)\, ds$: the *forced contribution*, accumulating the effect of the source f over time, each past instant s being evolved forward by the kernel G_{t-s}.

Well-Posedness: Existence, Uniqueness and Stability

Regularity (smoothing effect). A remarkable property of the heat equation is its **instantaneous smoothing**: even if the initial datum u_0 is merely in $L^2(\mathbb{R})$, the solution $u(t,\cdot)$ becomes real-analytic for every $t > 0$.

Proposition. For any $u_0 \in L^2(\mathbb{R})$ and any $t > 0$, the function $u(t,\cdot) = G_t * u_0$ is real-analytic on $\mathbb{R}$.

Sketch of proof. The heat kernel $G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$ is real-analytic, in fact entire as a function of complex x, for every fixed $t > 0$. Since the convolution of an L^2 function with an analytic function is analytic, provided the convolution is well defined, $G_t * u_0$ is analytic for all $t > 0$. □

This is in sharp contrast with the wave equation, where singularities in the initial data propagate indefinitely. The heat equation *instantly* regularises any L^2 initial datum to a C^∞, in fact analytic, function.

Uniqueness --- first approach (Fourier). In frequency space, the equation obtained above with $f = 0$ is a first-order linear ODE in t with a unique solution once the initial condition $\widehat{u}(0,\xi) = \widehat{u_0}(\xi)$ is given. Since the Fourier transform is a bijection on $L^2(\mathbb{R})$, by Plancherel's theorem, the solution u is unique.

Uniqueness --- second approach (energy method, preferred by the lecturer). Suppose u and v are two solutions of the homogeneous heat equation with the same initial datum u_0, and set $w = u - v$. Then w solves

$$\partial_t w - \partial_x^2 w = 0, \qquad w(0,\cdot) = 0, \tag{41}$$

and Duhamel's formula gives $w(t,\cdot) = G_t * w_0 = G_t * 0 = 0$. Hence $u = v$.

Stability and the Young convolution inequality. To make the stability argument rigorous we use the **Young convolution inequality**: for $1 \leq p, q, r \leq \infty$ satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$,

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \, \|g\|_{L^q} \, . \tag{42}$$

This also implies the pointwise bound $|(f * g)(x)| \leq \int_{\mathbb{R}} |f(x-y)| \, |g(y)| \, dy$.

Theorem (Stability). Let u and v be two solutions of the heat equation with initial data u_0 and v_0 respectively, both in $L^2(\mathbb{R})$. Then for all $t \geq 0$,

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} \leq \|u_0 - v_0\|_{L^2} \, . \tag{43}$$

Proof. Let $w = u - v$, so that $w(t,\cdot) = G_t * w_0$. Applying Young's inequality with $p = 1$, $q = 2$, $r = 2$,

$$\|w(t,\cdot)\|_{L^2} \leq \|G_t\|_{L^1} \, \|w_0\|_{L^2} \, . \tag{44}$$

A direct computation shows that $\|G_t\|_{L^1} = 1$ for all $t > 0$:

$$\int_{\mathbb{R}} G_t(x) \, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \, dx = 1, \tag{45}$$

which is simply the normalisation of the Gaussian. Therefore $\|w(t,\cdot)\|_{L^2} \leq \|w_0\|_{L^2}$, which gives stability and uniqueness simultaneously. □

Energy Identity

An alternative, purely PDE-based approach to stability and regularity is provided by the **energy method**. Multiplying the heat equation $\partial_t u - \partial_x^2 u = 0$ by u and integrating by parts:

$$\begin{aligned} u \cdot \left(\partial_t u - \partial_x^2 u\right) &= 0, \\ \partial_t \left(\frac{u^2}{2}\right) - \partial_x(u \, \partial_x u) + (\partial_x u)^2 &= 0. \end{aligned} \tag{46}$$

Integrating over $x \in \mathbb{R}$ and using the boundary condition $u(t, \pm\infty) = 0$, which kills the boundary term $\partial_x(u \, \partial_x u)$ upon integration, we obtain

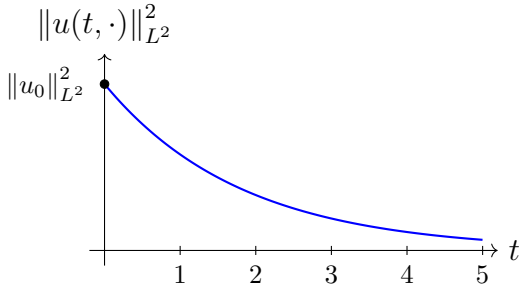
$$\boxed{\frac{1}{2}\frac{d}{dt} \|u(t,\cdot)\|_{L^2}^2 + \|\partial_x u(t,\cdot)\|_{L^2}^2 = 0.} \tag{47}$$

This identity has a clear physical interpretation:

- the term $\frac{1}{2}\frac{d}{dt} \|u\|_{L^2}^2$ is the rate of change of the *thermal energy* of the system;

- the term $\|\partial_x u\|_{L^2}^2 \geq 0$ is the *dissipation*: energy is continuously lost through diffusion, heat flowing from hot to cold regions.

In particular, the identity shows that $t \mapsto \|u(t,\cdot)\|_{L^2}^2$ is non-increasing, giving a direct proof that $\|u(t,\cdot)\|_{L^2} \leq \|u_0\|_{L^2}$ for all $t \geq 0$.



The L^2 energy $\|u(t,\cdot)\|_{L^2}^2$ is monotonically decreasing in t, decaying to 0 as $t \to \infty$.

The Heat Semigroup

The solution operator of the heat equation defines a **semigroup** of bounded linear operators on $L^2(\mathbb{R})$:

$$S_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad S_t u_0 := u(t,\cdot) = G_t * u_0, \quad t > 0. \tag{48}$$

This operator can also be written formally as $S_t = e^{t\partial_x^2}$, which is justified in frequency space by

$$\mathcal{F}(S_t u_0)(\xi) = e^{-4\pi^2\xi^2 t}\,\widehat{u_0}(\xi) = \mathcal{F}\Big(e^{t\partial_x^2} u_0\Big)(\xi). \tag{49}$$

The family $\{S_t\}_{t\geq 0}$ satisfies the semigroup law $S_t \circ S_s = S_{t+s}$, which corresponds to the composition of Gaussians $G_t * G_s = G_{t+s}$, together with $S_0 = \mathrm{Id}$.

Maximum Principle

Theorem (Maximum Principle). Let $u_0 \in L^p(\mathbb{R}) \cap C(\mathbb{R})$ for some $1 \leq p \leq \infty$. Then the solution $u(t,\cdot) = S_t u_0$ satisfies

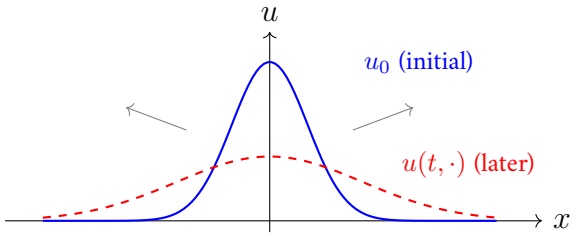
$$\min_{x\in\mathbb{R}} u_0(x) \;\leq\; u(t,x) \;\leq\; \max_{x\in\mathbb{R}} u_0(x) \qquad \forall\, t \geq 0,\ x \in \mathbb{R}. \tag{50}$$

In particular the L^p norm is controlled:

$$\|u(t,\cdot)\|_{L^p} \leq \|u_0\|_{L^p} \qquad \forall\, t \geq 0. \tag{51}$$

Discussion (raised by Oussama). A physical explanation of the heat equation was requested, with reference to the graph of the solution. Consider a rod with an initial temperature profile $u_0(x)$; the heat equation models the evolution of this temperature over time.

- **The Laplacian $\partial_x^2 u$ as a curvature detector.** At a local maximum of u (a ``hot spot''), $\partial_x^2 u < 0$, so $\partial_t u < 0$ and the temperature decreases. At a local minimum (a ``cold spot''), $\partial_x^2 u > 0$, so $\partial_t u > 0$ and the temperature increases. The equation drives the system towards a uniform temperature.
- **The role of the kernel G_t.** The value $u(t,x)$ is a weighted average of the initial profile u_0 over a neighbourhood of x of radius $\sim \sqrt{t}$. As t grows, this averaging region grows and the profile flattens.



Heat diffuses outward from the peak: the solution flattens over time, while the total ``heat'' $\int u\,dx$ is conserved.

Decay Estimates

We now quantify how fast the solution decays in time when the initial datum has better integrability.

Theorem (Decay in L^2). Let $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the solution $u(t,\cdot) = S_t u_0$ satisfies

$$\|u(t,\cdot)\|_{L^2}^2 \;\le\; \|u_0\|_{L^1}^2 \cdot C\,t^{-1/2} \;\xrightarrow[t\to\infty]{}\; 0. \tag{52}$$

Proof. By Plancherel's theorem,

$$\begin{aligned} \|u(t,\cdot)\|_{L^2}^2 = \|\widehat{u}(t,\cdot)\|_{L^2}^2 &= \int_{\mathbb{R}} |\widehat{u}(t,\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}} |\widehat{u_0}(\xi)|^2 \, e^{-2(2\pi\xi)^2 t} \, d\xi. \end{aligned} \tag{53}$$

Using the bound $|\widehat{u_0}(\xi)| \le \|u_0\|_{L^1}$, which follows from $|\widehat{u_0}(\xi)| = \left|\int u_0(x) e^{-2\pi i \xi x}\,dx\right| \le$

$\|u_0\|_{L^1}$, we get

$$\int_{\mathbb{R}} |\widehat{u_0}(\xi)|^2 \, e^{-2(2\pi\xi)^2 t} \, d\xi \le \|u_0\|_{L^1}^2 \int_{\mathbb{R}} e^{-8\pi^2\xi^2 t} \, d\xi$$
$$= \|u_0\|_{L^1}^2 \cdot \frac{1}{\sqrt{8\pi^2 t}} =: \|u_0\|_{L^1}^2 \cdot C\, t^{-1/2}. \tag{54}$$

Since $Ct^{-1/2} \to 0$ as $t \to \infty$, we conclude that $\|u(t,\cdot)\|_{L^2} \to 0$. □

The rate $t^{-1/2}$ for the squared L^2 norm is **sharp**: it matches the decay rate of the heat kernel itself, $\|G_t\|_{L^2} = (8\pi t)^{-1/4} \sim t^{-1/4}$, so that $\|G_t\|_{L^2}^2 \sim t^{-1/2}$. The assumption $u_0 \in L^1$ is essential: without it the solution need not decay at all, as the example $u_0 \equiv 1$ shows.

Summary of Main Results

Property	Statement	Tool
Existence	$u_0 \in L^2 \Rightarrow$ solution exists for all $t \ge 0$	Fourier / Parseval
Regularity	$u(t,\cdot)$ analytic for all $t > 0$	Smoothness of G_t
Uniqueness	Fourier is a bijection; the ODE has a unique solution	Fourier / Energy
Stability	$\|u - v\|_{L^2} \le \|u_0 - v_0\|_{L^2}$	Young + $\|G_t\|_{L^1} = 1$
Energy identity	$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 = 0$	Integration by parts
Maximum principle	$\min u_0 \le u(t,x) \le \max u_0$	PDE comparison
L^p **contraction**	$\|S_t u_0\|_{L^p} \le \|u_0\|_{L^p}$	Maximum principle
Decay	$u_0 \in L^1 \cap L^2 \Rightarrow \|u(t)\|_{L^2} \le Ct^{-1/4}\|u_0\|_{L^1}$	Plancherel + Gaussian

تقرير عشية اليوم الخامس

مداح أحمد عدنان

Geometric Group Theory --- Cayley Graphs, Quasi-Isometry and the Gromov Boundary

Organisational note. This session continued Lamine's course on the geometric point of view of groups. After a short recall of finitely generated groups and group presentations, the talk moved on to Cayley graphs, the word metric, quasi-isometry, Gromov hyperbolicity, hyperbolic groups, and finally the Gromov boundary.

Recall: Finitely Generated Groups and Presentations

A **finitely generated group** G is a group that can be written as a group presentation

$$G = \langle S \mid R \rangle, \tag{55}$$

where

- S is a finite set of **generators**;
- R is a set of **relations**, that is, words in $S \cup S^{-1}$ which are identified with the identity element $e \in G$.

The group G is then the quotient of the free group $F(S)$ by the normal closure of R:

$$G \;\cong\; F(S) \Big/ \langle\langle R \rangle\rangle. \tag{56}$$

Example. The integers: $\mathbb{Z} = \langle a \rangle$ (one generator, no relations). The free group on two generators: $F_2 = \langle a, b \rangle$ (two generators, no relations). The cyclic group of order n: $\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$.

Cayley Graphs

Definition

Definition (Cayley Graph). Let $G = \langle S \mid R \rangle$ be a finitely generated group. The **Cayley graph** $\mathrm{Cay}(G, S)$ is the directed graph (V, E) defined by

$$V = G, \qquad (g_1, g_2) \in E \iff \exists\, s \in S \text{ such that } g_2 = g_1 s.$$

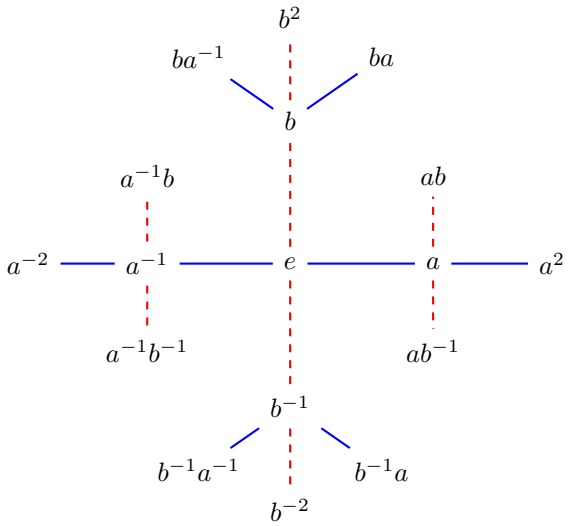
Each edge $(g_1, g_1 s)$ is labelled by the generator $s \in S$. The graph is equipped with the **word metric**: the distance $d_S(g_1, g_2)$ between two vertices is the length of the shortest path (word in $S \cup S^{-1}$) connecting g_1 to g_2.

Important properties.

- G acts on $\mathrm{Cay}(G, S)$ by left multiplication, $g \cdot h = gh$. This action is free, transitive on vertices, and preserves the graph structure; in particular the Cayley graph is *vertex-transitive*.
- The graph is *regular* of degree $|S| + |S^{-1}|$, counting inverses as separate generators when they are not already in S.
- Different choices of generating set S can yield very different-looking Cayley graphs, even for the same group G.

Example: the Free Group $F_2 = \langle a, b \rangle$

The Cayley graph of F_2 with generators $\{a, b\}$ is an infinite 4-**regular tree**: each vertex has exactly four edges, labelled a, a^{-1}, b, b^{-1}. The figure below shows the neighbourhood of the identity e up to distance 2.



Blue solid edges: multiplication by $a^{\pm 1}$. Red dashed edges: multiplication by $b^{\pm 1}$.

Example: $\mathbb{Z}$ with Two Different Generating Sets

Consider the two presentations of the integers:

1. $\mathbb{Z} = \langle a \rangle$: the edges only go $\cdots \to X_{-1} \xrightarrow{a} X_0 \xrightarrow{a} X_1 \xrightarrow{a} X_2 \to \cdots$

2. $\mathbb{Z} = \langle a, b \mid b = a^2 \rangle$: we add the generator b, acting as a jump of $+2$, so that in addition to the a-edges we have the b-edges $\cdots,\ X_{-1} \xrightarrow{b} X_1,\ X_0 \xrightarrow{b} X_2,\ X_1 \xrightarrow{b} X_3,\ \cdots$

The two Cayley graphs are *not isometric* as metric spaces, since the distances change, and yet they encode the same underlying group. This motivates a coarser notion of equivalence: **quasi-isometry**.

Exercise. Let $G = \langle S_1 \mid R_1 \rangle = \langle S_2 \mid R_2 \rangle$ be two presentations of the same group G. Show that $\mathrm{Cay}(G, S_1)$ and $\mathrm{Cay}(G, S_2)$ are quasi-isometric. *Hint*: consider the identity map on G, viewed as a map between the two metric spaces, and estimate the distortion in terms of the word lengths of the generators of S_1 expressed in S_2 and vice versa.

Quasi-Isometry

Definition

Definition (Quasi-Isometry). Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is a (λ, C, k)-**quasi-isometry**, with constants $\lambda \geq 1$, $C \geq 0$, $k \geq 0$, if the following two conditions hold.

1. **Coarse Lipschitz condition:**
$$\lambda^{-1} d_X(x, y) - C \ \leq\ d_Y\Big(f(x), f(y)\Big) \ \leq\ \lambda\, d_X(x, y) + C \qquad \forall\, x, y \in X. \tag{57}$$

2. **Coarse surjectivity:**
$$N_k\Big(f(X)\Big) = Y, \tag{58}$$
where $N_k(A) = \{y \in Y \mid \exists\, a \in A,\ d_Y(y, a) < k\}$ denotes the open k-neighbourhood of A in Y.

Two metric spaces X and Y are said to be **quasi-isometric**, written $X \overset{\text{Q.I.}}{\sim} Y$, if there exists a quasi-isometry between them.

Quasi-isometry is an equivalence relation on the class of metric spaces. The first condition allows for uniform multiplicative and additive distortion of distances, while the second ensures that f is ``almost surjective" up to a bounded error. Crucially, quasi-isometry ignores all bounded-scale geometry and only captures the *large-scale* (coarse) structure of a space.

Examples

- **Bounded spaces.** Any bounded metric space is quasi-isometric to a single point $\{\star\}$. Indeed, the constant map $f(x) = \star$ satisfies the coarse Lipschitz condition

trivially, since all distances on the right are 0 and all distances on the left are bounded by the diameter D of X, and coarse surjectivity holds with $k > 0$.

- $\mathbb{Z}^n$ **and** $\mathbb{R}^n$**.** The inclusion $\iota : \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a quasi-isometry. The coarse Lipschitz condition holds with $\lambda = 1$ and $C = 0$, and coarse surjectivity holds with $k = \sqrt{n}/2$, since every point of $\mathbb{R}^n$ lies within distance $\sqrt{n}/2$ of some lattice point.
- $\mathbb{Z}^n$ **and** F_2 **are *not* quasi-isometric.** The free group F_2 has **exponential** volume growth --- the ball of radius r in $\mathrm{Cay}(F_2, \{a, b\})$ contains $\sim 3 \cdot 2^r$ elements --- whereas $\mathbb{Z}^n$ has **polynomial** growth, the ball of radius r containing $\Theta(r^n)$ elements. Since quasi-isometries preserve the type of volume growth (polynomial versus exponential), $\mathbb{Z}^n$ and F_2 are not quasi-isometric.

Key Theorem (Milnor--Švarc Lemma). If a group G acts properly and cocompactly by isometries on a proper geodesic metric space X, then G is finitely generated and $G \overset{\text{Q.I.}}{\sim} X$. In particular, any two Cayley graphs of the same finitely generated group are quasi-isometric to each other.

Hyperbolic Spaces in the Sense of Gromov

Definition

Definition (δ-Hyperbolic Space). A geodesic metric space (X, d) is δ-**hyperbolic** (in the sense of Gromov), for some $\delta \geq 0$, if for every triple of points $x, y, z \in X$ the geodesic segment $[x, z]$ is contained in the δ-neighbourhood of the union of the other two sides of the geodesic triangle:

$$[x, z] \;\subseteq\; N_\delta\Big([x, y] \cup [y, z]\Big). \tag{59}$$

Equivalently, every geodesic triangle in X is δ-**slim**: each side lies within a δ-neighbourhood of the union of the other two sides.

Intuition. In classical hyperbolic geometry $\mathbb{H}^n$, geodesic triangles are ``thin'': they look more and more like tripods as their perimeter grows. Gromov's definition abstracts this thinness condition to a combinatorial and metric setting, which makes it applicable to graphs and to finitely generated groups.

Quasi-Isometry Invariance

Proposition. Gromov hyperbolicity is a quasi-isometry invariant among proper geodesic metric spaces. That is, if (X, d_X) is δ-hyperbolic and $f : X \to Y$ is a quasi-isometry, then (Y, d_Y) is δ'-hyperbolic for some $\delta' \geq 0$ depending only on δ and on the quasi-isometry constants.

This invariance is fundamental: it means that hyperbolicity is a *group-theoretic* property, not an artefact of the choice of generating set, and it is what makes the definition below well posed.

Examples of Hyperbolic Spaces

- **Bounded spaces.** Any bounded metric space is δ-hyperbolic with $\delta = \operatorname{diam}(X)$, since all geodesic triangles fit inside a ball of that radius.
- $\mathbb{Z}$ **is hyperbolic.** The integers with the standard metric form a geodesic space quasi-isometric to $\mathbb{R}$, which is 0-hyperbolic: all geodesic triangles are degenerate, being simply intervals on the real line.
- **Scaling construction.** Let X be a metric space, for instance a graph. Define a sequence of rescaled spaces

$$X_i = (X, d_i), \qquad d_i(x, y) = \frac{d(x, y)}{2^i}. \tag{60}$$

  The union $\bigcup_i (X_i, d_i)$, with an appropriate gluing, yields a new hyperbolic space. This construction is related to the definition of the Gromov boundary via ultralimits.
- **Simplicial trees.** Any simplicial tree T --- a connected acyclic graph, viewed as a metric space in which each edge has length 1 --- is 0-hyperbolic: every geodesic triangle in a tree is a tripod, so each side is literally contained in the union of the other two, with $\delta = 0$.

Hyperbolic Groups

Definition (Hyperbolic Group). A finitely generated group G is **(Gromov) hyperbolic** if its Cayley graph $\operatorname{Cay}(G, S)$ is δ-hyperbolic for some $\delta \geq 0$. By the quasi-isometry invariance of hyperbolicity together with the Milnor--Švarc lemma, this definition does not depend on the choice of the finite generating set S.

Examples.

- **Finite groups:** any finite group is hyperbolic, since its Cayley graph is bounded.
- $\mathbb{Z}$**:** hyperbolic, as its Cayley graph is quasi-isometric to $\mathbb{R}$.
- **Free groups** F_n**:** their Cayley graphs are trees, hence 0-hyperbolic.
- **Fundamental groups of closed hyperbolic surfaces:** the surface group $\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] \rangle$ is hyperbolic, since it acts properly and cocompactly on the hyperbolic plane $\mathbb{H}^2$, by the Milnor--Švarc lemma.

- **Non-example:** $\mathbb{Z}^2$ is *not* hyperbolic, as geodesic triangles in $\mathbb{Z}^2$ can be arbitrarily fat.

The Gromov Boundary of a Hyperbolic Group

Motivation

The **Gromov boundary** ∂X of a hyperbolic space X is a compact topological space capturing the ``ideal boundary" of X, that is, the set of directions in which one can go to infinity. It is a quasi-isometry invariant up to homeomorphism, and it plays a central role in the large-scale geometry and the dynamics of hyperbolic groups.

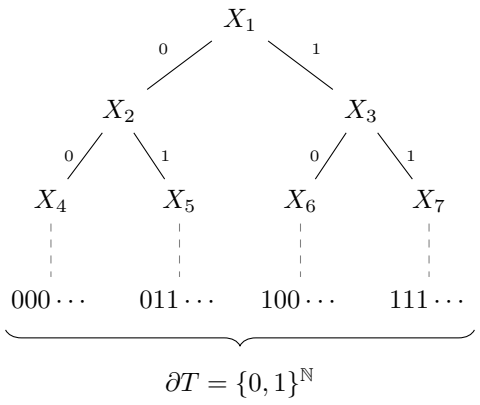
Boundary of a Simplicial Tree: the Cantor Set Structure

We illustrate the construction of the Gromov boundary through the concrete example of a **full binary tree** T.

Labelling. Root the tree and label the vertices level by level: the root is X_1, and the two children of X_i are X_{2i} (left, edge labelled 0) and X_{2i+1} (right, edge labelled 1):

$$X_i \xrightarrow{0} X_{2i}, \qquad X_i \xrightarrow{1} X_{2i+1}. \tag{61}$$

The first three levels of the tree, together with four infinite boundary rays, are depicted below.



Encoding paths. Every infinite path starting from the root is encoded by an infinite binary sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$, read off by following the $0/1$ labels at each level. The Gromov boundary is therefore

$$\partial T = \{0,1\}^{\mathbb{N}} = \left\{ \sum_{i=0}^{\infty} a_i 2^i \;\middle|\; a_i \in \{0,1\} \right\}, \tag{62}$$

which is homeomorphic to the **Cantor set**.

Metric on the boundary. Given two boundary points $\varepsilon_1, \varepsilon_2 \in \{0,1\}^{\mathbb{N}}$, let $c(\varepsilon_1, \varepsilon_2)$ denote the length of their longest common prefix, that is, the depth of the deepest common ancestor in the tree. The metric on ∂T is

$$d(\varepsilon_1, \varepsilon_2) = 2^{-c(\varepsilon_1, \varepsilon_2)}. \tag{63}$$

Two boundary points are close if and only if the corresponding infinite paths agree for a long time, that is, share a deep common ancestor. This is an **ultrametric**: it satisfies $d(\varepsilon_1, \varepsilon_3) \leq \max\{d(\varepsilon_1, \varepsilon_2), d(\varepsilon_2, \varepsilon_3)\}$, a strengthening of the triangle inequality.

General Gromov boundary. For any δ-hyperbolic group G, the Gromov boundary ∂G is a compact metrizable space which

- is empty if and only if G is finite;
- consists of two points if and only if G is virtually cyclic, that is, quasi-isometric to $\mathbb{Z}$;
- is a Cantor set if G is a non-elementary free group or a surface group;
- is homeomorphic to the sphere S^{n-1} if G is the fundamental group of a closed hyperbolic n-manifold.

Summary of Key Concepts

Concept	Definition / Condition	Key example
Cayley graph $\mathrm{Cay}(G, S)$	$(g_1, g_2) \in E \iff g_2 = g_1 s$, $s \in S$	F_2: infinite 4-regular tree
Word metric d_S	Length of the shortest word connecting g_1 and g_2	$d_{\{a\}}(n, m) = \lvert n - m \rvert$
Quasi-isometry	Bilipschitz up to an additive constant, and coarsely surjective	$\mathbb{Z}^n \overset{\text{Q.I.}}{\sim} \mathbb{R}^n$
δ**-hyperbolicity**	Every geodesic triangle is δ-slim	Trees: $\delta = 0$
Hyperbolic group	$\mathrm{Cay}(G, S)$ is δ-hyperbolic	F_n, surface groups
Gromov boundary ∂G	Infinite geodesic rays modulo finite Hausdorff distance	$\partial T = \{0,1\}^{\mathbb{N}} \cong$ Cantor set

تقرير صبيحة اليوم الأخير

حبي عبد الرحمن

Quantum Physics Session

The session began at 8:50 and quickly immersed us into one of the central ideas of quantum mechanics: the non-commutativity of observables.

We considered three operators corresponding to spin measurements:

$$S_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad S_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad S_y = \frac{1}{2}\sigma_y,$$

and we were told that:

$$[S_x, S_z] \neq 0.$$

This simple relation already carries a deep meaning: measurements in different directions are incompatible.

We then focused on the eigenstates of S_z. The system can only take two possible values:

$$|+\rangle_z = (1,0), \quad |-\rangle_z = (0,1),$$

with corresponding eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$.

However, we considered a state aligned in a different direction:

$$|\psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}(1,1).$$

Rewriting this state in the z-basis gives:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z).$$

This decomposition revealed an important phenomenon: a state that is perfectly determined in the x direction becomes a superposition in the z direction.

We then computed the expected value of S_z:

$$\langle S_z\rangle = \frac{1}{2}P_+ + \left(-\frac{1}{2}\right)P_- = 0,$$

since both outcomes occur with equal probability.

However, the possible measurement outcomes remain:

$$+\frac{1}{2} \quad \text{or} \quad -\frac{1}{2}.$$

We then calculated the variance:

$$\text{Var}(S_z) = \langle S_z^2 \rangle = \frac{1}{4}, \quad \Delta S_z = \sqrt{\text{Var}(S_z)} = \frac{1}{2}.$$

This showed that the state is not dispersion-free: there is an intrinsic uncertainty in the measurement.

intuition = even if the average is zero, the system is fluctuating between two extremes

This naturally led to the uncertainty principle:

$$\text{Var}(A)\text{Var}(B) \geq \frac{1}{4}|\langle [A, B] \rangle|^2.$$

This inequality expresses a fundamental limitation: certain pairs of observables cannot be simultaneously known with arbitrary precision.

intuition = non-commuting quantities cannot be sharply defined at the same time

A natural question was then raised: does there exist a deeper description of reality using hidden variables, where all quantities would be dispersion-free?

This question led us to consider a physical example: a particle that exists only for a very short time before decaying into two particles, such as e^+ and e^-. The system is described by an entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle).$$

This state cannot be separated into two independent parts; it encodes correlations between the two particles.

At this point, the discussion turned to the famous EPR paradox (Einstein--Podolsky--Rosen), which questions whether quantum mechanics provides a complete description of reality.

The paradox suggests that if quantum mechanics is correct, then measuring one particle instantly affects the other, no matter how far apart they are. This leads to the idea of wave function collapse.

intuition = measuring one part of the system instantly determines the other

The session ended with more questions than answers, leaving us with a mix of fascination and confusion about the true nature of reality in quantum mechanics.

Partial Differential Equations Session

The session began at 12:27 and immediately focused on the classic problem: the heat equation in one space dimension.

We considered an initial condition $u_0 \in L^2(\mathbb{R})$ and a solution given formally by convolution with the heat kernel:

$$U(t, \cdot) = U_0 * G_t,$$

which satisfies

$$u(t, x) > 0 \quad \text{for all } t > 0,\ x \in \mathbb{R}.$$

The PDE under study was:

$$\partial_t u - \partial_x^2 u = 0, \quad (t, x) \in [0, 1] \times [0, 1].$$

We then discretized the domain:

- Let $N \in \mathbb{N}^*$ with $\Delta x = 1/N$, and $x_j = j\Delta x$, $j = 0, 1, \ldots, N - 1$.
- Let $M \in \mathbb{N}^*$ with $\Delta t = 1/M$, and $t^n = n\Delta t$, $n = 0, 1, \ldots, M$.
- Denote the discrete solution by $u_j^n \approx u(t^n, x_j)$.

The forward-time centered-space (FTCS) scheme was introduced:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}.$$

We discussed the **CFL condition** for stability:

$$\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$

This ensures that errors do not grow:

$$\text{errors}^{n+1} \leq \text{errors}^{n}, \quad \forall n.$$

Next, we considered convergence in the limit:

$$N, M \to \infty, \quad u(t^n, x_j) \to u(t, x),$$

with an error estimate

$$u(t^n, x_j) = O(\Delta t) + O(\Delta x^2),$$

showing **first-order accuracy in time** and **second-order accuracy in space**.

Finally, we implemented a **simulation of the heat equation**. Watching the diffusion of an initial peak was particularly satisfying: the solution spreads smoothly, exactly as the theory predicts.

intuition = heat flows from peaks to valleys, and the discrete scheme mirrors this diffusion if the CFL cond

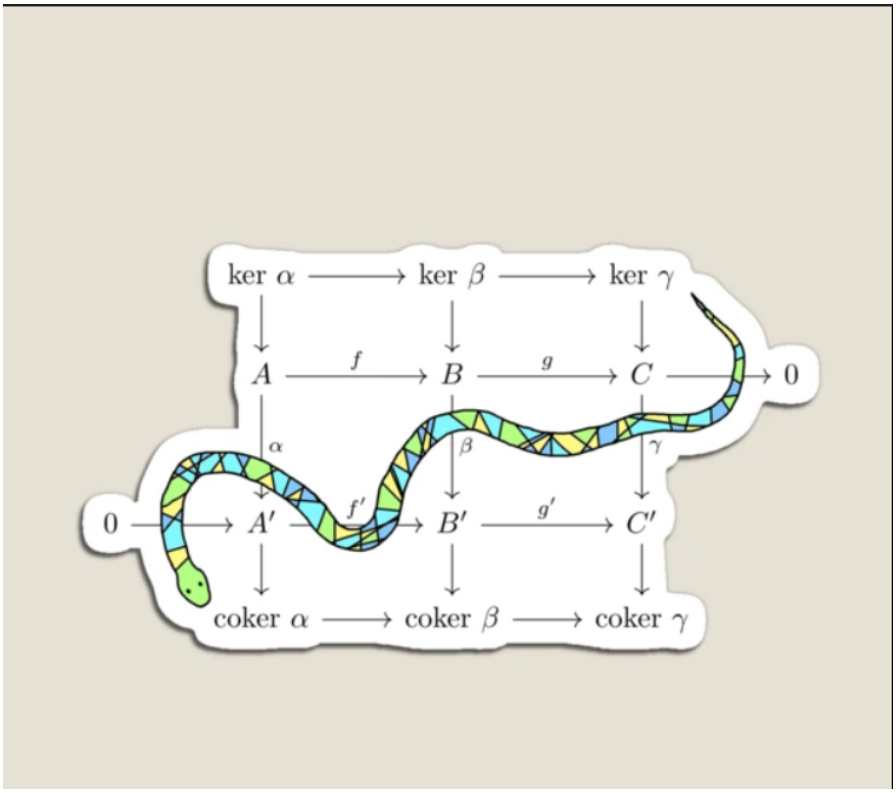
The session ended with discussions about:

- How the limit of the scheme satisfies the continuous heat equation.
- The balance between Δt and Δx to ensure convergence.
- The visual intuition provided by the simulation.

Category Theory Session

The session began at 10:41 with a clear goal: to understand and prove the **Snake Lemma**.

We started by drawing the snake lemma diagram on the board:



After that, we had some time to try proving the lemma ourselves. It was challenging but very insightful — the exercisecise forced us to think carefully about **exact sequences** and **connecting morphisms**.

Once we struggled with parts of the proof, the instructor guided us through a formal proof. We referenced standard sources online and rewrote the proofs in our notes. The key idea was:

- Start with a commutative diagram of exact sequences
- Construct the connecting morphism carefully
- Verify exactness at each step

Proof of the Snake Lemma

Statement: Given a commutative diagram with exact rows in an abelian category:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

there exists a long exact sequence:

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \xrightarrow{\delta} \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0.$$

Proof (diagram chasing):

1. The maps $\ker \alpha \to \ker \beta \to \ker \gamma$ are induced naturally by the diagram, and exactness at the first two positions follows from the exactness of the original rows.

2. To define the connecting homomorphism $\delta : \ker \gamma \to \operatorname{coker} \alpha$, take $x \in \ker \gamma \subset C$. Since $g : B \to C$ is surjective (top row exact), pick $y \in B$ with $g(y) = x$. Then
$$g'(\beta(y)) = \gamma(g(y)) = \gamma(x) = 0,$$
so $\beta(y) \in \ker g' = \operatorname{im} f'$ (bottom row exact). Write $\beta(y) = f'(z)$ for $z \in A'$, and define
$$\delta(x) := z + \operatorname{im} \alpha \in \operatorname{coker} \alpha.$$

3. Check well-definedness: if y' is another preimage of x, then $y - y' \in \ker g = \operatorname{im} f$, say $y - y' = f(w)$ with $w \in A$. Then
$$\beta(y') = \beta(y - f(w)) = \beta(y) - f'(\alpha(w)) = f'(z) - f'(\alpha(w)),$$
so the class in $\operatorname{coker} \alpha$ is independent of the choice of y.

4. Exactness at each step of the long sequence follows from a careful diagram chase, using exactness of the original rows.

This completes the proof of the Snake Lemma.

Next, we moved on to the **Freyd–Mitchell Embedding Theorem**. The statement was:

> If $\mathcal{C}$ is a small abelian category, there exists a ring R (with 1) and a full and faithful functor
> $$\mathcal{C} \longrightarrow R\text{-Mod}.$$

We discussed the meaning of "full" and "faithful":

- Let $F : \mathcal{C}_1 \to \mathcal{C}_2$
- F is **full** if for all $A, B \in \text{Obj}(\mathcal{C}_1)$:
$$F : \text{Mor}(A, B) \to \text{Mor}(F(A), F(B)) \quad \text{is surjective.}$$
- F is **faithful** if the map above is injective.

Finally, we discussed R-**modules**. If R is a ring, an R-module $(M, +)$ is an abelian group equipped with a scalar multiplication map

$$R \times M \longrightarrow M, \quad (r, m) \mapsto r \cdot m,$$

satisfying the familiar axioms:

$$\begin{aligned} r \cdot (s \cdot m) &= (rs) \cdot m, \\ r \cdot (m + n) &= r \cdot m + r \cdot n, \\ (r + s) \cdot m &= r \cdot m + s \cdot m, \\ 1_R \cdot m &= m. \end{aligned}$$

Intuition: modules are "vectors with rings instead of fields", capturing abelian structure plus scalar multiplication.

The session closed with discussions about how **exact sequences, functors, and modules** interact, laying the groundwork for a deep understanding of homological algebra.

تقرير عشية اليوم الأخير

زياني نزيم زين الدين

تاريخ النشاط: الأحد 29 مارس **توقيت الحفل:** من 30 : 17 إلى 30 : 18 مساءً

مقدمة

يُعد المخيم السادس للرياضيات من بين أبرز المبادرات العلمية التي تهدف إلى جمع الطلبة الشغوفين بعلم الرياضيات من مختلف أنحاء الوطن، في إطار علمي وتفاعلي يسعى إلى تنمية مهاراتهم الفكرية وتعزيز روح التعاون بينهم.

وقد شكل هذا الحدث فرصة مميزة لتبادل المعارف والخبرات، وكذا خلق بيئة محفزة على الإبداع والتفكير النقدي، مما جعله تجربة ثرية بكل المقاييس.

أجواء الاختتام

كان اليوم الأخير، أو بالأحرى اللحظات الأخيرة، مساء الأحد 29 من أبريل.

ترى بعض الحزن على وجوه الحضور من طلبة، أساتذة، دكاترة ومؤطرين. كيف لا وهم يعيشون اللحظات الأخيرة من المخيم السادس للرياضيات، فبعد حماس اللقاء وحميميته لا بد أن يفترقوا للقاء آخر.

ورغم هذا الحزن، فقد طغت مشاعر الفخر والاعتزاز بما تم تحقيقه خلال هذا المخيم، حيث عبر العديد من المشاركين عن رضاهم الكبير عن التجربة.

سير المراسم

انطلقت مراسم الاختتام على الساعة 30 : 17 مساءً، حيث ابتدأت بجلسة تكريمية حضرها عدد من الضيوف.

افتتح الجلسة الأستاذ علي سلاطنه مرحباً بالحضور، ولم يقصر في إرسال تحيته لمن غاب عن الأمسية لظروف خاصة، كالأستاذ زغيب والفنان عبد الرحمن غزلي.

وصادف ذلك اليوم وفاة الرئيس الأسبق للدولة الجزائرية لمين زروال، فكان لا بد من القيام بوقفة صمت لذكراه، في جو من الاحترام والتقدير.

الكلمات والتدخلات

انتقلت الكلمة مباشرة للسيد مسعود عبد اللاوي، الذي قدم شكره وتقديره للجميع، وخص بالذكر جهود الدكتور علاء وجمعية الخوارزمية، مشيدًا بالدور الكبير الذي قاموا به في إنجاح هذا الحدث.

بعد ذلك، ألقى الدكتور علاء كلمة عبّر فيها عن امتنانه لوجود مثل هذه المناسبات التي تجمع الشباب الشغوف بالرياضيات من مختلف أنحاء الوطن، مؤكّدًا أن مثل هذه المبادرات تساهم بشكل كبير في تنمية هذا الشغف وتعزيزه.

كما انتقلت الكلمة إلى الأستاذ حمزة خليف، الذي رفض في البداية تواضعًا وخجلاً أن يقف على منصة التشريف، غير أن إلحاح الحضور دفعه إلى تلبية الطلب، حيث ألقى كلمات تحفيزية مميزة، أشاد فيها بالإقبال الكبير، وأكد على ضرورة استمرار مثل هذه المبادرات، معبرًا عن أمله في أن تكون هذه نقطة بداية لا نهاية.

وأشاد كذلك بالسيدة كريمة سايح التي غابت عن الحضور لكنها تركت أثرًا واضحًا في المخيم، وتأثرت بدورها بهذه التجربة.

وختم مداخلته بعبارة عميقة تستحق التدبر:

> لا يزال المرء يطلب العلم حتى إذا ظن أنه صار عالمًا فقد جهل.

فقرة التكريمات

انتقل الحضور بعد ذلك إلى فقرة التكريمات، حيث تم تكريم عدد من الشخصيات التي ساهمت في إنجاح هذا الحدث.

فتم تكريم ممثل الوادي ومدير الجامعة السيد فرحاتي، وكذا خليفة رئيس المجلس الشعبي من طرف الأستاذ سمير.

كما تم تكريم السيد محمودي أحمد، مدير التكوين، والذي كان غائبًا عن الحضور، فتسلمها ممثل الوادي نيابة عنه، وذلك من طرف السيد عبد اللاوي.

وشملت التكريمات أيضًا كل من:

- السيد عثمان، مدير مركز التمهيين
- السيد بالغزالة ياسين، مدير دار الثقافة الذي استقبل الوفود دون ترخيص
- محمد رضا، رئيس المجلس الشعبي الولائي

وقد تمت هذه التكريمات من طرف الدكتور علاء، عرفانًا بالجهود المبذولة.

الانطباعات والتقييم

خلال هذا المخيم، برزت عدة جوانب إيجابية، من أهمها:

- التنظيم الجيد للمحاضرات والأنشطة
- التفاعل الكبير بين المشاركين
- المستوى العلمي المتميز للمحتوى المقدم
- روح التعاون والألفة بين الطلبة

كما أبدى العديد من المشاركين رغبتهم في تكرار مثل هذه المبادرات، وتوسيع نطاقها لتشمل عددًا أكبر من الطلبة.

الخاتمة

مرت حوالي ساعة منذ بداية الجلسة، لتصل المراسم إلى نهايتها بصورة جماعية من أمام المدرج، تخللتها كلمات عفوية للطالب نصرو بالمدرسة الوطنية العليا للرياضيات، أبدى من خلالها كيف

أن هذا المخيم، الذي كان من المفترض أن يكون متواضعًا، قد فاق تصوراته.

وفي الختام، يمكن القول إن هذا المخيم لم يكن مجرد نشاط علمي عابر، بل تجربة إنسانية وعلمية متكاملة، تركت أثرًا عميقًا في نفوس المشاركين، وستبقى ذكراها محفورة في أذهانهم، في انتظار لقاءات قادمة أكثر إشراقًا وتميزًا...